

YALE DIRECTED READING PROGRAM, SPRING 2016

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ABSTRACT. This note gathers the project descriptions for the Directed Reading Program in Mathematics at Yale University in the Spring 2016, coordinated by Thomas Hille and Lam Pham.

The Directed Reading Program pairs undergraduate students with graduate student mentors to read and work through a mathematics text over the course of one semester. The pairs meet once each week for one hour, with the undergraduates expected to do about 4 hours of independent reading per week. At the end of the semester, undergraduates either give a talk to their peers or prepare a short exposition of some of the material from the semester. Undergraduates are expected to have a high level of mathematical maturity and eagerness to learn the topic.

There are 15 projects proposed with 8 graduate student mentors. Note that for graduate students with more than one project listed, not all projects may be offered.

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1. EXACTLY SOLVED MODELS IN STATISTICAL MECHANICS (BY EFIM ABRIKOSOV)

1.1. **Description.** Statistical Mechanics is a big branch of modern physics that studies properties of macroscopic systems, typically having large number of degrees of freedom, for example, gases or fluids. In such situation as one can suspect it is too difficult to give precise answers about microscopic behaviour of systems. However, some macroscopic characteristics still can be investigated and that amounts to finding certain probability distributions. The book we will follow is [Bax89].

1.2. **Background and prerequisites.** Measure Theory (Math 320) and Probability Theory (Stat 600), some familiarity with analysis. Be ready to learn new techniques on the way.

1.3. **Syllabus.** It would be too ambitious to understand the whole book in this program, and I expect that several digressions to more elementary textbooks will be necessary. I think that developing of understanding of two-dimensional Ising model and its main properties, as well as how general ideas of statistical mechanics work in this particular case is already a very good goal for the program.

This program is mainly suggested to junior and senior students interested in mathematical physics

1.4. **Development and topics covered.** The book [Bax89] is a classical mathematical text, that focuses on study of so called two-dimensional lattice models. In this case rigorous mathematical foundations can be set up and a lot of beautiful mathematics emerge. Topics covered include

- Ising model
- Phase transition
- Potts model
- Stochastic processes on graphs

2. INTRODUCTION TO MATHEMATICAL PHYSICS (BY DYLAN ALLEGRETTI)

2.1. Description. Main Reference: [BM94]. I was a rising senior when I first read [BM94]. I had taken courses in analysis and abstract algebra, and I had studied vector calculus in my physics courses. I don't think the student necessarily needs all this background, but a certain amount of mathematical maturity and interest in physics is essential.

2.2. Background and prerequisites. Main prerequisites are linear algebra and vector calculus, at the level of Math 225-250 or Math 230-231.

2.3. Syllabus and primary goal. It should be possible to customize the syllabus based on the interests and ability of the student. The first few chapters of the book cover basic differential geometry, including the theory of manifolds, vector fields, and differential forms. These concepts are used to formulate Maxwell's equations on arbitrary spacetime manifolds. The second part of the book presents the theory of vector bundles and connections and uses these concepts to discuss gauge theory and its relation to knots. The final part of the book explains Riemannian geometry and its applications in general relativity.

Ideally, I would like to at least get to the section on knot theory, but in principle we could stop anywhere and it would still be a satisfying experience for the student. Actually, I think there's a danger that we might finish the book too soon. If that happens, there are plenty of online materials and texts that I can share with the student.

2.4. Development and topics covered. The topics covered in this project could include any of the following:

- Basic differential geometry
- Representations of Lie groups and Lie algebras
- Vector bundles and connections
- Yang-Mills theory
- Knots and Chern-Simons theory
- Riemannian geometry
- General relativity

If we finish all of those topics, we could also talk about

- Topological quantum field theory
- Three dimensional gravity

3. ARITHMETIC GEOMETRY (BY VESSELIN DIMITROV)

3.1. Description. The reference is [BG06]. The book has 600 pages including its three long appendices, and I will give the applicant some choice and flexibility. Many of its parts can be studied independently, and the exposition is extremely clear – and self-contained.

3.2. Background and prerequisites. Prerequisites include a solid background in abstract algebra (preferably Math 380-381), including rudiments of Galois theory, and a strong interest in number theory.

3.3. Syllabus and primary goal. This project is a rigorous introduction to modern arithmetic geometry. No previous exposure to algebraic geometry is required, although that would be helpful. We will start with the appendix on algebraic geometry and set up the Weil heights in the opening two chapters and then, depending on the student's interests and background, we can follow one (or two) of the following topics:

- Abelian varieties, Chevalley-Weil theorem, Mordell-Weil theorem (chapters 8 and 10);
- Roth's theorem and Schmidt's Subspace theorem (chapters 6 and 7);
- Roth's theorem and Faltings's theorem (chapters 6, 8, 9, 10, and 11);
- Topics on the unit equation (chapter 5 and related papers);
- Small points and equidistribution (chapters 3 and 4, followed by papers from the literature);
- The abc conjecture and related topics (chapter 12 and related papers). Assuming a good background in complex analysis this may be followed by chapter 13 on Nevanlinna's valued distribution theory and chapter 14 on the Vojta conjectures, which is in many ways the culmination and synthesis of the book.

3.4. Development and topics covered. Topics include:

- Basic algebraic geometry
- Heights
- Weil heights
- Roth's theorem
- Mordell-Weil theorem
- The abc conjecture

4. INTRODUCTION TO STOCHASTIC CALCULUS (BY THOMAS HILLE)

4.1. Description. Following [KS91], the aim of this project is to become familiar with two of the main concepts in probability theory, namely Markov processes and martingales. Our main example of both concepts will be Brownian motion in \mathbb{R}^d . One of the main applications of the notion of martingales is its connection to partial differential equations, which leads to the study of integration with respect to stochastic processes and in turn to the study of so-called stochastic differential equations.

4.2. Background and prerequisites. Measure Theory (Math 320) and Functional Analysis (Math 325). Some familiarity with stochastic processes in discrete time would be desirable, but not required. More precisely we will need a good understanding of the Riemann-Stieltjes integral for functions of bounded variation, Radon-Nikodym derivatives, Hahn-Banach theorem(s), the Riesz representation theorem (for locally compact spaces), etc.

4.3. Syllabus and primary goal. We will start by reviewing Chapter 1 of [KS91]. Following this, we will give a construction of Brownian motion, study the classical Wiener space and study some of the main properties of Brownian motion. After these two basic chapters, we will be able to start constructing the stochastic integral and study the change-of-variable formula for this integral, the Ito Lemma. Chapter 4 and 5 give connections between martingales and PDE's, most importantly, the Feynman-Kac formula.

4.4. Development and topics covered.

- Stochastic processes and Filtrations in continuous time. Stopping times.
- Martingales: convergence results and optional sampling theorem. Doob-Meyer decomposition result.
- Brownian motion and Wiener space. Canonical Brownian filtration, Blumenthal's 0-1 law, nowhere differentiability, LIL, modulus of continuity for Brownian motion.
- Construction of the stochastic integral for continuous semimartingales.
- The Ito lemma.
- The Girsanov theorem.
- Stochastic differential equations. Strong solutions and weak solutions.
- The martingale problem.
- Feynman-Kac representation.

If we finish all of these topics we can also talk about (non-continuous) stochastic differential equations on Fréchet spaces, e.g. stochastic differential equations on the space of measures. In this case we will study general Banach-valued semimartingales, the notion of measurability of Banach-valued random variables (which is drastically different from the finite-dimensional case) and the celebrated result of Métivier on the representation of semimartingales.

5. INTRODUCTION TO LIE GROUPS AND LIE ALGEBRAS (BY THOMAS HILLE AND LAM PHAM)

5.1. Description. This is an introduction to Lie groups and Lie algebras. We start with the book [Kir08] which gives a nice introduction to Lie groups and Lie algebras. We will review the necessary differential geometry along the way. We then move onto Lie algebras, and time permitting, we will study in more depth Lie groups using [Kna02].

5.2. Background and prerequisites. Familiarity with Differential Geometry, Abstract Algebra (Math 350).

5.3. Syllabus and primary goal. Ideally, since [Kir08] is quite short, we would cover most of it.

We start with the basic definitions of Lie groups and Lie algebras. We then follow with basic representation theory of Lie groups and Lie algebras, and structure theory of Lie algebras and root systems.

The goal is to build a good knowledge of general Lie groups (as opposed so “only” matrix Lie groups, even though these will be our most important examples).

5.4. Development and topics covered.

- Lie algebras. Ideals and commutant. Nilpotent and solvable Lie algebras. Lie’s and Engel’s theorems.
- Semisimple and reductive Lie algebras. Killing form and Cartan’s criterion for semisimplcity. Jordan decomposition.
- Representations of $\mathfrak{sl}_2(\mathbb{C})$
- Finite-dimensional representations. Weights, highest weight, Verma modules.
- Lie groups. Review of basic differential geometry.
- Compact Lie groups. Peter-Weyl theory.
- Structure theory of semisimple groups. Cartan decomposition, Iwasawa decomposition.
- Reductive Lie groups. KAK decomposition and Bruhat decomposition.
- Haar measure for Lie groups

If we cover all of these topics, we can study real rank 1 subgroups, parabolic subgroups, Cartan subgroups and Harish-Chandra decomposition.

6. $SL_2(\mathbb{R})$ (BY THOMAS HILLE)

6.1. Description. The main reference for this will be [Lan85] and it is mainly an introduction through $SL_2(\mathbb{R})$ to the infinite dimensional representation theory of semisimple Lie groups. We don't need any knowledge of Lie theory here.

6.2. Background and prerequisites. Measure theory and functional analysis mainly. More precisely we need a good understanding of the Haar measure and different versions of spectral theorems, which can be reviewed throughout the project. Other (less important) prerequisites include complex analysis

6.3. Syllabus and primary goal. The primary goal is to become acquainted with analysis on groups by focusing on the example of $SL_2(\mathbb{R})$, it's (discrete) subgroups and homogeneous spaces. We will start by reviewing the Haar measure, the associated modular function, mollifiers on manifolds, (unitary) representations, the spectral theorem for compact operators and trace class operators. We will go through these topics very quickly, so make sure you have seen 4 out of those 6 topics. We then will look at the representation theory over compact groups, first in the context of $SO(2)$ and then for general compact groups. We then study (the Haar measure on) $SL_2(\mathbb{R})$ by decomposing it as a product of certain closed subgroups, the main example being the Iwasawa decomposition and associated spherical functions. This will lead to the study of spherical function and the spherical transform, which give a close connection to (well-known integrals from) complex analysis. This will conclude the first half of the project, which is mainly (functional) analytic. After this, we will start to deal with the analytic structure of $SL_2(\mathbb{R})$ and study how a representation of $SL_2(\mathbb{R})$ induces a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ (this is a little bit different from the usual case, as we are dealing with general Banach or Hilbert spaces) and we will classify all unitary irreducible representations. We then study the Plancherel formula for $SL_2(\mathbb{R})$, if necessary we can review more facts about trace class operators. This would finalize the main techniques for this project. Once we reach this point, I would suggest studying representations on $L^2(\Gamma \backslash G)$, but we could also study the spectral decomposition of the Laplace operator on $\Gamma \backslash \mathbb{H}$.

6.4. Development and topics covered.

- Compact groups.
- Integration on coset spaces.
- Spherical functions and spherical transform. Harish and Mellin transform.
- Derived representation.
- Plancherel formula.
- Representation on $L^2(\Gamma \backslash G)$ or spectral decomposition of the Laplace operator on $\Gamma \backslash \mathbb{H}$.

7. ESSENTIALS OF STOCHASTIC PROCESSES (BY OANH NGUYEN)

7.1. Description and main goal. The main reference is [Dur12]. Stochastic processes constitute an important subject in probability theory and have strong connection with ergodic theory, analysis, theoretical computer science, etc.

7.2. Prerequisites. Familiarity with probability would be helpful.

7.3. Syllabus. We read the book Essentials of Stochastic Processes by Richard Durrett. This book is designed for an introductory course in stochastic processes for undergraduate and graduate students. The material is illustrated well by a lot of examples. We try to understand the processes and some classical results about them.

7.4. Development and topics covered. Topics include the study of different stochastic processes:

- Markov chains
- Poisson processes
- Renewal processes
- Martingales

If time permits, we will learn some applications in mathematical finance.

8. ABSTRACT HARMONIC ANALYSIS (BY LAM PHAM)

8.1. **Description.** We will follow [Fol15] and supplement with [BdlHV08] (which is available for free on the authors' website).

8.2. **Background and prerequisites.** Topology, Measure theory, functional analysis, abstract algebra.

8.3. **Syllabus and primary goal.** The primary goal is to become familiar with the non-commutative Fourier transform which is a very powerful tool. There are beautiful theories developed and we will see some of the special cases, in particular in the abelian case and the compact case. We will start with some review of topological groups and functional analysis (Banach algebras, spectral theory). Depending on the familiarity of the student, we can move quite quickly to unitary representations and functions of positive type before getting to our two main cases of study: analysis on locally compact abelian and compact groups.

8.4. **Development and topics covered.** The topics covered include

- Topological Groups and Haar measure
- Banach Algebras and Spectral Theory
- Convolutions and Group representations
- Unitary Representations and the non-commutative Fourier transform
- Pontryagin duality and analysis on locally compact abelian groups
- Homogeneous Spaces
- Analysis on Compact groups and the Peter-Weyl Theorem
- Induced representations

If we finish all this, we can talk about

- C^* -algebras
- Fell topology and Kazhdan's property (T)
- Amenability

9. KAZHDAN'S PROPERTY (T) (BY LAM PHAM)

9.1. Description. The main reference is [BdlHV08]. D. Kazhdan introduced Property (T) in 1967. This property has been very useful and has been used among other things by G. Margulis to give the first explicit construction of expander graphs, and many famous lecture notes are available for supplementary reading.

9.2. Background and prerequisites. Since Kazhdan's property (T) is a rather advanced notion, mathematical maturity and the will to learn is a must. We will keep the prerequisites to a minimum (basic algebra and topology), however, good knowledge of measure theory and functional analysis is important. Basic knowledge of representation theory is useful, although we will not delve on the tools used here. Group Theory and Probability theory are also important.

9.3. Syllabus and primary goal. The goal of this reading program is a survey of Kazhdan's property (T) and its applications in mathematics. We will start with Chapter 1 and then Chapters 4, 5, 6. If time allows, we will review some material from the appendix. We will introduce this topic in full generality, but in some cases, we will focus on finitely-generated discrete countable groups.

9.4. Development and topics covered. This is an advanced topic, and as such, we will touch upon many topics:

- Banach-Tarski paradox
- Unitary representations
- Amenability
- Kazhdan's property (T)
- Relative property (T) for semi-direct products
- Lubotzky's property (τ)
- Bounded generation
- Growth of groups (polynomial, exponential, intermediate)
- Uniformity issues
- Expander graphs
- Kesten's probabilistic criterion for amenability

10. PROBABILITY ON LOCALLY COMPACT GROUPS AND COMPACT LIE GROUPS (BY LAM PHAM)

10.1. **Description.** The main texts are [App14] and [Hey77]. The latter is a classic reference but more advanced.

10.2. **Background and prerequisites.** Differential geometry, abstract algebra, basic knowledge of Lie groups, measure theory, functional analysis, probability theory, representation theory.

10.3. **Syllabus and primary goal.** The goal is to become familiar with the analysis tools available for treating probability on groups. We will follow [App14]. Quickly reviewing some basics of Lie groups (Chapter 1), unitary representations and Peter-Weyl theory of compact Lie groups (Chapters 2 and 3), before moving on to the main part of the reading which is the study of probability measures on groups (Chapters 4-6).

10.4. **Development and topics covered.**

- Topological groups and Haar measure
- Haar measure
- Basic theory of Lie groups
- Representation theory of compact groups
- Weights and Root systems
- Representation theory of $SU(2)$
- Non-commutative Fourier transform
- Convolutions
- Infinitely divisible semigroups of measures
- Martingales and Lévy processes

10.5. **Development and topics covered.** Many classical (an big!) results are important in their own right, and we will prove many of these results along the way. Additionnally, we will try to get an idea of how these are studied nowadays, and what areas of research they give access to.

11. ADDITIVE COMBINATORICS (BY LAM PHAM)

11.1. Description. Arithmetic Combinatorics (also formerly known as *Additive Combinatorics*) is a very popular area of research. In my opinion, what makes it extremely interesting is the very strong interaction between numerous fields such as: number theory, combinatorics, group theory, harmonic analysis, ergodic theory, and many more. There was tremendous progress in recent years, and both classical and modern viewpoints will be studied. Historically, additive combinatorics studies additive properties of subsets of the integers, or more generally, abelian groups. In the past 10 years, it has been partially extended to the non-commutative case, with many important applications in group theory and number theory.

The reference for this is [TV10].

11.2. Background and prerequisites. Analysis, Discrete Mathematics (Math 244) Modern Combinatorics (Math 345), Basic Group Theory (at the level of Math 350).

11.3. Syllabus and primary goal. We will focus on purely additive combinatorics, studying the basic notions of the field, and some of the major results (Szemerédi's Theorem, Erdős-Szemerédi conjecture, Green-Tao Theorem, Freiman's Theorem, etc) and techniques involved in proving these (graph theoretic techniques, character theory and Fourier analysis, ergodic theory). The main text for this would be [TV10], and we would read Chapters 2, 3, 4, 5, 6.

11.4. Development and topics covered.

- Sum set theory
- Balog-Szemerédi-Gowers theorem
- Generalized arithmetic progressions
- Convex geometry
- Fourier analysis
- Inverse theorems in additive combinatorics
- Freiman homomorphisms
- Incidence geometry, sum-product theorems.

12. APPROXIMATE GROUPS (BY LAM PHAM)

12.1. Description. Arithmetic Combinatorics (also formerly known as *Additive Combinatorics*) is a very popular area of research. In my opinion, what makes it extremely interesting is the very strong interaction between numerous fields such as: number theory, combinatorics, group theory, harmonic analysis, ergodic theory, and many more. There was tremendous progress in recent years, and both classical and modern viewpoints will be studied. Historically, additive combinatorics studies additive properties of subsets of the integers, or more generally, abelian groups. In the past 10 years, it has been partially extended to the non-commutative case, with many important applications in group theory and number theory.

The reference for this is [TV10]

12.2. Background and prerequisites. Analysis, Discrete Mathematics (Math 244) Modern Combinatorics (Math 345), Basic Group Theory (at the level of Math 350).

12.3. Syllabus and primary goal. We will start with a quick introduction to additive combinatorics using [TV10] and to move on directly to the non-commutative case using [Tao14] and to some extent, the seminal paper [Tao08]. This is mostly the study of the so-called *approximate groups*.

12.4. Development and topics covered.

- Topological groups, Haar measure
- Lie groups, Lie algebras
- Baker-Campbell-Hausdorff formula
- Peter-Weyl theory of compact groups
- Gromov's theorem on polynomial growth
- Nilpotent groups
- Non-standard analysis

13. EXPANDER GRAPHS (BY LAM PHAM)

13.1. **Description.** The text is [Tao15], based on a course given by Terry Tao at UCLA. We will learn about the theory of expander graphs, isoperimetry in graphs, quasirandomness, probability on groups, arithmetic combinatorics, and applications to number theory.

13.2. **Background and preliminaries.** Analysis, Discrete Mathematics (Math 244) Modern Combinatorics (Math 345), Basic Group Theory (at the level of Math 350). Familiarity with probability theory and representation theory.

13.3. **Syllabus and primary goal.** We will follow the book closely. The goal is to learn about explicit construction of expander graphs. This was pioneered by Margulis, and then by Lubotzky-Phillips-Sarnak. Recently, Bourgain-Gamburd came up with a breakthrough method, bearing the name of *Bourgain-Gamburd expansion machine*, which is a very powerful and general method, and yet, relies on very simple concepts. After introducing the general theory of expander graphs, we will develop the main tools required to study the Bourgain-Gamburd expansion machine. These use crucial concepts from representation theory (quasirandomness), arithmetic combinatorics (approximate groups), and probability on groups (random walks, mixing, non-concentration).

13.4. **Development and topics covered.**

- Expander graphs
- Isoperimetry
- Kazhdan's property (T)
- Random walks on groups and graphs
- Approximate groups
- Product theorems
- Anti-concentration
- Quasirandomness
- Representation theory

14. INTRODUCTION TO MATRIX LIE GROUPS (BY ARSENIY SHEYDVASSER)

14.1. **Description.** The main reference is [Hal15]. Some additional reading from freely available textbooks/paper may be assigned.

The theory of Lie groups and Lie algebras is not only beautiful – it is of great importance in physics, where the representation theory of these objects is used to study the symmetries of the laws of motion and similar.

14.2. **Background and prerequisites.** Knowledge of multivariable calculus (preferably vector calculus), linear algebra, and group theory. Familiarity with manifolds and differential forms would be helpful for some of the more advanced topics this project can cover, but are not necessary.

14.3. **Syllabus and primary goal.** The full theory of Lie groups requires the theory of manifolds and differential forms, but most of the interesting examples and much of the underlying theory can be studied by restricting only to the case of matrix Lie groups, which is the perspective that Brian Hall takes in his book. I will, of course, be more than happy to extend to the full theory for the interested and prepared student.

The precise direction for this project will depend on the background and interests of the student, but it will start with a definition of a Lie group along with some important examples. From there, we will construct the Lie algebra associated to a particular Lie group. I would like to get to the representation theory of Lie groups and Lie algebras, and in particular consider the classification of semi-simple Lie algebras (although I would be happy to just get the classification of representations of $SU(3)$).

14.4. **Development and topics covered.** Topics include

- matrix Lie groups
- Intrinsic definition of abstract Lie groups
- Lie algebra of a Lie group
- Logarithm and Exponential maps
- Baker-Campbell-Hausdorff formula
- Basic representation theory
- Representation theory of $SU(3)$
- Representation theory of semi-simple Lie algebras

15. NAIVE LIE THEORY (BY GABRIEL BERGERON-LEGROS)

15.1. **Description.** The main reference is [Sti08]. Stillwell does an amazing job of introducing Lie theory using nothing more. He introduces the abstract group theory and the differential geometry that are needed for the book. Everything he does can easily be understood by following elementary computations.

15.2. **Prerequisites.** Only Calculus and Linear Algebra.

15.3. **Syllabus and main goal.** Matrix groups, quaternions, Lie groups, Lie algebras and the relationship between the two. Simplicity and semisimplicity. Exponential and logarithm of matrices. If time allows, compactness, simple connectedness and the Baker-Campbell-Hausdorff theorem.

15.4. **Development and topics covered.** The goal is to develop an intuitive understanding of Lie theory and its importance via Matrix groups.

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