THE CLASSICAL TRILOGARITHM,
ALGEBRAIC $K$-THEORY OF FIELDS,
AND DEDEKIND ZETA FUNCTIONS

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ABSTRACT. In this paper we show how to express the values of $\zeta_F(3)$ for arbitrary number field $F$ in terms of the trilogarithms (D. Zagier's conjecture) and how to relate this result to algebraic $K$-theory.

1. The classical polylogarithm function

The classical polylogarithm function

$$\text{Li}_p(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^p} (z \in \mathbb{C}, |z| \leq 1, p \in \mathbb{N})$$

during the last 200 years was the subject of much research—see [L]. Using the inductive formula $\text{Li}_p(z) = \int_0^z \text{Li}_{p-1}(t) t^{p-2} dt$, $\text{Li}_1(z) = -\log(1-z)$, the $p$-logarithm can be analytically continued to a multivalued function on $\mathbb{C} \setminus \{0, 1\}$. However, D. Wigner and S. Bloch introduced [B1] the single-valued cousin of the dilogarithm, namely

$$D_2(z) := \text{Im}(\text{Li}_2(z)) + \arg(1-z) \cdot \log |z|.$$  

Of course, for $\text{Li}_1$ such function is $-\log |z|$. Analogous functions $D_p(z)$ for $p \geq 3$ were introduced in [R] and computed explicitly in [Z]. Let us consider the slightly modified function

$$\mathcal{L}_p(z) := \text{Re} \left[ \text{Li}_3(z) - \log |z| \cdot \text{Li}_2(z) + \frac{1}{2} \log^2 |z| \cdot \text{Li}_1(z) \right].$$

Such modified functions were considered also for all $p$ by D. Zagier, A. A. Beilinson and P. Deligne [Z3, Be1]. $\mathcal{L}_p(z)$ is real-analytic on $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$ and continuous on $\mathbb{C}P^1$.

Let $F$ be a field. Let $F^1$ be the projective line over $F$, and let $\mathbb{Z}(F^1 \setminus \{0, 1, \infty\}$ be the free abelian group generated by symbols $\{x\}$, where $x \in F^1 \setminus \{0, 1, \infty\}$. 

Received by the editors February 9, 1990 and, in revised form, June 15, 1990. 
The proofs for this paper were reviewed by the editor.

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0273-0979/91 $1.00 + .25 per page

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We may consider \( \mathcal{L}_3 \) as defining a homomorphism

\[
(1.4) \quad \mathcal{L}_3^*: \mathbb{Z}[P^1_C\setminus\{0, 1, \infty\}] \rightarrow \mathbb{R}, \quad \mathcal{L}_3: \Sigma \eta \{x_i\} \mapsto \Sigma \mathcal{L}_3(x_i).
\]

We can do the same for any other real-valued function on \( P^1_C\setminus\{0, 1, \infty\} \), in particular for \( D_2 \).

2. Formula for \( \zeta(3) \)

Now let \( F \) be an arbitrary algebraic number field, \( d_F \) the discriminant of \( F \), \( r_1 \) and \( r_2 \) the number of real and complex places, \( \sigma_j \) all possible embeddings \( F \hookrightarrow \mathbb{C} \), \( 1 \leq j \leq r_1 + 2r_2 \), and \( \sigma_j^r = \sigma_{r_1 + r_2 + k} \). Set \( A_\mathbb{Q} := A \otimes \mathbb{Q} \). Let us consider the homomorphism

\[
\Delta: \mathbb{Q}[P^1_F\setminus\{0, 1, \infty\}] \rightarrow (\Lambda^2 F^* \otimes F^*)_\mathbb{Q},
\]

\[
\Delta: (x) \mapsto (1 - x) \wedge x \otimes x.
\]

Theorem 1. Let \( \zeta_F(s) \) be the Dedekind zeta function of \( F \). Then there exist \( y_1, \ldots, y_1 + r_2 \in \ker \Delta \subset \mathbb{Q}[P^1_F\setminus\{0, 1, \infty\}] \) such that \( \zeta_F(3) \) is equal to \( \pi^{3r_2} \cdot |d_F|^{-1/2} \) times the \( (r_1 + r_2) \)-determinant \( \det_{i,j} (\sigma_j(y_i)) \) \( (1 \leq j \leq r_1 + r_2) \).

For \( s = 2 \) a similar result was proved in [Z2]. It also follows directly from results of [Bo, B1, Su]. D. Zagier conjectured that an analogous fact should be valid for all integers \( s \geq 3 \) [Z3].

To prove Theorem 1 we give an explicit formula expressing the Borel regulator \( r_j: K_j(\mathbb{C}) \rightarrow R \) by \( \mathcal{L}_3^*(z) \), and then use the Borel theorem [Bo]. Below we indicate some ingredients of the proof which are of independent interest.

3. Generic 3-variable functional equation for \( \mathcal{L}_3^*(z) \)

The dilogarithm satisfies a remarkable 2-variable functional equation, discovered in the 19th century by W. Spence, N. H. Abel and others [L]. Its version for \( D_2(z) \) is as follows. Let \( r(x_1, \ldots, x_4) \) be the crossratio of a 4-tuple of different points on \( P^1 \). For every five different points on \( P^1 \) set

\[
R_3(x_0, \ldots, x_4) := \sum_{j=0}^{4} (-1)^j r(x_0, \ldots, \hat{x}_j, \ldots, x_4) \in \mathbb{Z}[P^1\setminus\{0, 1, \infty\}].
\]

Then \( D_2(R_3(x_0, \ldots, x_4)) = 0 \) in the sense of formula (1.4). Note that (3.1) depends actually on two variables because of the \( PGL_2 \)-
invariance of the crossratio. It seems that any other functional equation for \( D_3(z) \) can be deduced formally from this one.

It turns out that the analogous functional equation for \( \zeta_F^3(z) \) corresponds to a special configuration of seven points in the plane. Namely, let \( x_1, x_2, x_3 \) be vertices of a triangle in \( P^2_F \) (i.e., these points are not on a line); \( y_1, y_2, y_3 \) points on its "sides" \( x_1 x_2, x_2 x_3, \) and \( x_3 x_1 \), and \( z \) a point in generic position (see Figure 1). Further, denote by \( (y_1, y_2, y_3, x_3, z) \) the configuration of four points on a line obtained by projection of points \( y_2, y_3, x_3, z \) with center at the point \( y_1 \). Set

\[
R_3(x_i, y_i, z) := (1 + \tau + \tau^2) \\
\circ \{ r(y_1 y_2, y_3, x_2, z) - r(y_2 y_3, x_3, y_3, z) \} \\
+ \{ r(x_1 y_3, y_1, x_1, y_2) + r(z y_1, y_1, x_1, y_2) \} \\
+ \{ r(z x_2, x_1, y_2) \} \\
+ \{ r(z x_2, x_3, x_1, y_2) - r(z x_3, x_1, y_1, y_2) \} \\
+ \{ r(y_1 y_2, y_3, x_2, x_3) \} - 3 \{ 1 \}
\]

where \( \tau: x_i \rightarrow x_{i+1}, y_i \rightarrow y_{i+1} \) (indices modulo 3) (for example, \( \tau^2 \circ \{ r(y_1 y_2, y_3, x_2, z) \} = \{ r(y_2 y_3, y_1, x_1, z) \} \)) and, by definition, \( \{ 1 \} = \{ x \} + \{ 1 - x \} + \{ 1 - x^{-1} \} \) for any \( x \in F^* \setminus 1 \). As we will see below the choice of \( x \) is inessential for our purposes.

**Theorem 2.** In the case \( F = \mathbb{C} \), \( \zeta_F^3(R_3(x_i, y_i, z)) = 0 \). Note, that \( \zeta_F^3(\{ x \} - \{ x^{-1} \}) = 0 \) and \( \zeta_F^3(\{ x \} + \{ 1 - x \} + \{ 1 - x^{-1} \}) = \zeta_{\mathbb{Q}}(3) \).

![Figure 1](image-url)
A configuration \((x_1, x_2, x_3, y_1, y_2, y_3, z)\) of seven points in \(P_F^2\) depends on three parameters. Consider a specialization of this configuration, when \(z\) lies on the line \(\overline{x_3y_1}\). It depends on two parameters, and the corresponding functional equation coincides with the classical Spence-Kummer functional equation for the trigonarithmetic, discovered by Spence in 1809 [S] and, independently, by E. Kummer in 1840 [K] (see Chapter VI in [L]).

It is also possible to deduce the Spence-Kummer equation formally from Theorem 2 (as a linear combination of relations \(\mathcal{L}_2^2(R_3(x_1, y_1, z)) = 0\)). The validity of the inverse statement is an interesting problem.

**Conjecture 1.** Any functional equation for \(\mathcal{L}_2^2(x)\) can be formally deduced from Theorem 2.

4. **Algebraic K-theory of a field**

Now let \(F\) be an arbitrary field. Set \(B_2(F) := \mathbb{Z}[P_F^1 \setminus \{0, 1, \infty\}] / R_2\), where \(R_2\) is generated by elements \(x_5, \ldots, x_k\)—see (3.1). Then there is the well-known Bloch complex \(B_2(F) \xrightarrow{\delta} \Lambda^2 F^*\), where \(\delta(x) = (1 - x) \wedge x\). (It is not hard to prove that \(\delta(R_2) = 0\).) Thanks to Matsumoto, we know that \(\text{Coker} \delta = K^2_2(F)((M))\). Using some ideas of S. Bloch [B1], A. Suslin proved that \(K_3^{\text{ind}}(F) := \text{Coker}(K^2_3(F) \to K_3(F))\) coincides with \(\ker \delta\) modulo torsion [Su].

Note also that \(K_2(F) = F^*\) has an interpretation in the same spirit: \(F^* = \mathbb{Z}[1, 0, 1, \infty] / R_1\), where \(R_1\) is generated by expressions \([x] + [y] - [xy]\), reminiscent of the functional equation for \(L(\cdot, \cdot)\).

Let us define a complex \(Q(3)\) as follows:

\[
(4.1) \quad Q(P_F^1 \setminus \{0, 1, \infty\}) / R_3 \xrightarrow{\delta} (B_2(F) \otimes F^*) Q \xrightarrow{\delta} (\Lambda^3 F^*) Q
\]

(the left group placed in degree 1), where \(\delta_{i}([x] \otimes y) = (1 - x) \wedge x \wedge y\), \(\delta_{i}([x]) = [x] \otimes x\), and the subgroup \(R_3\) is generated by \(\{x\} - \{x^{-1}\}\), \(\{(x) + \{1 - x\} + \{1 - x^{-1}\}\} - \{(y) + \{1 - y\} + \{1 - y^{-1}\}\}\) and \(R_3(x_1, y_1, z)\) (see Equation 3.2).

**Theorem 2'.** \(\delta_{i}(R_3) = 0\) in \(B_2(F) \otimes F^*\).

Hence the complex \(Q(3)\) is well defined. Recall, that \(K_n(F) := \pi_n(BGL(F)^*),\) where \(BGL(F)^*\) is an \(H\)-space. Hence, by the Milnor-Moore theorem [MM] \(K_n(F) \otimes Q = \text{Prim} H_n(GL(F), Q)\).
A. Suslin proved [Su2] that $H_n(GL_n(F), \mathbb{Z}) = H_n(GL(F), \mathbb{Z})$. Therefore $K_n(F) \otimes \mathbb{Q} = \text{Prim } H_n(GL_n(F), \mathbb{Q})$. So $\text{Im}(H_n(GL_n, -)) \to H_n(GL_n))$ gives a canonical filtration $K_n(F) \supset K_n^{(1)}(F) \supset \ldots$. Set $K_n^{[m]}(F)_Q := K_n^{(m)}(F)_Q / K_n^{(m+1)}(F)_Q$.

**Theorem 3.** There are canonical maps

$$c_1: K_2^{[2]}(F)_Q \to H^1(Q(3), \mathcal{O})$$

$$c_1: K_2^{[1]}(F)_Q \to H^2(Q(3), \mathcal{O}).$$

**Conjecture 2.** $c_1$ and $c_2$ are isomorphisms.

Note, that according to [Su2]

$$K_3^{[0]}(F)_Q = H^3(Q(3), \mathcal{O}) \equiv K_3^M(F)_Q.$$

(A. A. Beilinson and S. Lichtenbaum conjectured that there should exist complexes $Q(j)\mathcal{O}$ computing all $K_n(F)$—see [Be2, Li].)

5. **The group $B_3(F)$**

For a $G$-space $X$, points of $G \backslash X \times \ldots \times X$ are called configurations. Let $Z(C_6(P^2_F))$ be the free abelian group generated by all possible configurations $(l_0, \ldots, l_5)$ of 6 points in $P^2_F$.

Let us define a homomorphism $L_3: \mathbb{Z}[P^1_F \backslash \{0, 1, \infty\}] \to Z(C_6(P^2_F))$ as follows: $L_3(x) = (x_1, x_2, x_3, y_1, y_2, y_3)$, where $r(y_1, x_1, x_2, y_2, y_3) = x$ (this configuration was described in §3). The (unique) configuration where $y_1, y_2, y_3$ are on a line will be denoted $\eta_3$.

**Definition.** $B_3(F)$ is the quotient of the group $Z(C_6(P^2_F))$ by the following relations

(R1) $(l_0, \ldots, l_5) = 0$, if two of the points $l_i$ coincide or four lie on a line.

(R2) (The seven-term relation.) For any seven points $(l_0, \ldots, l_6)$ in $P^2_F$

$$\sum_{i=0}^{6} (-1)^i(l_0, \ldots, \tilde{l}_i, \ldots, l_6) = 0.$$
(R3) Let \((m_0, \ldots, m_5)\) be a configuration of six points in \(P_F^2\), such that \(m_2 = \overline{m_0 m_1} \cap \overline{m_3 m_4}\) and \(m_5\) is in generic position—see Fig. 2. Then if \(L'_3(x) := -L_3(x) - 2L_3(1-x)\),

\[
(m_0, \ldots, m_5)
= \frac{1}{4} \sum_{i=0}^{4} (-1)^i L'_3 (r(m_3|m_0, \ldots, \hat{m}_i, \ldots, m_4)) + \frac{1}{3} \eta_3.
\]

**Lemma.** In the group \(B_3(F)\) we have

\[
(l_0, \ldots, l_5) = (-1)^{\sigma(i)} (l_{\sigma(0)}, \ldots, l_{\sigma(5)}).
\]

**Remark.** The configurations from (R1) are just the unstable ones in the sense of D. Mumford.

**Theorem 4.** The homomorphism \(L_3: \mathbb{Z}[P_F^1 \setminus 0, 1, \infty] \rightarrow \mathbb{Z}[C_6(P_F^2)]\) induces an isomorphism modulo 6-torsion.

\[L_3: \mathbb{Z}[P_F^1 \setminus 0, 1, \infty]/R_3 \cong B_3(F) \otimes \mathbb{Z}.
\]

(It is easy to check using (R2) and (R3) that \(L_3\) is onto; the 7-term relation for a configuration \((x_1, x_2, x_3, y_1, y_2, y_3, z)\) then coincides with \((L_3(R_3(x_i, y_i, z)))\).

Let us denote by \(M_3\) the inverse homomorphism. Then the composition \(L_3 \circ M_3: B_3(C) \rightarrow Q[P_C^3 \setminus 0, 1, \infty] \rightarrow \mathbb{R}\) defines a measurable function on configurations of six points in \(CP^2\), satisfying functional relations (R1) through (R3). So for \(x \in P_C^2\), \((L_3 \circ M_3)(x, g_1 x, \ldots, g_5 x)\) is a measurable cocycle. Let us prove that its cohomology class lies in \(\text{Im}(H^{\ast}_c(GL_3(C), R) \rightarrow H^{\ast}_c(GL_3(C), R))\), where \(H^{\ast}_c(G, R)\) is continuous cohomology.

![Figure 2](image-url)
Consider the complex
\[ \text{Meas } C_{2n-1}(\mathbb{C}P^{n-1}) \xrightarrow{d^{*}_{2n-1}} \text{Meas } C_{2n}(\mathbb{C}P^{n-1}) \xrightarrow{d^{*}_{2n}} \text{Meas } C_{2n+1}(\mathbb{C}P^{n-1}) \]
where \( C_{m}(\mathbb{C}P^{n}) \) is the space of all configurations of \( m \) points in \( \mathbb{C}P^{n} \), \( \text{Meas}(X) \) is the space of all measurable functions on the space \( X \), \( d_{m}^{*} : (l_{0}, \ldots, l_{m}) \mapsto \sum_{i=0}^{m}(-1)^{i}(l_{0}, \ldots, \hat{l}_{i}, \ldots, l_{m}) \) and \( d_{m}^{*} \) is the induced map.

**Theorem 5.** \( \text{Ker } d_{2n}^{*}/\text{Im } d_{2n-1}^{*} \) is canonically isomorphic to the indecomposable part of \( H_{\text{et}}^{2n-1}(GL_{n}(\mathbb{C}), R) \).

For \( n = 2 \) this was proved in [B1]. See also closely related work [HM].

**Conjecture 3.** There exists a canonical element in \( \text{Ker } d_{2n}^{*} \) that can be expressed by classical \( n \)-logarithm \( \mathcal{L}_{n}(x) \) and represents the Borel class in \( H_{\text{et}}^{2n-1}(GL_{n}(\mathbb{C}), R) \).

I would like to thank A. A. Beilinson for stimulating discussions and interest and M. L. Kontsevich for useful remarks.

**References**


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