Multiple ζ-Values, Galois Groups, and Geometry of Modular Varieties

Alexander B. Goncharov

Abstract. We discuss two *arithmetical* problems, at first glance unrelated:

1) The properties of the multiple ζ-values

$$\zeta(n_1, \ldots, n_m) := \sum_{0<k_1<k_2<\cdots<k_m} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_m^{n_m}} \qquad n_m > 1 \qquad (1)$$

and their generalizations, multiple polylogarithms at N-th roots of unity.

2) The action of the absolute Galois group on the pro-l completion

$$\pi_1^{(l)}(X_N) := \pi_1^{(l)}(\mathbb{P}^1 \backslash \{0, \mu_N, \infty\}, v)$$

of the fundamental group of $X_N := \mathbb{P}^1 \backslash \{0, \infty$ and all N-th roots of unity$\}$.

These problems are the Hodge and l-adic sites of the following one:

3) Study the Lie algebra of the image of motivic Galois group acting on the motivic fundamental group of $\mathbb{P}^1 \backslash \{0, \mu_N, \infty\}$.

We will discuss a surprising connection between these problems and *geometry* of the modular varieties

$$Y_1(m : N) := \Gamma_1(m; N) \backslash GL_m(\mathbb{R}) / O_m \cdot \mathbb{R}^*$$

where $\Gamma_1(m; N)$ is the subgroup of $GL_m(\mathbb{Z})$ stabilizing $(0, \ldots, 0, 1)$ mod N.

In particular using this relationship we get precise results about the Lie algebra of the image of the absolute Galois group in $\mathrm{Aut}\pi_1^{(l)}(X_N)$, and sharp estimates on the dimensions of the $\mathbb{Q}$-vector spaces generated by the multiple polylogarithms at N-th roots of unity, *depth* m and *weight* $w := n_1 + \cdots + n_m$.

The simplest case of the problem 3) is related to the classical theory of cyclotomic units. Thus the subject of this lecture is *higher cyclotomy.*

1. The Multiple ζ-Values

1.1. The algebra of multiple ζ-values and its conjectural description

Multiple ζ-values (1) were invented by L. Euler [8]. Euler discovered that the numbers $\zeta(m, n)$, when $w := m + n$ is odd, are $\mathbb{Q}$-linear combinations of $\zeta(w)$ and $\zeta(k)\zeta(w - k)$. Then these numbers were neglected. About 10 years ago they were resurrected as the coefficients of Drinfeld's associator [7], rediscovered by D. Zagier [33], appeared in works of M. Kontsevich on knot invariants and the author [10, 9] on mixed Tate motives over Spec ($\mathbb{Z}$). More recently they showed up in quantum field theory [27, 3], deformation quantization and so on.

We say that $w := n_1 + \cdots + n_m$ is the *weight* and m is the *depth* of (1).

Let $\mathcal{Z}$ be the space of $\mathbb{Q}$-linear combinations of multiple ζ's. It is a commutative algebra over $\mathbb{Q}$. For instance

$$\zeta(m) \cdot \zeta(n) = \zeta(m,n) + \zeta(n,m) + \zeta(m+n) \tag{2}$$

because

$$\sum_{k_1,k_2>0} \frac{1}{k_1^m k_2^n} = \Big(\sum_{0<k_1<k_2} + \sum_{0<k_1=k_2} + \sum_{k_1>k_2>0} \Big) \frac{1}{k_1^m k_2^n}\,. \tag{3}$$

The only known results about the classical ζ-values are the following:

$$\zeta(2n) = (-1)^{n-1}(2\pi)^{2n} \cdot \frac{B_{2n}}{2\cdot(2n)!} \quad \text{(Euler)}; \qquad \zeta(3) \notin \mathbb{Q} \quad \text{(Apery)}\,. \tag{4}$$

(where B_k are the Bernoulli numbers: $\frac{t}{e^t-1} = \sum B_k t^k/k!$) and recent results of T. Rivoal: dimension of the $\mathbb{Q}$-space spanned by $\zeta(3)$, $\zeta(5)$, ..., $\zeta(2n+1)$ grows at least as $C \log n$.

To describe the hypothetical structure of the algebra $\mathcal{Z}$ we introduce a free graded Lie algebra $\mathcal{F}(3,5,\dots)_\bullet$, which is freely generated by elements e_{2n+1} of degree $-(2n+1)$ where $n \geq 1$. Let

$$U\mathcal{F}(3,5,\dots)_\bullet^\vee := \oplus_{n\geq 1}\Big(U\mathcal{F}(3,5,\dots)_{-(2n+1)}\Big)^\vee$$

be the graded dual to its universal enveloping algebra. It is $\mathbb{Z}_+$-graded.

Conjecture 1.1. *a) The weight provides a grading on the algebra $\mathcal{Z}$.*

b) One has an isomorphism of graded algebras over $\mathbb{Q}$

$$\mathcal{Z}_\bullet = \mathbb{Q}[\pi^2] \otimes_{\mathbb{Q}} U\mathcal{F}(3,5,\dots)_\bullet^\vee \qquad \deg \pi^2 := 2\,. \tag{5}$$

Part a) means that relations between ζ's of different weight, like $\zeta(5) = \lambda \cdot \zeta(7)$ where $\lambda \in \mathbb{Q}$, are impossible. For motivic interpretation/formulation of conjecture 1.1 see Section 12 in [10]. For its l-adic version see conjecture 2.1 below.

Theorem 1.2. *One has* $\dim \mathcal{Z}_k \leq \dim(\mathbb{Q}[\pi^2] \otimes U\mathcal{F}(3,5,\dots)^\vee)_k$.

Both the origin of conjecture 1.1 and proof of this theorem are based on theory of mixed Tate motives over $Spec(\mathbb{Z})$: multiple ζ-values are periods of framed mixed Tate motives over $\mathbb{Z}$, and one can *prove* that the framed mixed Tate motives over $\mathbb{Z}$ form an algebra which is isomorphic to the one appearing on the right hand side of (5) (see [17]). This gives theorem 1.2. Conjecture 1.1 just means that *every* such a period is given by multiple ζ-values.

For the definition of the *abelian* category of mixed Tate motives over a number field convenient for our approach see chapter 5 in [14]. It has all the expected properties and based on V. Voevodsky's construction of the triangulated category of motives [31]. Another approach to mixed motives has been developed by M. Levine [28], [30]. A construction of the abelian category of mixed Tate motives over the ring $\mathcal{O}_{F,S}$ of S-integers in a number field F see in ch. 3 of [17].

It is difficult to estimate $\dim\mathcal{Z}_k$ from below: we believe that $\zeta(5) \notin \mathbb{Q}$ but nobody can prove it.

One may reformulate conjecture 1.1 as a hypothetical description of the $\mathbb{Q}$-vector space $\mathcal{PZ}$ of primitive multiple ζ's:

$$\mathcal{PZ}_\bullet := \frac{\mathcal{Z}_{>0}}{\mathcal{Z}_{>0}\cdot\mathcal{Z}_{>0}} \stackrel{?}{=} <\pi^2> \oplus \mathcal{F}(3,5,\dots)^\vee_\bullet\,. \tag{6}$$

Here $<\pi^2>$ is a 1-dimensional $\mathbb{Q}$-vector space generated by π^2, and $\mathcal{Z}_{>0}$ is generated by π^2 and $\zeta(n_1,\dots,n_m)$.

Example 1.3. *There are* 2^{10} *convergent multiple* ζ*'s of the weight* 12*. However according to theorem 1.2* $\dim\mathcal{Z}_{12} \leq 12$*. One should have* $\dim\mathcal{PZ}_{12} = 2$ *since* $\mathcal{F}(3,5,\dots)_{-12}$ *is spanned over* $\mathbb{Q}$ *by* $[e_5,e_7]$ *and* $[e_3,e_9]$*. The* $\mathbb{Q}$*-vector space of* decomposable *multiple* ζ*'s of the weight* 12 *is supposed to be generated by*

$$\pi^{12},\quad \pi^6\zeta(3)^2,\quad \pi^4\zeta(3)\zeta(5),\quad \pi^4\zeta(3,5),\quad \pi^2\zeta(3)\zeta(7),\quad \pi^2\zeta(5)^2,\quad \pi^2\zeta(3,7),$$
$$\zeta(3)^4,\quad \zeta(5)\zeta(7),\quad \zeta(3)\zeta(9)\,.$$

The algebra $U\mathcal{F}(3,5,\dots)^\vee_\bullet$ is commutative. It is isomorphic to the space of noncommutative polynomials in variables f_{2n+1}, $n = 1,2,3,\dots$ with the algebra structure given by the shuffle product.

Let $\mathcal{F}(2,3)_\bullet$ be the free graded Lie algebra generated by two elements of degree -2 and -3. Its graded dual $U\mathcal{F}(2,3)^\vee_\bullet$ is isomorphic as a graded vector space to the space of noncommutative polynomials in two variables p and g_3 of degrees 2 and 3. There is canonical isomorphism of *graded vector spaces*

$$\mathbb{Q}[\pi^2]\otimes U\mathcal{F}(3,5,\dots)^\vee_\bullet = U\mathcal{F}(2,3)^\vee_\bullet\,.$$

The rule is clear from the pattern $(\pi^2)^3 f_3(f_7)^3(f_5)^2 \longrightarrow p^3 g_3(g_3p^2)^3(g_3p)^2$.

In particular if $d_k := \dim\mathcal{Z}_k$ then one should have $d_k = d_{k-2} + d_{k-3}$. This rule has been observed in computer calculations of D. Zagier for $k \leq 12$. Later on extensive computer calculations, confirming it, were made by D. Broadhurst [3].

1.2. The depth filtration

Conjecture 1.1, if true, would give a very simple and clear picture for the structure of the multiple ζ-values algebra. However this algebra has an additional structure: the depth filtration, and conjecture 1.1 tells us nothing about it. The study of the depth filtration moved the subject in a completely unexpected direction: towards geometry of modular varieties for GL_m.

To formulate some results about the depth filtration consider the algebra $\overline{\mathcal{Z}}$ spanned over $\mathbb{Q}$ by the numbers

$$\overline{\zeta}(n_1,\dots,n_m) := (2\pi i)^{-w}\zeta(n_1,\dots,n_m)\,.$$

It is filtered by the weight and depth. Since $\overline{\zeta}(2) = -1/24$, there is no weight grading anymore. Let $\mathrm{Gr}^{W,D}_{w,m}\mathcal{P}\overline{\mathcal{Z}}$ be the associated graded. We assume that 1 is of depth 0. Denote by $d_{w,m}$ its dimension over $\mathbb{Q}$.

Euler's classical computation of $\zeta(2n)$ (see (4)) tells us that $d_{2n,1} = 0$. Generalizing this it is not hard to prove that $d_{w,m} = 0$ if $w + m$ is odd.

Theorem 1.4. *a)*

$$d_{w,2} \leq \left[\frac{w-2}{6}\right] \quad \textit{if } w \textit{ is even.} \tag{7}$$

b)

$$d_{w,3} \leq \left[\frac{(w-3)^2-1}{48}\right] \quad \textit{if } w \textit{ is odd.}$$

The part a) is due to Zagier; the dimension of the space of cusp forms for $SL_2(\mathbb{Z})$ showed up in his investigation of the double shuffle relations for the depth two multiple ζ's, ([33]). The part b) has been proved in [12]. Moreover we proved that, assuming some standard conjectures in arithmetic algebraic geometry, these estimates are exact, see also corollary 2.5 and theorem 7.5.

Problem 1.5. *Define explicitly a depth filtration on the Lie coalgebra $\mathcal{F}(3,5,\dots)^\vee$ which under the isomorphism (6) should correspond to the depth filtration on the space of primitive multiple ζ-values.*

The cogenerators of the Lie coalgebra $\mathcal{F}(3,5,\dots)^\vee$ correspond to $\zeta(2n+1)$. So a naive guess would be that the dual to the lower central series filtration on $\mathcal{F}(3,5,\dots)$ coincides with the depth filtration. However then one should have $d_{12,2} = 2$, while according to formula (7) $d_{12,2} = 1$. Nevertheless $\dim\mathcal{PZ}_{12} = 2$, but the new transcendental number appears only in the depth 4.

1.3. A heuristic discussion

Conjecture 1.1 in the form (6) tells us that the space of primitive multiple ζ's should have a Lie coalgebra structure. How to determine its coproduct δ in terms of the multiple ζ's? Here is the answer for the depth 1 and 2 cases. (The general case later on). Consider the generating series

$$\zeta(t) := \sum_{m>0} \zeta(m)t^{m-1}, \qquad \zeta(t_1,t_2) := \sum_{m,n>0} \zeta(m,n)t_1^{m-1}t_2^{n-1}$$

Then $\delta\zeta(t) = 0$, i.e. $\delta\zeta(n) = 0$ for all n, and

$$\delta\zeta(t_1,t_2) = \zeta(t_2)\wedge\zeta(t_1) + \zeta(t_1)\wedge\zeta(t_2-t_1) - \zeta(t_2)\wedge\zeta(t_1-t_2)\,. \tag{8}$$

To make sense out of this we have to go from the numbers $\zeta(n_1,\dots,n_m)$ to their more structured counterparts: framed mixed Tate motives $\zeta_{\mathcal{M}}(n_1,\dots,n_m)$, or their Hodge or l-adic realizations, (see [17]). The advantage is immediately seen: the coproduct $\delta_{\mathcal{M}}$ is well defined by the general formalism (see Section 10 in [10] or [17]), one easily proves not only that $\zeta_{\mathcal{M}}(2n) = 0$ (motivic version of Euler's theorem) as well as $\zeta_{\mathcal{M}}(1) = 0$, but also that $\zeta_{\mathcal{M}}(2n+1) \neq 0$, and there are no linear relations between $\zeta_{\mathcal{M}}(2n+1)$'s! Hypothetically we loose no information:

linear relations between the multiple ζ's should reflect linear relations between their motivic avatars. Using $\zeta_{\mathcal{M}}(2n) = 0$ we rewrite formula (8) as

$$\delta_{\mathcal{M}} : \zeta_{\mathcal{M}}(t_1, t_2) \longmapsto \Big(1 + U + U^2\Big)\zeta_{\mathcal{M}}(t_1) \wedge \zeta_{\mathcal{M}}(t_2) \tag{9}$$

where U is the linear operator $(t_1, t_2) \longmapsto (t_1 - t_2, t_1)$. For example $\delta_{\mathcal{M}}$ sends the subspace of weight 12 double $\zeta_{\mathcal{M}}$'s to a one dimensional $\mathbb{Q}$-vector space generated by $3\zeta_{\mathcal{M}}(3) \wedge \zeta_{\mathcal{M}}(9) + \zeta_{\mathcal{M}}(5) \wedge \zeta_{\mathcal{M}}(7)$. One can identify the cokernel of the map (9), restricted to the weight w subspace, with $H^1(GL_2(\mathbb{Z}), S^{w-2}V_2 \otimes \varepsilon_2)$ where V_2 is the standard GL_2-module, and $\otimes\varepsilon_2$ is the twist by the determinant, i.e. with the space of weight w cusp forms for $GL_2(\mathbb{Z})$. Moreover, one can prove that $\mathrm{Ker}\delta_{\mathcal{M}}$ is spanned by $\zeta_{\mathcal{M}}(2n+1)$'s: this is a much more difficult result which uses all the machinery of mixed motives. Thus an element of the depth 2 associated graded of the space of primitive double ζ's is zero if and only if its coproduct is 0. So formula (9) provides a complete description of the space of double ζ's. In particular $d_{12,2} = 1$.

For the rest of this paper we suppress the motives working mostly with the l-adic side of the story and looking at the Hodge side for motivations.

2. Galois Symmetries of the pro-l Completion of the Fundamental Group of $\mathbb{P}^1\backslash\{0, \mu_N, \infty\}$

2.1. The Lie algebra of the image of the Galois group

Let X be a regular curve, $\overline{X}$ the corresponding projective curve, and v a tangent vector at a point $x \in \overline{X}$. According to Deligne [4] one can define the geometric profinite fundamental group $\widehat{\pi}_1(X, v)$ based at v. If X, x and v are defined over a number field F then the group $\mathrm{Gal}_F := \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ acts by automorphisms of $\widehat{\pi}_1(X, v)$.

If $X = \mathbb{P}^1\backslash\{0, \mu_N, \infty\}$ there is a tangent vector v_∞ corresponding to the inverse t^{-1} of the canonical coordinate t on $\mathbb{P}^1\backslash\{0, \mu_N, \infty\}$. Denote by $\pi^{(l)}$ the pro-l-completion of the group π. We will investigate the map

$$\Phi_N^{(l)} : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow \mathrm{Aut}\pi_1^{(l)}(\mathbb{P}^1\backslash\{0, \mu_N, \infty\}, v_\infty)\,. \tag{10}$$

When $N = 1$ it was studied by Grothendieck [19], Deligne [4], Ihara (see [22, 24]), Drinfeld [7], and others (see [23]), but for $N > 1$ it was not investigated.

Denote by $H(m)$ the lower central series for the group H. Then the quotient $\pi_1^{(l)}(X_N)/\pi_1^{(l)}(X_N)(m)$ is an l-adic Lie group. Taking its Lie algebra and making the projective limit over m we get a pronilpotent Lie algebra over $\mathbb{Q}_l$:

$$\mathbb{L}_N^{(l)} := \varprojlim Lie\Big(\frac{\pi_1^{(l)}(X_N)}{\pi_1^{(l)}(X_N)(m)}\Big)\,.$$

Similarly one defines an l-adic pronilpotent Lie algebra $\mathbb{L}^{(l)}(X, v)$ corresponding to the geometric fundamental group $\widehat{\pi}_1(X, v)$ of a variety X with a base at v.

For the topological reasons $\mathbb{L}_N^{(l)}$ is a free pronilpotent Lie algebra over $\mathbb{Q}_l$ with $n+1$ generators corresponding to the loops around 0 and N-th roots of unity.

Let $\mathbb{Q}(\zeta_n)$ be the field generated by n-th roots of unity. Set $\mathbb{Q}(\zeta_{l^\infty N}) := \cup \mathbb{Q}(\zeta_{l^a N})$. We restrict map (10) to the Galois group $\mathrm{Gal}_{\mathbb{Q}(\zeta_{l^\infty N})}$. Passing to Lie algebras we get a homomorphism

$$\phi_N^{(l)}\colon \mathrm{Gal}_{\mathbb{Q}(\zeta_{l^\infty N})} \longrightarrow \mathrm{Aut}(\mathbb{L}_N^{(l)})\,.$$

Let us linearize the image of this map. Let $\mathbb{L}_N^{(l)}(m)$ be the lower central series for the Lie algebra $\mathbb{L}_N^{(l)}$. There are homomorphisms to l-adic Lie groups

$$\phi_{N;m}^{(l)}\colon \mathrm{Gal}_{\mathbb{Q}(\zeta_{l^\infty})} \longrightarrow \mathrm{Aut}\Big(\mathrm{L}_N^{(l)}/\mathbb{L}_N^{(l)}(m)\Big)\,.$$

The main hero of this story is the pronilpotent Lie algebra

$$\mathcal{G}_N^{(l)} := \lim_{\longleftarrow \atop m} Lie\Big(Im\phi_{N;m}^{(l)}\Big) \hookrightarrow \mathrm{Der}\mathbb{L}_N^{(l)}\,.$$

When $N=1$ we denote it by $\mathcal{G}^{(l)}$.

Conjecture 2.1. *$\mathcal{G}^{(l)}$ is a free Lie algebra with generators indexed by odd integers ≥ 3.*

It has been formulated, as a question, by Deligne [4] and Drinfeld [7].

2.2. The weight and depth filtration on $\mathbb{L}_N^{(l)}$

There are two increasing filtrations by ideals on the Lie algebra $\mathbb{L}_N^{(l)}$, indexed by negative integers.

The weight filtration $\mathcal{F}_\bullet^W$. It coincides with the lower central series for $\mathbb{L}_N^{(l)}$:

$$\mathbb{L}_N^{(l)} = \mathcal{F}_{-1}^W \mathbb{L}_N^{(l)}; \qquad \mathcal{F}_{-n-1}^W \mathbb{L}_N^{(l)} := [\mathcal{F}_{-n}^W \mathbb{L}_N^{(l)}, \mathbb{L}_N^{(l)}]\,.$$

The depth filtration $\mathcal{F}_\bullet^D$. The natural inclusion

$$\mathbb{P}^1 \backslash \{0, \mu_N, \infty\} \hookrightarrow \mathbb{P}^1 \backslash \{0, \infty\}$$

provides a morphism of the corresponding fundamental Lie algebras

$$p\colon \mathbb{L}_N^{(l)} \longrightarrow \mathbb{L}^{(l)}(\mathbb{P}^1 \backslash \{0,\infty\}) = \mathbb{Q}_l(1)\,.$$

Let $\mathcal{I}_N$ be the kernel of this projection. Its powers give the depth filtration:

$$\mathcal{F}_0^D \mathbb{L}_N^{(l)} = \mathbb{L}_N^{(l)}, \quad \mathcal{F}_{-1}^D \mathbb{L}_N^{(l)} = \mathcal{I}_N, \quad \mathcal{F}_{-n-1}^D \mathbb{L}_N^{(l)} = [\mathcal{I}_N, \mathcal{F}_{-n}^D \mathbb{L}_N^{(l)}]\,.$$

2.3. The Galois Lie algebra and its shape

These filtrations induce two filtrations on the Lie algebra $\mathrm{Der}\mathbb{L}_N^{(l)}$ and hence on the Lie algebra $\mathcal{G}_N^{(l)}$. The associated graded Lie algebra $\mathrm{Gr}\mathcal{G}_{\bullet\bullet}^{(l)}(\mu_N)$, which we call the level N Galois Lie algebra, is bigraded by the weight $-w$ and depth $-m$. The weight filtration can be defined by a grading. Moreover one can define it in a way compatible the depth filtration and the subspace $\mathcal{G}_N^{(l)} \subset \mathrm{Der}\mathbb{L}_N^{(l)}$. Therefore

$$\mathrm{Gr}\mathcal{G}_{\bullet\bullet}^{(l)}(\mu_N) \hookrightarrow \mathrm{Gr}_{\bullet\bullet}\mathrm{Der}\mathbb{L}_N^{(l)} .$$

The depth m quotients. For any $m \geq 1$ there is the depth $\geq -m$ (we will also say depth m) quotient Galois Lie algebra:

$$\mathrm{Gr}\mathcal{G}_{\bullet,\geq -m}^{(l)}(\mu_N) := \frac{\mathrm{Gr}\mathcal{G}_{\bullet,\bullet}^{(l)}(\mu_N)}{\mathrm{Gr}\mathcal{G}_{\bullet,<-m}^{(l)}(\mu_N)} . \tag{11}$$

It is a nilpotent graded Lie algebra of the nilpotence class $\leq m$.

The diagonal Lie algebra. Notice that $\mathrm{Gr}\mathcal{G}_{-w,-m}^{(l)}(\mu_N) = 0$ if $w < m$. We define the diagonal Galois Lie algebra $\mathrm{Gr}\mathcal{G}_{\bullet}^{(l)}(\mu_N)$ as the Lie subalgebra of $\mathrm{Gr}\mathcal{G}_{\bullet\bullet}^{(l)}(\mu_N)$ formed by the components with $w = m$. It is graded by the weight. It can be defined as the Lie subalgebra of $\mathrm{Gr}^W\mathcal{G}^{(l)}(\mu_N)$ by imposing the *weight* = *depth* condition. Thus we do not need to take the associated graded for the depth filtration for its definition. Therefore the diagonal Galois Lie algebra is isomorphic, although non canonically, to a Lie subalgebra of $\mathcal{G}^{(l)}(\mu_N)$.

The picture below exibites all possibly non zero components of the Galois Lie algebra and indicates its depth 2 quotient and the diagonal Lie subalgebra.

2.4. The mysterious correspondence

Let V_m be the standard m-dimensional representation of GL_m. Our key point ([9]-[13]) is that

the structure of Galois Lie algebra $\mathrm{Gr}\mathcal{G}^{(l)}_{\bullet,\geq -m}(\mu_N)$	*is related to*	geometry of local systems with fibers $S^{\bullet-m}V_m$ over (the closure of) modular variety $Y_1(m;N)$, which is defined for $m>1$ as $\Gamma_1(m;N)\backslash GL_m(\mathbb{R})/O_m\cdot\mathbb{R}^*$.

The adelic approach to modular varieties shows that for $m=1$ we have

$$Y_1(1;N) := \quad S_N := \quad \mathrm{Spec}\mathbb{Z}[\zeta_N][\frac{1}{N}] \tag{12}$$

In particular the diagonal level N Galois Lie algebra is related to the geometry of the modular varieties $Y_1(m;N)$.

Both the dual to the Galois Lie algebras (11) and the modular varieties $\overline{Y}_1(m;N)$ form inductive systems with respect to m. The correspondence is compatible with these inductive structures.

Recall the standard cochain complex of a Lie algebra $\mathcal{G}$

$$\mathcal{G}^\vee \stackrel{\delta}{\longrightarrow} \Lambda^2\mathcal{G}^\vee \stackrel{\delta}{\longrightarrow} \Lambda^3\mathcal{G}^\vee \longrightarrow \cdots$$

where the first differential is dual to the commutator map $[,]\colon \Lambda^2\mathcal{G}\longrightarrow\mathcal{G}$, and the others are obtained using the Leibniz rule. The condition $\delta^2=0$ is equivalent to the Jacobi identity.

For a precise form of this correspondence see section 6. It relates the depth m, weight w part of the standard cochain complex of the Lie algebra (11) with

$$(\text{rank } m \text{ modular complex}) \otimes_{\Gamma_1(m;N)} S^{w-m}V_m\,. \tag{13}$$

The rank m modular complex is a complex of $GL_m(\mathbb{Z})$-modules constructed purely combinatorially. It has a *geometric realization* in the symmetric space $\mathbb{H}_m := GL_m(\mathbb{R})/O(m)\cdot\mathbb{R}^*$, see section 7 and [15]. It is well understood only for $m\leq 4$.

2.5. Examples of this correspondence

Strangely enough it is more convenient to describe the structure of the bigraded Lie algebra

$$\mathrm{Gr}\widehat{\mathcal{G}}^{(l)}_{\bullet\bullet}(\mu_N) := \mathrm{Gr}\mathcal{G}^{(l)}_{\bullet\bullet}(\mu_N)\oplus\mathbb{Q}_l(-1,-1)$$

where $\mathbb{Q}_l(-1,-1)$ is a one dimensional Lie algebra of weight and depth -1. Its motivic or Galois-theoretic meaning is non clear: it should correspond to $\zeta(1)$.

a) The depth 1 case. The depth -1 quotient $\mathrm{Gr}_{\bullet,-1}\mathcal{G}^{(l)}_N$ is an abelian Lie algebra. Its structure is described by the following theorem.

Theorem 2.2. *There is a natural isomorphism of $\mathbb{Q}_l$-vector spaces*

$$\mathrm{Hom}\Big(K_{2n-1}(\mathbb{Z}[\zeta_N,N^{-1}]),\mathbb{Q}_l\Big) \quad \stackrel{=}{\longrightarrow} \quad \mathrm{Gr}_{-n,-1}\mathcal{G}^{(l)}_N\,. \tag{14}$$

According to the Borel theorem one has

$$\dim K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Q} = \begin{cases} 0 & n\text{: even} \\ 1 & n > 1\text{: odd} \end{cases} \tag{15}$$

$$\dim K_{2n-1}(\mathbb{Z}[\zeta_N, N^{-1}]) \otimes \mathbb{Q} = \begin{cases} \frac{\varphi(N)}{2} & N > 2, n > 1 \\ \frac{\varphi(N)}{2} + p(N) - 1 & N > 2, n = 1 \\ 1 & N = 2 \end{cases} . \tag{16}$$

where $p(N)$ is the number of prime factors of N.

Theorem 2.2 for $N = 1$ is known thanks to Soulé, Deligne [4], and Ihara [23]. The general case can be deduced from the motivic theory of classical polylogarithms developed by Deligne and Beilinson [4, 1]. In the case $n = 1$ there is canonical isomorphism justifying the name "higher cyclotomy" for our story:

$$\mathrm{Gr}\mathcal{G}^{(l)}_{-1,-1}(\mu_N) = \mathrm{Hom}\Big(\text{group of the cyclotomic units in } \mathbb{Z}[\zeta_N][\tfrac{1}{N}], \quad \mathbb{Q}_l\Big) . \tag{17}$$

The level N modular variety for $GL_{1/\mathbb{Q}}$ is the scheme S_N, see (12). It has $\varphi(N)$ complex points parametrized by the primitive roots of unity ζ_N^α where $(\alpha, N) = 1$. Our correspondence for $m = 1$ is given by the isomorphism (where $+$ means invariants under the complex conjugation):

$$[\mathbb{Q}(n-1) - \text{valued functions on} \quad S_N \otimes \mathbb{C}]^+ \quad \stackrel{=}{\longrightarrow} \quad K_{2n-1}(S_N) \otimes \mathbb{Q}$$

provided by motivic classical polylogarithms: one associates to ζ_N^α the the cyclotomic element $\{\zeta_N^\alpha\}_n \in K_{2n-1}(S_N)$, whose regulator is computed via $Li_n(\zeta_N^\alpha)$.

b) The depth 2 case, $N = 1$. The structure of the depth ≥ -2 quotient of the Lie algebra $\mathrm{Gr}\widehat{\mathcal{G}}^{(l)}_{\bullet,\bullet}$ is completely described by the commutator map

$$[,]: \quad \Lambda^2 \mathrm{Gr}\widehat{\mathcal{G}}^{(l)}_{-1,\bullet} \longrightarrow \mathrm{Gr}\mathcal{G}^{(l)}_{-2,\bullet} . \tag{18}$$

Construction of the dual to complex (18). Look at the classical modular triangulation of the hyperbolic plane $\mathbb{H}_2$ where the central ideal triangle has vertices at $0, 1, \infty$:

The group $GL_2(\mathbb{R})$, acting on $\mathbb{C}\backslash\mathbb{R}$ by $z \longmapsto \frac{az+b}{cz+d}$ commutes, with $z \longmapsto \overline{z}$. We let $GL_2(\mathbb{R})$ act on $\mathbb{H}_2$ by identifying $\mathbb{H}_2$ with the quotient of $\mathbb{C}\backslash\mathbb{R}$ by complex conjugation. The subgroup $GL_2(\mathbb{Z})$ preserves the modular picture. Consider the chain complex of the modular triangulation placed in degrees $[1, 2]$:

$$M^*_{(2)} := \quad M^1_{(2)} \longrightarrow M^2_{(2)} . \tag{19}$$

It is a complex of $GL_2(\mathbb{Z})$-modules. The group $M^1_{(2)}$ is generated by the triangles, and $M^2_{(2)}$ by the geodesics. Let ε_2 be the one dimensional GL_2-module given by the determinant.

Lemma 2.3. *Let Γ be a finite index subgroup of $GL_2(\mathbb{Z})$ and V a GL_2-module over $\mathbb{Q}$. Then the complex $M^*_{(2)} \otimes_\Gamma V[1]$ computes the cohomology $H^*(\Gamma, V \otimes \varepsilon_2)$.*

Theorem 2.4. *The weight w part of the dual to complex (18) is* **canonically** *isomorphic to the complex*

$$\Big(M^*_{(2)} \otimes_{GL_2(\mathbb{Z})} S^{w-2}\mathrm{V}_2\Big) \otimes \mathbb{Q}_l\,. \qquad (20)$$

Motivic version of this theorem was obtained in section 7 of [9]. Its Hodge side provides a refined version of the story told in section 1.3.

According to the lemma complex (20) computes $H^{*-1}(GL_2(\mathbb{Z}), S^{w-2}\mathrm{V}_2 \otimes \varepsilon_2)$. Since these cohomology groups are known, we compute the Euler characteristic of the complex (18) and using theorem 2.2 and formula (15)‘ get the following result, the l-adic version of theorem 1.4, (see related results of Ihara and Takao in [24]).

Corollary 2.5.

$$\dim \mathrm{Gr}\mathcal{G}^{(l)}_{-w,-2} = \begin{cases} 0 & w: \ odd \\ \left[\frac{w-2}{6}\right] & w: \ even\,. \end{cases} \qquad (21)$$

c) $N = p$ is a prime, $w = m = -2$. The structure of the weight ≥ -2 quotient of the *diagonal* Galois Lie algebra $\mathrm{Gr}\mathcal{G}^{(l)}_\bullet(\mu_N)$ is described by the commutator map

$$[,]: \Lambda^2 \mathrm{Gr}\widehat{\mathcal{G}}^{(l)}_{-1,-1}(\mu_p) \longrightarrow \mathrm{Gr}\mathcal{G}^{(l)}_{-2,-2}(\mu_p) \qquad (22)$$

Projecting the modular triangulation of the hyperbolic plane onto the modular curve $Y_1(p) := \Gamma_1(p)\backslash\mathbb{H}_2$ we get the modular triangulation of $Y_1(p)$. The complex involution acts on the modular curve preserving the triangulation. Consider the following complex, where $+$ means invariants of the complex involution:

$$\Big(\text{the chain complex of the modular triangulation of } Y_1(p)\Big)^+ \otimes \mathbb{Q}_l\,. \qquad (23)$$

Theorem 2.6. *The dual to complex (22) is naturally* **isomorphic** *to complex (23).*

In particular there is canonical isomorphism

$$\mathbb{Q}_l[\text{triangles of the modular triangulation of } Y_1(p)]^+ = \Big(\mathrm{Gr}\mathcal{G}^{(l)}_{-2,-2}(\mu_p)\Big)^\vee\,. \qquad (24)$$

Computing the Euler characteristic of the complex (22) using (24) and (17) we get

$$\dim \mathrm{Gr}\mathcal{G}^{(l)}_{-2}(\mu_p) = \frac{(p-5)(p-1)}{12}\,.$$

Deligne proved [5] that the Hodge-theoretic version of $\mathcal{G}^{(l)}_N$ is free when $N = 2$, and recently extended the arguments to the case $N = 3, 4$ in [6]. The results above imply that it can not be free for sufficiently big N. For instance it is not free for a prime $N = p$ if the genus of $Y_1(p)$ is positive, i.e. $p > 5$. Indeed,

$$\text{the depth} \leq 2 \text{ part of } H^2(\mathcal{G}^{(l)}_p) = H^2_{(2)}(\mathcal{G}^{(l)}_\bullet(\mu_p)) = H^1(\Gamma_1(p), \varepsilon_2)\,.$$

2.6. Our strategy

To describe the structure of the Galois Lie algebras (11) in general we need the dihedral Lie algebra of the group μ_N ([12, 13]) recalled in section 4. To motivate to some extent its definition we turn in section 3 to the Hodge side of higher cyclotomy. As explained in section 5 both Galois and dihedral Lie algebra of μ_N act in a special way on the pronilpotent completion of $\pi_1(\mathbb{P}^1\backslash\{0, \mu_N, \infty\}, v_\infty)$, and the Galois is contained in the dihedral Lie algebra ([13]). In section 6 we relate the standard cochain complex of the dihedral Lie algebra of μ_N with the modular complex ([12]), whose canonical geometric realization in the symmetric space ([15]) is given in section 7. Thus we related the structure of the Galois Lie algebras with geometry of modular varieties.

3. Multiple Polylogarithms and higher Cyclotomy

3.1. Definition and iterated integral presentation

Multiple polylogarithms ([10, 9]) are defined as the power series

$$Li_{n_1,\dots,n_m}(x_1,\dots,x_m) = \sum_{0<k_1<k_2<\cdots<k_m} \frac{x_1^{k_1}x_2^{k_2}\cdots x_m^{k_m}}{k_1^{n_1}k_2^{n_2}\cdots k_m^{n_m}} \tag{25}$$

generalizing both the classical polylogarithms $Li_n(x)$ (if $m = 1$) and multiple ζ-values (if $x_1 = \cdots = x_m = 1$). These series are convergent for $|x_i| < 1$.

Recall a definition of iterated integrals. Let $\omega_1, \dots, \omega_n$ be 1-forms on a manifold M and $\gamma\colon [0,1] \to M$ a path. The iterated integral $\int_\gamma \omega_1 \circ \cdots \circ \omega_n$ is defined inductively:

$$\int_\gamma \omega_1 \circ \cdots \circ \omega_n \quad := \quad \int_0^1 \Big(\int_{\gamma_t} \omega_1 \circ \cdots \circ \omega_{n-1}\Big)\gamma^*\omega_n\,. \tag{26}$$

Here γ_t is the restriction of γ to the interval $[0,t]$ and $\int_{\gamma_t} \omega_1 \circ\cdots\circ \omega_{n-1}$ is considered as a function on $[0,1]$. We multiply it by the 1-form $\gamma_t^*\omega_n$ and integrate.

Denote by $I_{n_1,\dots,n_m}(a_1 : \cdots : a_m : a_{m+1})$ the iterated integral

$$\int_0^{a_{m+1}} \underbrace{\frac{dt}{a_1 - t} \circ \frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{n_1 \quad \text{times}} \circ \cdots \circ \underbrace{\frac{dt}{a_m - t} \circ \frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{n_m \quad \text{times}}\,. \tag{27}$$

Its value depends only on the homotopy class of a path connecting 0 and a_{m+1} on $\mathbb{C}^*\backslash\{a_1,\dots,a_m\}$. Thus it is a multivalued analytic function of $a_1,\dots,a_{m+1}$. The following result provides an analytic continuation of multiple polylogarithms.

Theorem 3.1. $Li_{n_1,\dots,n_m}(x_1,\dots,x_m) = I_{n_1,\dots,n_m}(1 : x_1 : x_1x_2 : \cdots : x_1\cdots x_m)$.

The proof is easy: develop $dt/(a_i - t)$ into a geometric series and integrate. If $x_i = 1$ we get the Kontsevich formula. In particular in the depth one case we recover the classical Leibniz presentation for $\zeta(n)$:

$$\zeta(n) = \int_0^1 \underbrace{\frac{dt}{1-t} \circ \frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{n \quad \text{times}}. \tag{28}$$

3.2. Multiple polylogarithms at roots of unity

Let $\overline{\mathcal{Z}}_{\leq w}(N)$ be the $\mathbb{Q}$-vector space spanned by the numbers

$$\overline{L}i_{n_1,...,n_m}(\zeta_N^{\alpha_1}, \ldots, \zeta_N^{\alpha_m}) := (2\pi i)^{-w} Li_{n_1,...,n_m}(\zeta_N^{\alpha_1}, \ldots, \zeta_N^{\alpha_m}); \qquad \zeta_N := e^{2\pi i/N}. \tag{29}$$

Here we may take any branch of $Li_{n_1,...,n_m}(x_1, \ldots, x_m)$. Similarly to (3) the space $\overline{\mathcal{Z}}(N) := \cup \overline{\mathcal{Z}}_{\leq w}(N)$ is an algebra *bifiltred* by the weight and by the depth. We want to describe this algebra and its associate graded for the weight and depth filtrations. Let us start from some relations between these numbers. Notice that

$$Li_1(\zeta_N^\alpha) = -\log(1 - \zeta_N^\alpha)$$

so the simplest case of this problem reduces to theory of cyclotomic units. By the Bass theorem all relations between the cyclotomic units $1 - \zeta_N^\alpha$ follow from the distribution relations and symmetry under $\alpha \to -\alpha$ valid modulo roots of unity.

3.3. Relations

The double shuffle relations. Consider the generating series

$$Li(x_1, \ldots, x_m | t_1, \ldots, t_m) := \sum_{n_i \geq 1} Li_{n_1,...,n_m}(x_1, \ldots, x_m) t_1^{n_1 - 1} \cdots t_m^{n_m - 1}.$$

Let $\Sigma_{p,q}$ be the subset of permutations of $p + q$ letters $\{1, \ldots, p+q\}$ consisting of all shuffles of $\{1, \ldots, p\}$ and $\{p+1, \ldots, p+q\}$. Similarly to (2) multiplying power series (25) we immediately get

$$Li(x_1, \ldots, x_p | t_1, \ldots, t_p) \cdot Li(x_{p+1}, \ldots, x_{p+q} | t_{p+1}, \ldots, t_{p+q}) =$$
$$= \sum_{\sigma \in \Sigma_{p,q}} Li(x_{\sigma(1)}, \ldots, x_{\sigma(p+q)} | t_{\sigma(1)}, \ldots, t_{\sigma(p+q)}) \quad + \quad \text{lower depth terms}. \tag{30}$$

To get the other set of the relations we multiply iterated integrals (27), and use theorem 3.1 plus the following product formula for the iterated integrals:

$$\int_\gamma \omega_1 \circ \cdots \circ \omega_p \cdot \int_\gamma \omega_{p+1} \circ \cdots \circ \omega_{p+q} \quad = \quad \sum_{\sigma \in \Sigma_{p,q}} \int_\gamma \omega_{\sigma(1)} \circ \cdots \circ \omega_{\sigma(p+q)}. \tag{31}$$

For example the simplest case of formula (31) is derived as follows:

$$\int_0^1 f_1(t) dt \cdot \int_0^1 f_2(t) dt \quad = \quad \Big(\int_{0 \leq t_1 \leq t_2 \leq 1} + \int_{0 \leq t_2 \leq t_1 \leq 1} \Big) f_1(t_1) f_2(t_2) dt_1 dt_2.$$

It is very similar in spirit to the derivation (3) of formula (2).

To get nice formulas consider the generating series

$$I^*(a_1 : \cdots : a_m : a_{m+1}|t_1, \ldots, t_m) :=$$
$$\sum_{n_i \geq 1} I_{n_1,\ldots,n_m}(a_1 : \cdots : a_m : a_{m+1}) t_1^{n_1-1}(t_1+t_2)^{n_2-1} \cdots (t_1 + \cdots + t_m)^{n_m-1} . \quad (32)$$

Theorem 3.2.

$$I^*(a_1 : \cdots : a_p : 1|t_1, \ldots, t_p) \cdot I^*(a_{p+1} : \cdots : a_{p+q} : 1|t_{k+1}, \ldots, t_{p+q}) =$$
$$= \sum_{\sigma \in \Sigma_{p,q}} I^*(a_{\sigma(1)}, \ldots, a_{\sigma(p+q)} : 1|t_{\sigma(1)}, \ldots, t_{\sigma(p+q)}) . \quad (33)$$

A sketch of the proof It is not hard to prove the following formula

$$I^*(a_1 : \cdots : a_m : 1|t_1, \ldots, t_m) = \int_0^1 \frac{s^{-t_1}}{a_1 - s} ds \circ \cdots \circ \frac{s^{-t_m}}{a_m - s} ds . \quad (34)$$

The theorem follows from this and product formula (31) for the iterated integrals.

For multiple ζ's these are precisely the relations of Zagier, who conjectured that, properly regularized, they provide all the relations between the multiple ζ's.

Distribution relations. From the power series expansion we immediately get

Proposition 3.3. *If $|x_i| < 1$ and l is a positive integer then*

$$Li(x_1, \ldots, x_m|t_1, \ldots, t_m) = \sum_{y_i^l = x_i} Li(y_1, \ldots, y_m|lt_1, \ldots, lt_m) . \quad (35)$$

If $N > 1$ the double shuffle plus distribution relations do not provide all relations between multiple polylogarithms at N-th roots of unity. However I conjecture they do give all the relations if N is a prime and we restrict to the weight = depth case.

3.4. Multiple polylogarithms at roots of unity and the cyclotomic Lie algebras

Denote by $UC_\bullet$ the universal enveloping algebra of a graded Lie algebra $C_\bullet$. Let $UC_\bullet^\vee$ be its graded dual. It is a commutative Hopf algebra.

Conjecture 3.4. *a) There exists a graded Lie algebra $C_\bullet(N)$ over $\mathbb{Q}$ such that one has an isomorphism $\overline{\mathcal{Z}}(N) = UC_\bullet(N)^\vee$ of filtered by the weight on the left and by the degree on the right algebras.*
b) $H^1_{(n)}(C_\bullet(N)) = K_{2n-1}(\mathbb{Z}[\zeta_N][\frac{1}{N}]) \otimes \mathbb{Q}$.
c) $C_\bullet(N) \otimes \mathbb{Q}_l = \mathcal{G}_N^{(l)}$ as filtered by the weight Lie algebras.

Here $H_{(n)}$ is the degree n part of H. Notice that $H^1_{(n)}(C_\bullet(N))$ is dual to the space of degree n generators of the Lie algebra $C_\bullet(N)$.

Examples 3.5. *i) If $N = 1$ the generators should correspond to $\overline{\zeta}(2n+1)$.*
ii) If $N > 1, n > 1$ the generators should correspond $\overline{L}i_n(\zeta_N^\alpha)$ where $(\alpha, N) = 1$.

A construction of the Lie algebra $C_\bullet(N)$ using the Hodge theory see in [17]. Similarly to theorem 1.2 one proves ([17]) that the algebra $\overline{\mathcal{Z}}(N)$ is a subalgebra of the universal enveloping algebra of the motivic Tate Lie algebra of the scheme S_N, i.e. free graded Lie algebra generated by $K_{2n-1}(S_N) \otimes \mathbb{Q}$ in degrees $n \geq 1$. However this estimate is not exact for sufficiently big N.

3.5. The coproduct

The iterated integral

$$I(a_0; a_1, \ldots, a_m; a_{m+1}) := \int_{a_0}^{a_{m+1}} \frac{dt}{t-a_1} \circ \cdots \circ \frac{dt}{t-a_m} \tag{36}$$

provides a framed mixed Hodge-Tate structure, denote by $I_{\mathcal{H}}(a_0; a_1, \ldots, a_m; a_{m+1})$, see [10, 11, 17]. The set of equivalence classes of framed mixed Hodge-Tate structures has a structure of the graded Hopf algebra over $\mathbb{Q}$ with the coproduct Δ.

Theorem 3.6. *For the framed Hodge-Tate structure corresponding to (36) we have:*

$$\Delta I_{\mathcal{H}}(a_0; a_1, a_2, \ldots, a_m; a_{m+1}) =$$

$$= \sum_{0=i_0<i_1<\cdots<i_k<i_{k+1}=m} I_{\mathcal{H}}(a_0; a_{i_1}, \ldots, a_{i_k}; a_{m+1}) \otimes \prod_{p=0}^{k} I_{\mathcal{H}}(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}}) \tag{37}$$

This formula also provides an explicit description of the variation of mixed Hodge-Tate structures whose period function is given by (36), see [11, 17]. Specializing it we get explicit formulas for the coproduct of all multiple polylogarithms (27). When a_i are N-th roots of unity and the lower depth terms are suppressed the result has a particular nice form. It is described, in an axiomatized form of the coproduct for dihedral Lie algebras, in the next section.

4. The Dihedral Lie Coalgebra of a Commutative Group G

Let G and H be two commutative groups or, better, commutative group schemes. Then, generalizing a construction given in [12, 13] one can define a graded Lie coalgebra $\mathcal{D}_\bullet(G|H)$, called the dihedral Lie coalgebra of G and H ([17]). In the special case when $H = \mathrm{Spec}\mathbb{Q}[[t]]$ is the additive group of the formal line it is a bigraded Lie coalgebra $\mathcal{D}_{\bullet\bullet}(G)$ called the dihedral Lie coalgebra of G. (The second grading is coming from the natural filtration on $\mathbb{Q}[[t]]$). We recall its definition below. The construction of $\mathcal{D}_\bullet(G|H)$ is left as an easy exercise.

4.1. Formal definitions ([12, 13])

Let G be a commutative group. We will define a bigraded Lie coalgebra $\mathcal{D}_{\bullet\bullet}(G) = \oplus_{w \geq m \geq 1} \mathcal{D}_{w,m}(G)$. The $\mathbb{Q}$-vector space $\mathcal{D}_{w,m}(G)$ is generated by the symbols

$$I_{n_1,\ldots,n_m}(g_1 : \cdots : g_{m+1}), \qquad w = n_1 + \cdots + n_m, \quad n_i \geq 1\,. \tag{38}$$

To define the relations we introduce the generating series

$$\{g_1 : \cdots : g_{m+1} | t_1 : \cdots : t_{m+1}\} := \sum_{n_i > 0} I_{n_1,\dots,n_m}(g_1 : \cdots : g_{m+1})(t_1 - t_{m+1})^{n_1 - 1} \cdots (t_m - t_{m+1})^{n_m - 1} . \quad (39)$$

We will also need two other generating series:

$$\{g_1 : \cdots : g_{m+1} | t_1, \dots, t_{m+1}\} := \{g_1 : \cdots : g_{m+1} | t_1 : t_1 + t_2 : \cdots : t_1 + \cdots + t_m : 0\} \quad (40)$$

where $t_1 + \cdots + t_{m+1} = 0$, and

$$\{g_1, \dots, g_{m+1} | t_1 : \cdots : t_{m+1}\} := \{1 : g_1 : g_1 g_2 : \cdots : g_1 \cdots g_m | t_1 : \cdots : t_{m+1}\} \quad (41)$$

where $g_1 \cdot \cdots \cdot g_{m+1} = 1$.

4.2. Relations

i) *Homogeneity.* For any $g \in G$ one has

$$\{g \cdot g_1 : \cdots : g \cdot g_{m+1} | t_1 : \cdots : t_{m+1}\} = \{g_1 : \cdots : g_{m+1} | t_1 : \cdots : t_{m+1}\} . \quad (42)$$

(Notice that the homogeneity in t is true by the very definition (39)).

ii) *The double shuffle relations* $(p + q = m, p \geq 1, q \geq 1)$.

$$\sum_{\sigma \in \Sigma_{p,q}} \{g_{\sigma(1)} : \cdots : g_{\sigma(m)} : g_{m+1} | t_{\sigma(1)}, \dots, t_{\sigma(m)}, t_{m+1}\} = 0, \quad (43)$$

$$\sum_{\sigma \in \Sigma_{p,q}} \{g_{\sigma(1)}, \dots, g_{\sigma(m)}, g_{m+1} | t_{\sigma(1)} : \cdots : t_{\sigma(m)} : t_{m+1}\} = 0 . \quad (44)$$

iii) *The distribution relations.* Let $l \in \mathbb{Z}$. Suppose that the l-torsion subgroup G_l of G is finite and its order is divisible by l. Then if $x_1, \dots, x_m$ are l-powers

$$\{x_1 : \cdots : x_{m+1} | t_1 : \cdots : t_{m+1}\} - \\ - \frac{1}{|G_l|} \sum_{y_i^l = x_i} \{y_1 : \cdots : y_{m+1} | l \cdot t_1 : \cdots : l \cdot t_{m+1}\} = 0$$

except the relation $I_1(e : e) = \sum_{y^l = e} I_1(y : e)$ which is not supposed to hold.

iv) $I_1(e : e) = 0$.

Denoted by $\widehat{\mathcal{D}}_{\bullet\bullet}(G)$ the bigraded space defined just as above except condition iv) is dropped, so $\widehat{\mathcal{D}}_{\bullet\bullet}(G) = \mathcal{D}_{\bullet\bullet}(G) \oplus \mathbb{Q}_{(1,1)}$ where $\mathbb{Q}_{(1,1)}$ is of bidgree $(1, 1)$.

Theorem 4.1. (Theorem 4.1 in [13]. *If $m \geq 2$ the double shuffle relations imply the dihedral symmetry relations, which include the cyclic symmetry*

$$\{g_1 : g_2 : ... : g_{m+1} | t_1 : t_2 : ... : t_{m+1}\} = \{g_2 : ... : g_{m+1} : g_1 | t_2 : ... : t_{m+1} : t_1\}$$

the reflection relation

$$\{g_1 : ... : g_{m+1} | t_1 : ... : t_{m+1}\} = (-1)^{m+1} \{g_{m+1} : ... : g_1 | - t_m : ... : -t_1 : -t_{m+1}\}$$

and the inversion relations

$$\{g_1 : ... : g_{m+1}|t_1 : ... : t_{m+1}\} = \{g_1^{-1} : ... : g_{m+1}^{-1}| - t_1 : ... : -t_{m+1}\}$$

4.3. Pictures for the definitions

We think about generating series (39) as a function of $m+1$ pairs $(g_1, t_1), \ldots, (g_{m+1}, t_{m+1})$ located cyclically on an oriented circle as follows. The oriented circle has slots, where the g's sit, and in between the consecutive slots, dual slots, where t's sit:

To make definitions (40) and (41) more transparent set $g'_i := g_i^{-1}g_{i+1}$, $t'_i := -t_{i-1} + t_i$ and put them on the circle together with g's and t's as follows:

Then it is easy to check that

$$\{g_1 : \cdots : g_{m+1}|t_1 : \cdots : t_{m+1}\} = \{g_1 : \cdots : g_{m+1}|t'_1, \ldots, t'_{m+1}\} = \\ = \{g'_1, \ldots, g'_{m+1}|t_1 : \cdots : t_{m+1}\}. \quad (45)$$

To picture any of three generating series (45) we leave on the circle only the two sets of variables among g's, g''s, t's, t''s which appear in this generating series:

The "$\{:\}$"-variables are outside, and the "$\{,\}$"-variables are inside of the circle.

4.4. Relation with multiple polylogarithms when $G = \mu_N$

Theorem 4.2. *(*see [12], [17]*) There is a well defined homomorphism of the* $\mathbb{Q}$*-vector spaces*

$$\mathcal{D}_{w,m}(\mu_N) \longrightarrow \mathrm{Gr}^{W,D}_{w,m}\mathcal{P}\overline{\mathcal{Z}}(N)$$

defined on the generators by

$$I_{n_1,\ldots,n_m}(a_1 : a_2 : \cdots : a_{m+1}) \longrightarrow (2\pi i)^{-w}\text{integral (27)}\,. \tag{46}$$

Here we consider iterated integrals (27) modulo the lower depth integrals and products of similar integrals. Formula (41) reflects theorem 3.1. Formula (40) reflects definition (32) of the generating series I^*. The shuffle relations (43) (resp. (44)) correspond to relations (30) (resp. (33)).

Remark 4.3. *Notice an amazing symmetry between g's and t's in the generating series (45), completely unexpected from the point of view of iterated integrals (27).*

4.5. The cobracket $\delta : \mathcal{D}_{\bullet\bullet}(G) \longrightarrow \Lambda^2\mathcal{D}_{\bullet\bullet}(G)$

It will be defined by

$$\begin{aligned}&\delta\{g_1 : \cdots : g_{m+1}|t_1 : \cdots : t_{m+1}\} =\\ &= -\sum_{k=2}^{m}\text{Cycle}_{m+1}\left(\{g_1 : \cdots : g_{k-1} : g_k|t_1 : \cdots : t_{k-1} : t_{m+1}\}\right.\\ &\left.\wedge\{g_k : \cdots : g_{m+1}|t_k : \cdots : t_{m+1}\}\right)\end{aligned} \tag{47}$$

where indices are modulo $m+1$ and $\text{Cycle}_{m+1}f(v_1,\ldots,v_m) := \sum_{i=1}^{m+1} f(v_i,\ldots,v_{i+m})$.

Each term of the formula corresponds to the following procedure: choose a slot and a dual slot on the circle. Cut the circle at the chosen slot and dual slot and make two oriented circles with the data on each of them obtained from the initial data. It is useful to think about the slots and dual slots as of little arcs, not points, so cutting one of them we get the arcs on each of the two new circles marked by the corresponding letters. The formula reads as follows:

$$\delta(47) = -\sum_{\text{cuts}}(\text{start at the dual slot}) \wedge (\text{start at the slot})$$

Theorem 4.4. *There exists unique map $\delta : \mathcal{D}_{\bullet\bullet}(G) \longrightarrow \Lambda^2\mathcal{D}_{\bullet\bullet}(G)$ for which (47) holds, providing a bigraded Lie coalgebra structure on $\mathcal{D}_{\bullet\bullet}(G)$.*

A similar result is true for $\widehat{\mathcal{D}}_{\bullet\bullet}(G)$. Moreover there is an isomorphism of bigraded Lie algebras $\widehat{\mathcal{D}}_{\bullet\bullet}(G) = \mathcal{D}_{\bullet\bullet}(G) \oplus \mathbb{Q}_{(1,1)}$.

5. The Dihedral Lie Algebra of μ_N and Galois Action on $\pi_1^{(l)}(X_N)$

5.1. Constraints on the image of the Galois group

Recall the homomorphism

$$\phi_N^{(l)} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}\mathbb{L}_N^{(l)} \tag{48}$$

It has the following properties.

i) *The action of* μ_N. The group μ_N acts on X_N by $z \longmapsto \zeta_N z$. This action does not preserve the base vector v_∞. However one can define an action of μ_N on $\pi_1^{(l)}(X_N, v_\infty) \otimes \mathbb{Q}_l$, and hence on $\mathbb{L}^{(l)}(X_N)$, commuting with the action of the Galois group $\mathrm{Gal}_{\mathbb{Q}(\zeta_{l^\infty N})}$ [13].

ii) *"Canonical generator" at* ∞. Recall the projection

$$\mathbb{L}^{(l)}(X_N) \longrightarrow \mathbb{L}^{(l)}(\mathbb{G}_m) = \mathbb{Z}_l(1) \tag{49}$$

Let X be a regular curve over $\overline{\mathbb{Q}}$, $\overline{X}$ the corresponding projective curve and v a tangent vector at $x \in \overline{X}$. Then there is a natural map of Galois modules ([D]):

$$\mathbb{Z}_l(1) = \pi_1^{(l)}(T_x\overline{X}\backslash 0, v) \longrightarrow \pi_1^{(l)}(X, v)$$

For $X = \mathbb{G}_m$, $x = \infty, v = v_\infty$ it is an isomorphism. So for $X = X_N$ it provides a splitting of (49):

$$X_\infty : \mathbb{Z}_l(1) = \mathbb{L}^{(l)}(\mathbb{G}_m) \hookrightarrow \mathbb{L}^{(l)}(X_N) \tag{50}$$

iii) *Special equivariant generators for* $\mathbb{L}_N^{(l)}$. For topological reasons there are well defined conjugacy classes of "loops around 0 or $\zeta \in \mu_N$" in $\pi_1(X_N, v_\infty)$. It turns out that there are μ_N-equivariant representatives of these classes providing a set of the generators for the Lie algebra $\mathbb{L}_N^{(l)}$:

Lemma 5.1. *There exist maps $X_0, X_\zeta : \mathbb{Q}_l(1) \longrightarrow \mathbb{L}_N^{(l)}$ which belong to the conjugacy classes of the "loops around $0, \zeta$" such that $X_0 + \sum_{\zeta\in\mu_N} X_\zeta + X_\infty = 0$ and the action of μ_N permutes X_ζ's (i.e. $\xi_* X_\zeta = X_{\xi\zeta}$) and fixes X_0, X_∞.*

To incorporate these constraints we employ a more general set up.

5.2. The Galois action and special equivariant derivations

Let G be a commutative group written multiplicatively. Let $L(G)$ be the free Lie algebra with the generators X_i where $i \in \{0\} \cup G$ (we assume $0 \notin G$). Set $X_\infty := -X_0 - \sum_{g\in G} X_g$.

A derivation D of the Lie algebra $L(G)$ is called special if there are elements $S_i \in L(G)$ such that

$$D(X_i) = [S_i, X_i] \quad \text{for any } i \in \{0\} \cup G, \quad \text{and} \quad D(X_\infty) = 0\,. \tag{51}$$

The special derivations of $L(G)$ form a Lie algebra, denoted $\mathrm{Der}^S L(G)$. Indeed, if $D(X_i) = [S_i, X_i]$, $D'(X_i) = [S'_i, X_i]$, then

$$[D, D'](X_i) = [S''_i, X_i], \quad \text{where} \quad S''_i := D(S'_i) - D'(S_i) + [S'_i, S_i]\,. \tag{52}$$

The group G acts on the generators by $h\colon X_0 \longmapsto X_0$, $X_g \longmapsto X_{hg}$. So it acts by automorphisms of the Lie algebra $L(G)$. A derivation D of $L(G)$ is called equivariant if it commutes with the action of G. Let $\mathrm{Der}^{SE}L(G)$ be the Lie algebra of all special equivariant derivations of the Lie algebra $L(G)$.

Denote by $\mathbb{L}(\mu_N)$ the pronilpotent completion of the Lie algebra $L(\mu_N)$. Lemma 5.1 provides a (non canonical) isomorphism $\mathbb{L}(\mu_N) \otimes \mathbb{Q}_l \longrightarrow \mathbb{L}_N^{(l)}$. Then it follows that $\mathcal{G}_N^{(l)}$ acts by special equivariant derivations of the Lie algebra $\mathbb{L}_N^{(l)}$, i.e.

$$\mathcal{G}_N^{(l)} \hookrightarrow \mathrm{Der}^{SE}\mathbb{L}_N^{(l)} . \tag{53}$$

5.3. Incorporating the two filtrations

The Lie algebra $L(G)$ is bigraded by the weight and depth. Namely, the free generators X_0, X_g are bihomogeneous: they are of weight -1, X_0 is of depth 0 and the X_g's are of depth -1. Each of the gradings induces a filtration of $L(G)$.

The Lie algebra $\mathrm{Der}L(G)$ is bigraded by the weight and depth. Its Lie subalgebras $\mathrm{Der}^{S}L(G)$ and $\mathrm{Der}^{SE}L(G)$ are compatible with the weight grading. However they are *not* compatible with the depth grading. Therefore they are graded by the weight, and *filtered* by the depth. A derivation (51) is of depth $-m$ if each S_j mod X_j is of depth $-m$, i.e. there are at least m X_i's different from X_0 in S_j mod X_j. The depth filtration is compatible with the weight grading. Let $\mathrm{GrDer}^{SE}_{\bullet\bullet}L(G)$ be the associated graded for the depth filtration.

Consider the following linear algebra situation. Let $W_\bullet L$ be a filtration on a vector space L. A splitting $\varphi\colon \mathrm{Gr}^W L \longrightarrow L$ of the filtration leads to an isomorphism $\varphi^*\colon \mathrm{End}(L) \longrightarrow \mathrm{End}(\mathrm{Gr}^W L)$. The space $\mathrm{End}(L)$ inherits a natural filtration, while $\mathrm{End}(\mathrm{Gr}^W L)$ is graded. The map φ^* respects the corresponding filtrations. The map $\mathrm{Gr}\varphi^*\colon \mathrm{Gr}^W(\mathrm{End}L) \longrightarrow \mathrm{End}(\mathrm{Gr}^W L)$ does not depend on the choice of the splitting. Therefore if $L = \mathbb{L}_N$ we get a *canonical* isomorphism

$$\mathrm{Gr}^W(\mathrm{Der}^{SE}\mathbb{L}_N) \stackrel{\sim}{=} \mathrm{Der}^{SE}L(\mu_N) \tag{54}$$

respecting the weight grading. Thus there is a *canonical* injective morphism

$$\mathrm{Gr}\mathcal{G}_{\bullet\bullet}^{(l)}(\mu_N) \hookrightarrow \mathrm{GrDer}^{SE}_{\bullet\bullet}\mathbb{L}_N^{(l)} \stackrel{\sim}{=} \mathrm{GrDer}^{SE}_{\bullet\bullet}L(\mu_N) \otimes \mathbb{Q}_l . \tag{55}$$

The Lie algebra $\mathcal{G}_N^{(l)}$ is isomorphic to $\mathrm{Gr}^W_\bullet\mathcal{G}_N^{(l)}$, but this isomorphism is not canonical. The advantage of working with $\mathrm{Gr}^W_\bullet\mathcal{G}_N^{(l)}$ is that, via isomorphism (54), it became a Lie subalgebra of $\mathrm{Der}^{SE}L(\mu_N) \otimes \mathbb{Q}_l$, which has natural generators provided by the canonical generators of $L(\mu_N)$. This gives canonical "coordinates" for description of $\mathrm{Gr}^W_\bullet\mathcal{G}_N^{(l)}$. The benefit of taking its associated graded for the *depth* filtration is an unexpected relation with the geometry of modular varieties for GL_m, where m is the depth.

5.4. Cyclic words and special differentiations ([7, 25])

Denote by $A(G)$ the free associative algebra generated by elements X_i where $i \in \{0\} \cup G$. Let $\mathcal{C}(A(G)) := A(G)/[A(G), A(G)]$ be the space of cyclic words in X_i.

Consider a map of linear spaces $\partial_{X_i} : \mathcal{C}(A(G)) \longrightarrow A(G)$ given on the generators by the following formula (the indices are modulo m):

$$\partial_{X_j}\mathcal{C}(X_{i_1}\cdots X_{i_m}) := \sum_{X_{i_k}=X_j} X_{i_{k+1}}\cdots X_{i_{k+m-1}}\,.$$

For example $\partial_{X_1}\mathcal{C}(X_1X_2X_1X_2^2) = X_2X_1X_2^2 + X_2^2X_1X_2$.

Define special derivations just as in (51), but with $S_i \in A(G)$. There is a map

$$\kappa : \mathcal{C}(A(G)) \longrightarrow \mathrm{Der}^S A(G), \quad \kappa\mathcal{C}(X_{i_1}\ldots X_{i_m})(X_j) := [\partial_{X_j}\mathcal{C}(X_{i_1},\ldots,X_{i_m}), X_j]\,.$$

It is easy to check that it is indeed a special derivation. Denote by $\widetilde{\mathcal{C}}(A(G))$ the quotient of $\mathcal{C}(A(G))$ by the subspace generated by the monomials X_i^n. Then one can show that the map κ provides an isomorphism of vector spaces

$$\kappa\colon \widetilde{\mathcal{C}}(A(G)) \longrightarrow \mathrm{Der}^S A(G)\,.$$

5.5. The dihedral Lie algebra as a Lie subalgebra of special equivariant derivations

Let G be a finite commutative group. We will use a notation Y for the generator X_0 of $A(G)$. So $\{Y, X_g\}$ are the generators of the algebra $A(G)$. Set

$$\mathcal{C}(X_{g_0}Y^{n_0-1}\cdot\ldots\cdot X_{g_m}Y^{n_m-1})^G := \sum_{h\in G}\mathcal{C}(X_{hg_0}Y^{n_0-1}\cdot\ldots\cdot X_{hg_m}Y^{n_m-1})\,.$$

Consider the following formal expression:

$$\xi_G := \sum \frac{1}{|\mathrm{Aut}\mathcal{C}|} I_{n_0,\ldots,n_m}(g_0:\cdots:g_m)\otimes\mathcal{C}(X_{g_0}Y^{n_0-1}\cdot\ldots\cdot X_{g_m}Y^{n_m-1})^G \quad (56)$$

where the sum is over all G-orbits on the set of cyclic words $\mathcal{C}$ in X_g, Y. The weight $1/|\mathrm{Aut}\mathcal{C}|$ is the order of automorphism group of the cyclic word $\mathcal{C}$.

Applying the map $\mathrm{Id}\otimes\mathrm{Gr}(\kappa)$ we get a bidegree $(0,0)$ element

$$\xi_G \in \mathcal{D}_{\bullet\bullet}(G)\widehat{\otimes}_{\mathbb{Q}}\mathrm{GrDer}^{SE}_{\bullet\bullet}A(G)\,.$$

Let $D_{-w,-m}(G) = \mathcal{D}_{w,m}(G)^\vee$. Then $D_{\bullet\bullet}(G) := \oplus_{w,m\geq 1}D_{-w,-m}(G)$ is a bigraded Lie algebra. Consider ξ_G as a map of bigraded spaces:

$$\xi_G \in Hom_{\mathbb{Q}-Vect}(D_{\bullet\bullet}(G), \mathrm{GrDer}^{SE}_{\bullet\bullet}A(G))\,. \quad (57)$$

Notice that $\mathrm{GrDer}^{SE}_{\bullet\bullet}L(G)$ is a Lie subalgebra of $\mathrm{GrDer}^{SE}_{\bullet\bullet}A(G)$.

Theorem 5.2. *The map ξ_G provides an injective Lie algebra morphism*

$$\xi_G\colon D_{\bullet\bullet}(G)\hookrightarrow\mathrm{GrDer}^{SE}_{\bullet\bullet}L(G)\,. \quad (58)$$

Theorem 5.3. $\mathrm{Gr}\mathcal{G}^{(l)}_{\bullet\bullet}(\mu_N) \hookrightarrow \xi_{\mu_N}(D_{\bullet\bullet}(\mu_N))\otimes_{\mathbb{Q}}\mathbb{Q}_l$.

If G is a trivial group we set $D_{\bullet\bullet} := D_{\bullet\bullet}(\{e\})$, and $\xi := \xi_{\{e\}}$. Denote by $D_\bullet(G)$ the quotient of $D_{\bullet\bullet}(G)$ by the components with $w \neq m$.

Conjecture 5.4. *a) One has $\xi(D_{\bullet\bullet})\otimes\mathbb{Q}_l = \mathrm{Gr}\mathcal{G}^{(l)}_{\bullet\bullet}$.*

b) Let p be a prime number. Then $\xi_{\mu_p}(D_\bullet(\mu_p)) = \mathrm{Gr}\mathcal{G}^{(l)}_{\bullet}(\mu_p)$.

Summarizing we see the following picture: both Lie algebras $D_{\bullet\bullet}(\mu_N)$ and $\mathrm{Gr}\mathcal{G}^{(l)}_{\bullet\bullet}(\mu_N)$ are realized as Lie subalgebras of the Lie algebra of special equivariant derivations $\mathrm{GrDer}^{SE}_{\bullet\bullet}L(G)$. The (image of) dihedral Lie algebra contains the (image of) Galois. Hypothetically they coincide when $N = 1$, or when weight=depth and N is prime. In general the gap between them exists, but should not be big.

Theorem 5.5. *Conjecture 5.4 is true for* $m = 1, 2, 3$.

The proof of this theorem is based on the following two ideas: the standard cochain complex of $D_{\bullet\bullet}(\mu_N)$ is related to the modular complex, and the modular complex has a geometric realization. We address them in the last two sections.

6. Modular Complexes and Galois Symmetries of $\pi_1^{(l)}(X_N)$

6.1. The modular complexes

Let L_m be a rank m lattice. The rank m modular complex $M^\bullet(L_m) = M^\bullet_{(m)}$ is a complex of $GL_m(\mathbb{Z})$-modules

$$M^1_{(m)} \stackrel{\partial}{\longrightarrow} M^2_{(m)} \stackrel{\partial}{\longrightarrow} ... \stackrel{\partial}{\longrightarrow} M^m_{(m)}$$

If $m = 2$ it is isomorphic to complex (19). In general it is defined as follows.

i) *The group* $M^1_{(m)}$. An extended basis of a lattice L_m is an $(m+1)$-tuple of vectors $v_1, ..., v_{m+1}$ of the lattice such that $v_1 + ... + v_{m+1} = 0$ and $v_1, ..., v_m$ is a basis. The extended bases form a principal homogeneous space over $GL_m(\mathbb{Z})$.

The abelian group $M^1(L_m) = M^1_{(m)}$ is generated by the extended bases. Denote by $< e_1, ..., e_{m+1} >$ the generator corresponding to the extended basis $e_1, ..., e_{m+1}$.

Let $u_1, ..., u_{m+1}$ be elements of the lattice L_m such that the set of elements $\{(u_i, 1)\}$ form a basis of $L_m \oplus \mathbb{Z}$. The lattice L_m acts on such sets by $l : \{(u_i, 1)\} \longmapsto \{(u_i + l, 1)\}$. We call the coinvariants of this action *homogeneous affine bases* of L_m and denote them by $\{u_1 : ... : u_{m+1}\}$.

To list the relations we need another sets of the generators corresponding to the homogeneous affine bases of L_m (compare with (40)):

$$< u_1 : ... : u_{m+1} > := < u'_1, u'_2, ..., u'_{m+1} >; \qquad u'_i := u_{i+1} - u_i$$

We will also employ the notation

$$[v_1, ..., v_k] := < v_1, ..., v_k, v_{k+1} >, \quad v_1 + ... + v_k + v_{k+1} = 0$$

Relations. One has $< v, -v > = < -v, v >$. For any $1 \leq k \leq m$, $m \geq 2$ one has (compare with (43) - (44)):

$$\sum_{\sigma \in \Sigma_{k,m-k}} < v_{\sigma(1)}, ..., v_{\sigma(m)}, v_{m+1} > \quad = \quad 0 \qquad (59)$$

$$\sum_{\sigma \in \Sigma_{k,m-k}} < u_{\sigma(1)} : ... : u_{\sigma(m)} : u_{m+1} > \quad = \quad 0 \qquad (60)$$

ii) *The group* $M^k_{(m)}$. It is the sum of the groups $M^1(L^1)\wedge\dots\wedge M^1(L^k)$ over all unordered lattice decompositions $L_m = L^1\oplus\dots\oplus L^k$. Thus it is generated by the elements $[A_1]\wedge\dots\wedge[A_k]$ where A_i is a basis of the sublattice L_i and $[A_i]$'s anticommute. Define a map $\partial\colon M^1_{(m)}\longrightarrow M^2_{(m)}$ by setting (compare with (47))

$$\partial\colon < v_1,\dots,v_{m+1} >\longmapsto -\mathrm{Cycle}_{m+1}\Big(\sum_{k=1}^{m-1}[v_1,\dots,v_k]\wedge[v_{k+1},\dots,v_m]\Big)$$

where indices are modulo $m+1$. We get the differential in $M^\bullet_{(m)}$ by Leibniz' rule:

$$\partial([A_1]\wedge[A_2]\wedge\dots) := \partial([A_1])\wedge[A_2]\wedge\dots - [A_1]\wedge\partial([A_2])\wedge\dots+\dots\,.$$

6.2. The modular complexes and the cochain complex of $D_{\bullet\bullet}(\mu_N)$

Denote by $\Lambda^*_{(m,w)}\mathcal{D}_{\bullet\bullet}(\mu_N)$ the depth m, weight w part of the standard cochain complex of the Lie algebra $D_{\bullet\bullet}(\mu_N)$.

Theorem 6.1. *a) For $m>1$ there exists canonical surjective map of complexes*

$$\mu^*_{m;w}\colon M^*_{(m)}\otimes_{\Gamma_1(m;N)} S^{w-m}\mathrm{V}_m \longrightarrow \Lambda^*_{(m,w)}\mathcal{D}_{\bullet\bullet}(\mu_N)\,. \qquad (61)$$

b) Let $N=1$, or $N=p$ is a prime and $w=m$. Then this map is an isomorphism.

The map (61) was defined in [12]. Here is the definition when $w=m$. Notice that

$$M^*_{(m)}\otimes_{\Gamma_1(m;N)}\mathbb{Q} = M^*_{(m)}\otimes_{GL_m(\mathbb{Z})}\mathbb{Z}[\Gamma_1(m;N)\backslash GL_m(\mathbb{Z})]\,.$$

The set $\Gamma_1(m;N)\backslash GL_m(\mathbb{Z})$ is identified with the set $\{(\alpha_1,\dots,\alpha_m)\}$ of all nonzero vectors in the vector space over $\mathbb{Z}/p\mathbb{Z}$. Then

$$\mu^1_{m;m}: [v_1,\dots,v_m]\otimes(\alpha_1,\dots,\alpha_m)\longrightarrow I_{1,\dots,1}(\zeta_N^{\alpha_1},\dots,\zeta_N^{\alpha_m})\,.$$

The other components $\mu^*_{m;m}$ are the wedge products of the maps $\mu^1_{k;k}$.

6.3. Modular complexes and cochain complexes of Galois Lie algebras

Combining theorems 6.1a) and 5.3 we get a surjective map of complexes

$$M^*_{(m)}\otimes_{\Gamma_1(m;N)} S^{w-m}\mathrm{V}_m\longrightarrow \Lambda^*_{(m,w)}\mathrm{Gr}\mathcal{G}^{(l)}_{\bullet\bullet}(\mu_N)^\vee\,. \qquad (62)$$

So using theorem 6.1b) we reformulate conjecture 5.4 as follows:

Conjecture 6.2. *Let $N=1$, or $N=p$ is a prime and $w=m$. Then the map (62) is an isomorphism.*

According to theorem 5.3 the cochain complex of $D_{\bullet\bullet}(\mu_N)$ projects onto the cochain complex of the level N Galois Lie algebra. Combining this with theorem 5.5 (and conjecture 6.2) we describe the structure of the Galois Lie algebras via the modular complexes. Since the modular complexes are defined very explicitly this leads to a precise description of Galois Lie algebras. However to get the most interesting results about them we need the geometric realization of modular complexes, which allows to express the structure of the Galois Lie algebra in terms of the *geometry and topology* of modular varieties.

7. Geometric Realization of Modular Complexes in Symmetric Spaces

As was emphasized before the rank m modular complex is purely combinatorial object. Surprisingly it has a *canonical* realization in the symmetric space $\mathbb{H}_n$. In the simplest case $m = 2$ it identifies the rank two modular complex with the chain complex of the modular triangulation of the hyperbolic plane.

7.1. Voronoi's cell decomposition of the symmetric space for $GL_m(\mathbb{R})$

Let

$$\mathbb{H}_m := GL_m(\mathbb{R})/O(m) \cdot \mathbb{R}^* \quad = \frac{> 0 \text{ definite quadratic forms on } V_m^*}{\mathbb{R}_+^*}$$

Let $L_m \subset V_m$ be a lattice. Any vector $v \in V_m$ defines a nonnegative definite quadratic form $\varphi(v) :=< v, \cdot >^2$ on V_m^*. The convex hull of the forms $\varphi(l)$, when l runs through all non zero primitive vectors of the lattice L_m, is an infinite polyhedra. Its projection into the closure of $\mathbb{H}_m$ defines a polyhedral decomposition of $\mathbb{H}_m$ invariant under the symmetry group of the lattice L_m.

Example 7.1. *If $m = 2$ we get the modular triangulation of the hyperbolic plane.*

Denote by $(V_\bullet^{(m)}, d)$ the chain complex of the Voronoi decomposition.

7.2. The relaxed modular complex

Consider a version $\widehat{M}_{(m)}^\bullet$ of the modular complex, called the relaxed modular complex, where the group $\widehat{M}_{(m)}^1$ is defined using the same generators $[v_1, \ldots, v_m]$ satisfying only the first shuffle relations (59) and the dihedral symmetry relations

$$< v_2, \ldots, v_{m+1}, v_1 >=< v_1, \ldots, v_{m+1} >= (-1)^{m+1} < v_{m+1}, \ldots, v_1 > \ .$$

The other groups are defined in a similar way. The differential is as before.

7.3. The geometric realization map

Denote by $\varphi(v_1, \ldots, v_k)$ the convex hull of the forms $\varphi(v_1), \ldots, \varphi(v_k)$ in the space of quadratic forms. Let $v_1, \ldots, v_{n_1}$ and $v_{n_1+1}, \ldots, v_{n_1+n_2}$ be two sets of lattice vectors such that the lattices they generate have zero intersection. Define the join $*$ by

$$\varphi(v_1, \ldots, v_{n_1}) * \varphi(v_{n_1+1}, \ldots, v_{n_1+n_2}) := \varphi(v_1, \ldots, v_{n_1+n_2})$$

and extend it by linearity. Make a homological complex $\widehat{M}_\bullet^{(m)}$ out of $\widehat{M}_{(m)}^\bullet$ by

$$\widehat{M}_i^{(m)} := \widehat{M}_{(m)}^{2m-1-i} \ .$$

Theorem 7.2. *There exists a canonical morphism of complexes*

$$\widehat{\psi}_\bullet^{(m)} : \widehat{M}_\bullet^{(m)} \longrightarrow V_\bullet^{(m)} \qquad \textit{such that}$$

$$\widehat{\psi}_\bullet^{(m)}\Big([A_1] \wedge \cdots \wedge [A_k]\Big) := \widehat{\psi}_\bullet^{(m)}([A_1]) * \cdots * \widehat{\psi}_\bullet^{(m)}([A_k]) \ . \tag{63}$$

In particular $\widehat{\psi}^{(m)}_{m-1}\Big([v_1]\wedge\cdots\wedge[v_m]\Big) = \varphi(v_1,\ldots,v_m)$.

To define such a morphism $\widehat{\psi}^{(m)}_{\bullet}$ one needs only to define $\widehat{\psi}^{(m)}_{2m-2}([v_1,\ldots,v_m])$ for vectors $v_1,\ldots,v_m$ forming a basis of the lattice L_m in such a way that the dihedral and the first shuffle relations go to zero and

$$d\widehat{\psi}^{(m)}_{2m-2}([v_1,\ldots,v_m]) = \widehat{\psi}^{(m)}_{2m-3}(\partial[v_1,\ldots,v_m])$$

where the right hand side is computed by (63) and the formula for ∂.

7.4. Construction of the map $\widehat{\psi}^{(m)}_{2m-2}$

A *plane tree* is a tree without self intersections located on the plane. The edges of a tree consist of legs (external edges) and internal edges. Choose a lattice L_m. A *colored tree* is a plane tree whose legs are in a bijective correspondence with the elements of an affine basis of the lattice L_m. In particular a colored 3-valent tree has $2m-1$ edges. We visualize it as follows:

The vectors $e_0,\ldots,e_m$ of an affine basis are located cyclically on an oriented circle and the legs of the tree end on the circle and labelled by $e_0,\ldots,e_m$. (The circle itself is not a part of the graph).

Construction. Each edge E of the tree T provides a vector $f_E\in L_m$ defined up to a sign. Namely, the edge E determines two trees rooted at E, see the picture

The union of the incoming (i.e. different from E) legs of these rooted trees coincides with the set of all legs of the initial tree. Take the sum of all the vectors e_i corresponding to the incoming legs of one of these trees. Denote it by f_E. If we choose the second rooted tree the sum will change the sign. So the degenerate quadratic form $\varphi(f_E)$ is well defined. Set

$$\widehat{\psi}^{(m)}_{2m-2}(< e_0,e_1,\ldots,e_m >) := \sum_{\text{plane 3-valent trees}} \mathrm{sgn}(E_1\wedge\cdots\wedge E_{2m-1})\cdot\varphi(f_{E_1},\ldots,f_{E_{2m-1}}) \quad (64)$$

Here the sum is over all plane 3-valent trees colored by $e_0,\ldots,e_m$. The sign is defined as follows. Let $V(E)$ be the $\mathbb{R}$-vector space generated by the edges of a tree.

An orientation of a tree is a choice of the connected component of $\det(V(E))\backslash 0$. A plane 3-valent tree has a canonical orientation. Indeed, the orientation of the plane provides orientations of links of each of the vertices. The sign in (64) is taken with respect to the canonical orientation of the plane 3-valent tree. Then one proves ([15]) that this map has all the required properties, so we get theorem 7.2.

Examples 7.3. *a) For $m = 2$ there is one plane 3-valent tree colored by e_0, e_1, e_2, so we get a modular triangle $\varphi(e_0, e_1, e_2)$ on the hyperbolic plane. The geometric realization in this case leads to an isomorphism of complexes $M_\bullet^{(2)} \longrightarrow V_\bullet^{(2)}$:*

$$[e_1, e_2] \longmapsto \varphi(e_0, e_1, e_2); \quad [e_1] \wedge [e_2] \longmapsto \varphi(e_1) * \varphi(e_2) = \varphi(e_1, e_2)$$

b) Let $f_{ij} := e_i + e_j$. For $m = 3$ there are two plane 3-valent trees colored by e_0, e_1, e_2, e_3, see the picture, so the chain is

$$\widehat{\psi}_4^{(3)}([e_1, e_2, e_3]) := \quad \varphi(e_0, e_1, e_2, e_3, f_{01}) - \varphi(e_0, e_1, e_2, e_3, f_{12})$$

The symmetric space $\mathbb{H}_3$ has dimension 5. The Voronoi decomposition consists of the cells of dimensions $5, 4, 3, 2$. All Voronoi cells of dimension 5 are $GL_3(\mathbb{Z})$-equivalent to the Voronoi simplex $\varphi(e_0, e_1, e_2, e_3, f_{01}, f_{12})$. The map $\widehat{\psi}_4^{(3)}$ sends the second shuffle relation (59) to the boundary of a Voronoi 5-simplex.

Theorem 7.4. *The geometric realization map provide quasiisomorphisms*

$$M_\bullet^{(3)} \longrightarrow \tau_{[4,2]}(V_\bullet^{(3)}); \qquad M_\bullet^{(4)} \longrightarrow \tau_{[6,3]}(V_\bullet^{(4)}) .$$

7.5. Some corollaries

a) Let $N = p$ be a prime. Take the geometric realization of the rank 3 relaxed modular complex. Project it onto the modular variety $Y_1(3; p)$. Take the quotient of the group of the 4-chains generated by $\widehat{\psi}_4^{(3)}(v_1, v_2, v_3)$ on $Y_1(3; p)$ by the subgroup generated by the boundaries of the Voronoi 5-cells. Then the complex we get is *canonically isomorphic* to the depth=weight 3 part of the standard cochain complex of the level p Galois Lie algebra. Therefore

$$H^i_{(3)}(\mathrm{Gr}\widehat{\mathcal{G}}_\bullet^{(l)}(\mu_p)) = H^i(\Gamma_1(3; p)) \quad i = 1, 2, 3 .$$

In particular we associate to each of the numbers $Li_{1,1,1}(\zeta_p^{\alpha_1}, \zeta_p^{\alpha_2}, \zeta_p^{\alpha_3})$, or to the corresponding Hodge, l-adic or motivic avatars of these numbers, a certain 4-cell on the 5-dimensional orbifold $Y_1(3; p)$. The properties of the framed motive encoded by this number, like the coproduct, can be read from the geometry of this 4-cell.

Similarly the map $\widehat{\psi}^{(m)}_{2m-2}$ provides a canonical $(2m-2)$-cell on $Y_1(m;p)$ corresponding to the framed motive with the period $Li_{1,\ldots,1}(\zeta_p^{\alpha_1},\ldots,\zeta_p^{\alpha_m})$.

b) $N=1$. Theorems 5.5, 6.1a) and 7.4 lead to the following

Theorem 7.5.

$$\dim \mathrm{Gr}\mathcal{G}^{(l)}_{-w,-3} = \begin{cases} 0 & w: \quad \textit{even} \\ \left[\frac{(w-3)^2-1}{48}\right] & w: \quad \textit{odd}. \end{cases} \tag{65}$$

Since, according to standard conjectures, this number should coincide with $d_{w,3}$ the estimate given in theorem 1.4 should be exact.

8. Multiple Elliptic Polylogarithms

The story above is related to the field $\mathbb{Q}$. I hope that for an arbitrary number field F there might be a similar story. The Galois group $\mathrm{Gal}(\overline{F}/F)$ should have a remarkable quotient G_F given by an extension of the maximal abelian quotient of $\mathrm{Gal}(\overline{F}/F)$ by a prounipotent group U_F:

$$0 \longrightarrow U_F \longrightarrow \mathrm{G}_F \longrightarrow \mathrm{Gal}(\overline{F}/F)^{\mathrm{ab}} \longrightarrow 0$$

Its structure should be related to modular varieties for $GL_m/_F$, for all m.

The group $\mathrm{G}_{\mathbb{Q}}$ is obtained from the motivic fundamental group of $\mathrm{G}_m - \{$"all" roots of unity$\}$. It turns out that for an imaginary quadratic filed K one can get a similar picture by taking the motivic fundamental group of the CM elliptic curve $E_K := \mathbb{C}/\mathcal{O}_K$ punctured at the torsion points. Below we construct the periods of the corresponding mixed motives, *multiple Hecke L-values*, as the values at torsion points of multiple elliptic polylogarithms. We define the multiple polylogarithms for arbitrary curves as correlators for certain Feynman integrals. We make sence out of these Feynman integrals by using the perturbation expansion via Feynman diagrams, which in this case are plane 3-valent trees. Unlike the Feynman integrals the coefficients of the perturbative expansion are given by convergent finite dimensional integrals, and so well defined. I leave to the reader the pleasure to penetrate the analogy between this construction and the geometric realization of modular complexes described in s. 7.4.

8.1. The classical Eisensten-Kronecker series.

Let E be an elliptic curve over $\mathbb{C}$ with the period lattice Γ, so that $E(\mathbb{C}) = \mathbb{C}/\Gamma$. The intersection form $\Lambda^2\Gamma \longrightarrow 2\pi i\mathbb{Z}$ leads to the pairing $\chi : E(\mathbb{C}) \times \Gamma \longrightarrow S^1$. So for $a \in E(\mathbb{C})$ we get a character $\chi_a : \Gamma \longrightarrow S^1$.

Consider the generating function for the classical Eisenstein-Kronecker series

$$G(a|t) := \quad \frac{\mathrm{vol}(\Gamma)}{\pi} {\sum_{\gamma\in\Gamma}}' \frac{\chi_a(\gamma)}{|\gamma - t|^2}$$

where $\sum'$ means the summation over all non zero vectors γ of the lattice. It depends on a point a of the elliptic curve and an element t in a formal neighborhood of zero in $H_1(E, \mathbb{R})$. It is invariant under the involution $a \longmapsto -a, t \longmapsto -t$. Expanding it into power series in t and $\bar{t}$ we get, as the coefficients, the classical Eisensten-Kronecker series:

$$G(a|t) = \sum_{p,q \geq 1} \Big(\frac{\mathrm{vol}(\Gamma)}{\pi} \sum_{\gamma \in \Gamma}{}' \frac{\chi_a(\gamma)}{\gamma^p \overline{\gamma}^q} \Big) t^{p-1} \bar{t}^{q-1}$$

When E is a CM curve their special values at the torsion points of E provide the special values of the Hecke L-series with Groessencharacters.

8.2. Multiple Eisenstein-Kronecker series: a description

We define them as the coefficients of certain generating functions. The generating function for the depth m multiple Eisenstein-Kronecker series is a function

$$G(a_1 : ... : a_{m+1} | t_1, ..., t_{m+1}), \quad t_1 + ... + t_{m+1} = 0$$

where a_i are points on the elliptic curve E and t_i are elements in a formal neighborhood of zero in $H_1(E, \mathbb{R})$. It is invariant under the shift $a_i \longrightarrow a_i + a$. Decomposing this function into the series in $t_i, \bar{t}_i$ we get the depth m multiple Eisenstein-Kronecker series.

Construction. Consider a plane trivalent tree T colored by $m + 1$ pairs consisting of points a_i on the elliptic curve E and formal elements $t_i \in H_1(E, \mathbb{R})$:

$$(a_1, t_1), ..., (a_{m+1}, t_{m+1}); \qquad t_1 + ... + t_{m+1} = 0 \tag{66}$$

Each oriented edge $\overrightarrow{E}$ of the tree T provides an element $t_{\overrightarrow{E}} \in H_1(E, \mathbb{R})$. Namely, as explained in s. 7.4 the edge E determines two trees rooted at E. An orientation of the edge E corresponds to the choice of one of them: take the tree obtained by going in the direction of the orientation of the edge. Then $t_{\overrightarrow{E}}$ is the sum of all t_i's corresponding to the legs of this tree different from E. Since $\sum t_i = 0$ changing the orientation of the edge E we get $-t_{\overrightarrow{E}}$.

Let X be a manifold and $\mathcal{A}^i(X)$ the space of i-forms on X. We define a map

$$\omega_m : \Lambda^{m+1} \mathcal{A}^0(X) \longrightarrow \mathcal{A}^m(X), \qquad \omega_m : \varphi_1 \wedge ... \wedge \varphi_{m+1} \longmapsto$$

$$\frac{1}{(m+1)!} \mathrm{Alt}_{m+1} \Big(\sum_{j=0}^{m+1} (-1)^j \varphi_1 \partial\varphi_2 \wedge ... \wedge \partial\varphi_k \wedge \overline{\partial}\varphi_{k+1} \wedge ... \wedge \overline{\partial}\varphi_{m+1} \Big)$$

If $\varphi_i = \log |f_i|$ it is the form used to define the Chow polylogarithms in [G10].

Every edge E of the tree T defines a function G_E on $E(\mathbb{C})^{\{\text{vertices of T}\}}$ depending on t_i. Namely, let v_1^E, v_2^E be the vertices of the edge E. Their order orients the edge E. Consider the natural projection

$$p_E : E(\mathbb{C})^{\{\text{vertices of the tree T}\}} \longrightarrow E(\mathbb{C})^{\{v_1^E, v_2^E\}} = E(\mathbb{C}) \times E(\mathbb{C}) \quad (67)$$

Then $G_E := p_E^* G\left(x_1^E - x_2^E | t_{\vec{E}}\right)$ where (x_1^E, x_2^E) is a point at the right of (67). This function does not depend on the orientation of the edge E.

Definition 8.1.

$$G(a_1 : ... : a_{m+1} | t_1, ..., t_{m+1}) := \quad (68)$$

$$\sum_{\textit{plane 3-valent trees } T} \mathrm{sgn}(E_1 \wedge ... \wedge E_{2m-1}) \cdot \int_{S^{m-1}E(\mathbb{C})} \mathrm{sym}^* \omega_{2m-2} \left(G_{E_1} \wedge ... \wedge G_{E_{2m-1}}\right)$$

The sum is over all plane 3-valent trees whose legs are cyclically labelled by (66). The correspondence $\mathrm{sym} : S^{m-1}E(\mathbb{C}) \longrightarrow E(\mathbb{C})^{\{\text{internal vertices of T}\}}$ is given by the sum of all $(m-1)!$ natural maps $E(\mathbb{C})^{m-1} \longrightarrow E(\mathbb{C})^{\{\text{internal vertices of T}\}}$.

Recall the CM elliptic curve E_K. Let $\mathcal{N}$ be an ideal of $\mathrm{End}(E_K)$. Denote by $K_{\mathcal{N}}$ the field generated by the $\mathcal{N}$-torsion points of E_K. If a_i are $\mathcal{N}$-torsion points of E_K we view the numbers obtained in the $t, \bar{t}$-expansion (68) as multiple Hecke L-values related to K. They are periods of mixed motives over the ring of integers in $K_{\mathcal{N}}$, with $\mathrm{Norm}(\mathcal{N})$ inverted. These are the motives which appear in the motivic fundamental group of $E - \{\mathcal{N} - \text{torsion points}\}$.

9. Multiple polylogarithms on curves, Feynman integrals and special values of L-functions

9.1. Polylogarithms on curves and special values of L-functions

Let X be a regular complex algebraic curve. We also suppose, for simplicity only, that it is a projective curve of genus $g \geq 1$. Choose a volume form on $X(\mathbb{C})$, and let $G(x, y)$ be the corresponding Green function. Set

$$\mathcal{H} := H_1(X, \mathbb{R}); \qquad \mathcal{H}_{\mathbb{C}} := \mathcal{H} \otimes \mathbb{C} = \mathcal{H}^{-1,0} \oplus \mathcal{H}^{0,-1}$$

For each integer $n \geq 1$ we define a 0-current $G_n(x, y)$ on $X \times X$ with values in

$$\mathrm{Sym}^{n-1}\mathcal{H}_{\mathbb{C}}(1) \quad = \quad \oplus_{a+b=n-2} S^{a-1}\mathcal{H}^{-1,0} \otimes S^{b-1}\mathcal{H}^{0,-1} \quad (69)$$

Then $G_1(x, y) := G(x, y)$. For $n > 1$ it is a function on $X(\mathbb{C}) \times X(\mathbb{C})$.

To define the function $G_n(x, y)$ we proceed as follows. Let

$$\overline{\Omega}_a = \omega_{\alpha_1} \cdot ... \cdot \omega_{\alpha_{a-1}} \in S^{a-1}\overline{\Omega}^1, \quad \Omega_b = \omega_{\beta_1} \cdot ... \cdot \omega_{\beta_{b-1}} \in S^{b-1}\Omega^1; \qquad \omega_* \in \Omega^1$$

Then $\overline{\Omega}_a \otimes \Omega_b$ is an element of the dual to (69). We are going to define the pairing $< G_n(x, y), \overline{\Omega}_a \otimes \Omega_b >$. Denote by $p_i : X^{n-1} \longrightarrow X$ the projection on i-th factor.

Definition 9.1. *The n-th polylogarithm function on the curve X is defined by*

$$< G_n(x,y), \overline{\Omega}_a \otimes \Omega_b >:=$$

$$\text{Alt}_{\{z_1,...,z_{n-1}\}}\Big(\int_{X^{n-1}(\mathbb{C})} \omega_{n-1}\Big(G(x,z_1)\wedge G(z_1,z_2)\wedge ... \wedge G(z_{n-2},z_{n-1})\wedge G(z_{n-1},y)\Big)\wedge$$

$$\Lambda_{i=1}^{a-1} p_i^* \overline{\omega}_{\alpha_i} \wedge \Lambda_{j=1}^{b-1} p_{a-1+j}^* \omega_{\beta_j}\Big)$$

We skewsymmetrized the integrand with respect to $z_1, ..., z_{n-1}$.

These functions provide a variation of $\mathbb{R}$-mixed Hodge structures on $X \times X - \Delta$ of motivic origin. If X is an elliptic curve it is given by Beilinson-Levin theory of elliptic polylogarithms [2].

In particular for a pair of distinct points x, y on X we get an $S^{n-1}\mathcal{H}(1)$-framed mixed motive (see [16] for the background) denoted $\{x,y\}_n$, whose period is given by $G_n(x,y)$. Its coproduct δ is given by $\{x,y\}_n \longmapsto \{x,y\}_{n-1} \wedge (x-y)$ where $(x-y)$ is the point of the Jacobian of X corresponding to the divisor $\{x\} - \{y\}$. If X is defined over a number field this leads to a very precise conjecture expressing the special value $L(S^{n-1}H^1(X), n)$ via the polylogarithms $G_n(x,y)$ - an analog of Zagier's conjecture. If X is an elliptic curve we are in the situation considered in [16], [32]. An especially interesting example appears when x, y are cusps on a modular curve. Then $(x-y)$ is a torsion point in the Jacobian, so $\delta\{x,y\}_n = 0$ and thus $G_n(x,y)$ is the regulator of an element of motivic $Ext^1(\mathbb{Q}(0), S^{n-1}\mathcal{H}(1))$!

In particular we can apply this construction in the case when $X = \mathbb{G}_m$, $G(x,y) = \log|x-y|$ and $\omega = d\log(z)$. Then the construction above boils down to the Chow polylogarithm ([18]) corresponding to the element

$$(x-z_1)\wedge(z_1-z_2)\wedge ... \wedge(z_{n-2}-z_{n-1})\wedge(z_{n-1}-y)\wedge z_1 \wedge ... \wedge z_{n-1} \in \Lambda^{2n-1}\mathcal{O}^*(\mathbb{G}_m^{n-1})$$

9.2. Multiple polylogarithms on curves

We package the polylogarithms $G_n(x,y)$ into the generating series

$$G(x,y|t_1,t_2), \quad t_i \in \mathcal{H}, t_1 + t_2 = 0$$

so that $G_n(x,y)$ emerges as the weight $-n-1$ component of the power series decomposition into $t_1, \overline{t}_1$. Then $G(x,y|t_1,t_2) = G(y,x|t_2,t_1)$. The construction of the previous section provides multiple polylogarithms $G(a_1, ..., a_{m+1}|t_1, ..., t_{m+1})$ on X, where $t_1 + ... + t_{m+1} = 0$. Indeed, for an edge E of a plane 3-valent tree T set $G_E := p_E^* G(x_1^E, x_2^E | t_{\vec{E}}, -t_{\vec{E}})$ and repeat the construction. We call the constant term in t's the multiple Green function on X.

Applying this construction in the case when $X = \mathbb{G}_m$ we get a single-valued version of multiple polylogarithms written as Chow polylogarithms. One can also interpret them as Grassmannian polylogarithms ([18]) on certain stratas.

9.3. Feynman integral for multiple Green functions

Let φ be a function and ψ a $(1,0)$-form on $X(\mathbb{C})$ with values in $N \times N$ complex skew symmetric matrices. We denote by $\overline{\varphi}$ and $\overline{\psi}$ the result of the action of complex conjugation. Then the multiple logarithm (=Green) function $G(a_1, .., a_{m+1})$ emerges as the leading term of the asymptotic when $N \to \infty$ of the following correlator:

$$\int \mathrm{Tr}\Big((\varphi + \overline{\varphi})(a_1) \cdot ... \cdot (\varphi + \overline{\varphi})(a_{m+1})\Big) e^{iS(\varphi,\psi)} \mathcal{D}\varphi \mathcal{D}\psi$$

where

$$S(\varphi, \psi) := \int_{X(\mathbb{C})} \mathrm{Tr}\Big(\varphi \overline{\partial} \psi + \overline{\varphi} \partial \overline{\psi} + \psi \overline{\psi} + \varphi[\psi, \overline{\psi}] + \overline{\varphi}[\overline{\psi}, \psi]\Big)$$

I conjecture that the special values $L(S^n H^1(X), n+m)$ can be expressed via the depth m multiple polylogarithms on X. So Feynman integrals provide construction of (periods of) mixed motives, which are in particular responsible for special values of L-functions. I hope this reflects a very general phenomena.

Acknowledgment. This work was partially supported by the NSF grant DMS-9800998.

References

[1] A. A. Beilinson, P. Deligne, *Motivic polylogarithms and Zagier's conjecture* , A manuscript.

[2] Beilinson A.A., Levin A.M. : *The elliptic polylogarithms*, Proc. Symp. in Pure Math, vol. 55, (1994), part 2, 126-196.

[3] D. J. Broadhurst, *On the enumeration of irreducible k-fold sums and their role in knot theory and field theory*, Preprint hep-th/9604128.

[4] P. Deligne, *Le group fondamental de la droite projective moine trois points*, In: Galois groups over $\mathbb{Q}$. Publ. MSRI, no. 16 (1989) 79–298.

[5] P. Deligne, *A letter to D. Broadhurst*, June 1997.

[6] P. Deligne, *Letter to the author*. July 2000.

[7] V. G. Drinfeld, *On quasi-triangular quasi-Hopf algebras and some group related to closely associated with Gal*($\overline{\mathbb{Q}}/\mathbb{Q}$), Leningrad Math. Journal, 1991. (In Russian).

[8] L. Euler, "Opera Omnia," Ser. 1, Vol XV, Teubner, Berlin 1917, 217–267.

[9] A. B. Goncharov, *Multiple ζ-numbers, hyperlogarithms and mixed Tate motives*, Preprint MSRI 058–93, June 1993.

[10] A. B. Goncharov, *Polylogarithms in arithmetic and geometry*, Proc. ICM-94, Zurich. 374–387.

[11] A. B. Goncharov, *The double logarithm and Manin's complex for modular curves*, Math. Res. Letters, vol. 4. N 5 (1997), pp. 617–636.

[12] A. B. Goncharov, *Multiple polylogarithms, cyclotomy and modular complexes*, Math. Res. Letters, vol. 5. (1998), pp. 497–516.

[13] A. B. Goncharov, *The dihedral Lie algebras and Galois symmetries of* $\pi_1^{(l)}(\mathbb{P}^1\backslash 0, \mu_N, \infty)$, Preprint MPI-1998-131 (1998); To appear in Duke Math. J. (2001). Math.AG/0009121

[14] A. B. Goncharov, *Volumes of hyperbolic manifolds and mixed Tate motives*, J. Amer. Math. Soc. 12 (1999) N2, 569–618

[15] A. B. Goncharov, *Galois groups, geometry of modular varieties and graphs*, Arbeitstagung, June 1999, preprint MPI 1999-50-f (http//www.mpim-bonn.mpg.de/).

[16] A. B. Goncharov, *Mixed elliptic motives*, in London Math. Soc. Lect. Note Series, 254, Cambridge Univ. Press, Cambridge, 1998, 147-221.

[17] A. B. Goncharov, *Multiple polylogarithms and mixed Tate motives*, Math.AG/0103059, and *Multiple polylogarithms and mixed Tate motives II*, (to appear).

[18] A. B. Goncharov, *Chow polylogarithms and regulators* Math. Res. Letters, vol. 2, N 1, (1995), pp. 95–112

[19] A. Grothendieck, *Esquisse d'un programme*, Mimeographed note (1984).

[20] Hain R., Matsumoto M.: *Weighted Completion of Galois Groups and Some Conjectures of Deligne.* Preprint June 2000.

[21] Hoffman M.: *Multiple harmonic series*, Pacific J. Math., 152 (1992), 275-290.

[22] Y. Ihara, *Profinite braid groups, Galois representations and complex multiplications*, Ann. Math. 123 (1986) 43–106.

[23] Y. Ihara, *Braids, Galois groups, and some arithmetic functions*, Proc. ICM-90, Kyoto, (1990).

[24] Y. Ihara, *Some arithmetical aspects of Galois action on the pro-p fundamental group of* $\widehat{\pi}_1(\mathbb{P}^1\backslash\{0,1,\infty\})$, Preprint RIMS-1229, 1999.

[25] M. Kontsevich, *Formal (non)commutative symplectic geometry*, The Gelfand mathematical seminars, Birkhauser, 1993, p. 173–187.

[26] M. Kontsevich, *Operads and motives in deformation quantization*, Lett. Math. Phys., 48 (1999) N1, 35–72.

[27] D. Kreimer, *Renormalization and knot theory*, J. Knot Theory Ramifications, 6, (1997) N4, 479–581.

[28] M. Levine, *Tate motives and the vanishing conjectures for algebraic K-theory*, In Algebriac K-theory and Algebraic topology, 167–188, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 407, Kluver, (1993).

[29] A. Levin, *Kronecker double series and the dilogarithm* Preprint MPI 2000-35 (2000).

[30] M. Levine, *Mixed motives*, Mathematical Surveys and Monographs, 57, AMS, Providence, RI, 1998.

[31] V. Voevodsky, *Triangulated categories of motives over a field.* Cycles, transfers, and motivic homology theories, 188–238, Ann. of Math. Stud., 143, Princeton Univ. Press, Princeton, NJ, 2000.

[32] J. Wildeshaus, *On an elliptic analogue of Zagier's conjecture* 87 (1997), 355–407.

[33] D. Zagier, *Values of zeta functions and their applications*, Proc. ECM-92, vol. 2, 497–512, in Progr. Math., 120, Birkhauser, Basel, 1994.

[34] D. Zagier, *Periods of modular forms, traces of Hecke operators, and multiple ζ-values*, Kokyuroku No. 843 (1993), 162–170 (in Japanese).

Department of Mathematics,
Brown University,
Providence RI 02912, USA
E-mail address: sasha@math.brown.edu