FUCHSIAN GROUPS AND COMPACT HYPERBOLIC SURFACES

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Abstract. We present a topological proof of the following theorem of Benoist-Quint: for a finitely generated non-elementary discrete subgroup Γ_1 of $\mathrm{PSL}(2,\mathbb{R})$ with no parabolics, and for a cocompact lattice Γ_2 of $\mathrm{PSL}(2,\mathbb{R})$, any Γ_1 orbit on $\Gamma_2\backslash \mathrm{PSL}(2,\mathbb{R})$ is either finite or dense.

1. Introduction

Let Γ_1 be a non-elementary finitely generated discrete subgroup with no parabolic elements of $\mathrm{PSL}(2,\mathbb{R})$. Let Γ_2 be a cocompact lattice in $\mathrm{PSL}(2,\mathbb{R})$. The following is the first non-trivial case of a theorem of Benoist-Quint [1].

Theorem 1.1. *Any Γ_1-orbit on $\Gamma_2\backslash \mathrm{PSL}(2,\mathbb{R})$ is either finite or dense.*

The proof of Benoist-Quint is quite involved even in the case as simple as above and in particular uses their classification of stationary measures [2]. The aim of this note is to present a short, and rather elementary proof.

We will deduce Theorem 1.1 from the following Theorem 1.2. Let

- $H_1 = H_2 := \mathrm{PSL}(2,\mathbb{R})$ and $G := H_1 \times H_2$;
- $H := \{(h,h) : h \in \mathrm{PSL}_2(\mathbb{R})\}$ and $\Gamma := \Gamma_1 \times \Gamma_2$.

Theorem 1.2. *For any $x \in \Gamma\backslash G$, the orbit xH is either closed or dense.*

Our proof of Theorem 1.2 is purely topological, and inspired by the recent work of McMullen, Mohammadi and Oh [5] where the orbit closures of the $\mathrm{PSL}(2,\mathbb{R})$ action on $\Gamma_0\backslash \mathrm{PSL}(2,\mathbb{C})$ are classified for certain Kleinian subgroups Γ_0 of infinite co-volume. While the proof of Theorem 1.2 follows closely the sections 8-9 of [5], the arguments in this paper are simpler because of the assumption that Γ_2 is cocompact. We remark that the approach of [5] and hence of this paper is somewhat modeled after Margulis's original proof of Oppenheim conjecture [4]. When Γ_1 is cocompact as well, Theorem 1.2 also follows from [6].

2. Horocyclic flow on convex cocompact surfaces

In this section we prove a few preliminary facts about unipotent dynamics involving only one factor H_1.

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We recall that Γ_1 is a non-elementary finitely generated discrete subgroup with no parabolic elements of the group $H_1 = \mathrm{PSL}_2(\mathbb{R})$, that is, Γ_1 is a convex cocompact subgroup. We will identify the boundary of the hyperbolic plane $\mathbb{H}^2 := \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ with the extended real line $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$ which is topologically a circle. Let S_1 denote the hyperbolic orbifold $\Gamma_1\backslash\mathbb{H}^2$, and let $\Lambda_{\Gamma_1} \subset \partial\mathbb{H}^2$ be the limit set of Γ_1. Let A_1 and U_1 be the subgroups of H_1 given by

$$A_1 := \{a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R}\} \text{ and } U_1 := \{u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R}\}.$$

The set

$$\Omega_{\Gamma_1} = \{x \in \Gamma_1\backslash H_1 : xA_1 \text{ is bounded}\}. \tag{2.1}$$

is called the renormalized frame bundle of Γ_1. As Γ_1 is a convex cocompact subgroup, Ω_{Γ_1} is a compact A_1-invariant subset and one has the equality

$$\Omega_{\Gamma_1} = \{[h] \in \Gamma_1\backslash H_1 : h(0), h(\infty) \in \Lambda_{\Gamma_1}\}.$$

The image of Ω_{Γ_1} in S_1 under the map $h \mapsto h(i)$ is equal to the convex core of S_1.

Definition 2.2. *Let $K > 1$. A subset $I \subset \mathbb{R}$ is called K-thick if, for any $t > 0$, I meets $[-Kt, -t] \cup [t, Kt]$.*

Lemma 2.3. *There exists $K > 1$ such that for any $x \in \Omega_{\Gamma_1}$, the subset $I(x) := \{t \in \mathbb{R} : xu_t \in \Omega_{\Gamma_1}\}$ is K-thick.*

Proof. Using an isometry, we may assume without loss of generality that $x = [e]$ where e corresponds to a downward unit vector at i in the identification of $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{T}^1(\mathbb{H}^2)$. As $x \in \Omega_{\Gamma_1}$, both points 0 and ∞ belong to the limit set Λ_{Γ_1}. Since $u_t(\infty) = \infty$ and $u_t(0) = t$, one has the equality $I(x) = \{t \in \mathbb{R} : t \in \Lambda_{\Gamma_1}\}$. Write $\mathbb{R} - \Lambda_{\Gamma_1}$ as the union $\cup J_\ell$ where J_ℓ's are maximal open intervals. Note that the minimum distance between the convex hulls

$$\delta := \inf_{\ell \neq m} d(\mathrm{hull}(J_\ell), \mathrm{hull}(J_m))$$

is positive, as 2δ is the length of the shortest closed geodesic of the double of the core of S_1. Choose the constant $K > 1$ so that for $t > 0$, one has

$$d(\mathrm{hull}[-Kt, -t], \mathrm{hull}[t, Kt]) = \delta/2.$$

Note that this choice of K is independent of t. If $I(x)$ does not intersect $[-Kt, -t] \cup [t, Kt]$ for some $t > 0$, then the intervals $[-Kt, -t]$ and $[t, Kt]$ must belong to two distinct intervals J_ℓ and J_m, since $0 \in \Lambda_{\Gamma_1}$. This contradicts to the choice of K. $\square$

Lemma 2.4. *Let $K > 1$ and let I be a K-thick subset of $\mathbb{R}$. For any sequence h_n in $H_1 \smallsetminus U_1$ converging to e, there exists a sequence $t_n \in I$ such that the sequence $u_{-t_n} h_n u_{t_n}$ has a non-trivial limit point in U_1.*

Proof. Write $h_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. We compute

$$q_n := u_{-t_n} h_n u_{t_n} = \begin{pmatrix} a_n - c_n t_n & (a_n - d_n - c_n t_n)t_n + b_n \\ c_n & d_n + c_n t_n \end{pmatrix}.$$

Since h_n does not belong to U_1, it follows that the $(1,2)$-entries $P_n := (a_n - d_n - c_n t_n)t_n + b_n$ are non-constant polynomials in t_n of degree at most 2 whose coefficients converge to 0. Hence we can choose $t_n \in I$ going to ∞ so that $1 \le |P_n| \le k$, for some positive constant k depending only on K. Then the product $c_n t_n$ must converge to 0 and the sequence q_n has a limit point in $U_1 - \{e\}$. □

Lemma 2.5. *Let U_1^+ be the semigroup $\{u_t : t \ge 0\}$. If Γ_1 is cocompact, any U_1^+-orbit is dense in $\Gamma_1 \backslash H_1$.*

Proof. Consider xU_1^+ for $x \in \Gamma_1\backslash H_1$. Set $x_n := xu_n$. We then have $x_n u_{-n} U_1^+ \subset xU_1^+$. Hence if z is a limit point of the sequence x_n, we have $zU \subset \overline{xU_1^+}$. By Hedlund's theorem [3], zU is dense, proving the claim. □

3. Proof of Theorems 1.1 and 1.2

In this section, using minimal sets and unipotent dynamics on the product space $\Gamma\backslash G$, we provide a proof of Theorem 1.2.

3.1. **Unipotent dynamics.** We recall the notation $G := \mathrm{PSL}_2(\mathbb{R})\times\mathrm{PSL}_2(\mathbb{R})$ and $\Gamma := \Gamma_1 \times \Gamma_2$. Set

- $H_1 = \{(h, e)\}$, $H_2 = \{(e, h)\}$, $H = \{(h, h)\}$;
- $U_1 = \{(u_t, e)\}$, $U_2 = \{(e, u_t)\}$, $U = \{(u_t, u_t)\}$;
- $A_1 = \{(a_t, e)\}$, $A_2 = \{(e, a_t)\}$, $A = \{(a_t, a_t)\}$;
- $X_1 = \Gamma_1\backslash H_1$, $X_2 = \Gamma_2\backslash H_2$, $X = \Gamma\backslash G = X_1 \times X_2$.

Recall that Γ_1 is a non-elementary finitely generated discrete subgroup of H_1 with no parabolic elements and that Γ_2 is a cocompact lattice in H_2,

For simplicity, we write $\widetilde{u}_t$ for (u_t, u_t) and $\widetilde{a}_t$ for (a_t, a_t). Note that the normalizer of U in G is AU_1U_2.

Lemma 3.1. *Let g_n be a sequence in $G \smallsetminus AU_1U_2$ converging to e, and let I be a K-thick subset of $\mathbb{R}$ for some $K > 1$. Then for any neighborhood G_0 of e in G, there exist sequences $s_n \in I$ and $t_n \in \mathbb{R}$ such that the sequence $\widetilde{u}_{-s_n} g_n \widetilde{u}_{t_n}$ has a non-trivial limit point $q \in AU_2 \cap G_0$.*

Proof. Fix $\varepsilon > 0$. Write $g_n = (g_n^{(1)}, g_n^{(2)})$ with $g_n^{(i)} = \begin{pmatrix} a_n^{(i)} & b_n^{(i)} \\ c_n^{(i)} & d_n^{(i)} \end{pmatrix}$. Then the products $q_n := \widetilde{u}_{-s_n} g_n \widetilde{u}_{t_n}$ are given by

$$q_n^{(i)} = u_{-s_n} g_n^{(i)} u_{t_n} = \begin{pmatrix} a_n^{(i)} - c_n^{(i)} s_n & (b_n^{(i)} - d_n^{(i)} s_n) - t_n(c_n^{(i)} s_n - a_n^{(i)}) \\ c_n^{(i)} & d_n^{(i)} + c_n^{(i)} t_n \end{pmatrix}.$$

Set $t_n = \frac{b_n^{(1)} - d_n^{(1)} s_n}{c_n^{(1)} s_n - a_n^{(1)}}$. The differences $q_n - e$ are now rational functions in s_n of the form $q_n - e = \frac{1}{c_n^{(1)} s_n - a_n^{(1)}} P_n$, where P_n is a polynomial in s_n of degree at most 2 with values in $M_2(\mathbb{R}) \times M_2(\mathbb{R})$. Since the elements g_n do not belong to AU_1U_2, these polynomials P_n are non-constants. Hence we can choose $s_n \in I$ going to ∞ so that $\varepsilon \leq \|P_n\| \leq k\varepsilon$ for some constant $k > 1$ depending only on K. We can also simultaneously impose that the denominator satisfy $1/2 \leq |c_n^{(1)} s_n - a_n^{(1)}| \leq k$ so that $\varepsilon/k \leq \|q_n - e\| \leq 2k\varepsilon$. By construction, when ε is small enough, the sequence q_n has a non trivial limit point q in $A_1A_2U_2 \cap G_0$.

We claim that this limit $q = (q^{(1)}, q^{(2)})$ belongs to the group AU_2. It suffices to check that the diagonal entries of $q^{(1)}$ and $q^{(2)}$ are equal. If not, the two sequences $c_n^{(i)} s_n$ converge to real numbers $c^{(i)}$ with $c^{(1)} \neq c^{(2)}$, and a simple calculation shows that the $(1,2)$- entries of $q_n^{(2)}$ are comparable to $\frac{c^{(2)} - c^{(1)}}{1 - c^{(1)}} s_n$ which tends to ∞. Contradiction. Hence q belongs to AU_2. □

3.2. ***H*-minimal and *U*-minimal subsets.** Let

$$\Omega := \Omega_{\Gamma_1} \times X_2$$

where Ω_{Γ_1} is the renormalized frame bundle of Γ_1 as in (2.1). Note that, since Γ_2 is cocompact, the renormalized frame bundle of Γ_2 is $\Omega_{\Gamma_2} = X_2$.

Let $x = (x_1, x_2) \in \Gamma\backslash G$ and consider the orbit xH. Note that xH intersects Ω non-trivially. Let Y be an H-minimal subset of the closure $\overline{xH}$ with respect to Ω, i.e., Y is a closed H-invariant subset of $\overline{xH}$ such that $Y \cap \Omega \neq \emptyset$ and the orbit yH is dense in Y for any $y \in Y \cap \Omega$. Since any H orbit intersects Ω, it follows that yH is dense in Y for any $y \in Y$. Let Z be a U-minimal subset of Y with respect to Ω. Since Ω is compact, such minimal sets Y and Z exist. Set

$$Y^* = Y \cap \Omega \quad \text{and} \quad Z^* = Z \cap \Omega.$$

In the following, we assume that

the orbit xH is not closed

and aim to show that xH is dense in X.

Lemma 3.2. *For any $y \in Y$, the identity element e is an accumulation point of the set $\{g \in G \smallsetminus H : yg \in \overline{xH}\}$.*

Proof. If y does not belong to xH, there exists a sequence $h_n \in H$ such that xh_n converges to y. Hence there exists a sequence $g_n \in G$ converging to e such that $xh_n = yg_n$. These elements g_n do not belong to H; hence proving the claim.

Suppose now that y belongs to xH. If the claim does not hold, then for a sufficiently small neighborhood G_0 of e in G, the set $yG_0 \cap Y$ is included in the orbit yH. This implies that the orbit yH is an open subset of Y. The minimality of Y implies that $Y = yH$, contradicting the assumption that the orbit yH is not closed. □

Lemma 3.3. *There exists a non-trivial element $v \in U_2$ such that $Zv \subset \overline{xH}$.*

Proof. Choose a point $z = (z_1, z_2) \in Z^*$. By Lemma 3.2, there exists a sequence g_n in $G \smallsetminus H$ converging to e such that $zg_n \in \overline{xH}$. We may assume without loss of generality that g_n belongs to H_2. If g_n belongs to U_2 for some n, the Lemma follows. Suppose that g_n does not belong to U_2. Then, since the set $I(z_1)$ is K-thick for some $K > 1$ by Lemma 2.3, it follows from Lemma 2.4 that there exist a sequence $t_n \to \infty$ in $I(z_2)$ such that, after extraction, the products $\widetilde{u}_{-t_n} g_n \widetilde{u}_{t_n}$ converge to non-trivial element $v \in U_2$.

Since the points $z\widetilde{u}_{t_n}$ belong to Ω, this sequence has a limit point $z' \in Z^*$. Since one has the equality

$$z'v = \lim_{n\to\infty} z\widetilde{u}_{t_n}(\widetilde{u}_{-t_n} g_n \widetilde{u}_{t_n})$$

the point $z'v$ belongs to $\overline{xH}$. Since v commutes with U and Z is U-minimal with respect to Ω, one has the equality $Zv = \overline{z'vU}$, hence the set Zv is included in $\overline{xH}$. □

Lemma 3.4. *For any $z \in Z^*$, there exists a sequence g_n in $G \smallsetminus U$ converging to e such that $zg_n \in Z$.*

Proof. Since the group Γ_2 is cocompact, it does not contain unipotent elements and hence the orbit zU is not compact. Since the orbit zU is recurrent in Z^*, the set $Z^* \smallsetminus zU$ contains at least one point. Call it z'. Since the orbit $z'U$ is dense in Z, there exists a sequence $\widetilde{u}_{t_n} \in U$ such that $z = \lim z'\widetilde{u}_{t_n}$. Hence one can write $z'\widetilde{u}_{t_n} = zg_n$ with g_n in $G \smallsetminus U$ converging to e. □

Proposition 3.5. *There exists a one-parameter semi-group $L^+ \subset AU_2$ such that $ZL^+ \subset Z$.*

Proof. It suffices to find, for any neighborhood G_0 of e, a non-trivial element q in $AU_2 \cap G_0$ such that the set Zq is included in Z; then writing $q = \exp w$ for an element w of the Lie algebra of G, we can take L^+ to be the semigroup $\{\exp(sw_\infty) : s \geq 0\}$ where w_∞ is a limit point of the elements $\frac{w}{\|w\|}$ when the diameter of G_0 shrinks to 0.

Fix a point $z = (z_1, z_2) \in Z^*$. According to Lemma 3.4 there exists a sequence $g_n \in G \smallsetminus U$ converging to e such that $zg_n \in Z$.

Suppose first that g_n belongs to AU_1U_2 for infinitely many n; then one can find $\widetilde{u}_{t_n} \in U$ such that the product $q_n := g_n\widetilde{u}_{t_n}$ belongs to AU_2 and is non-trivial, and zq_n belongs to Z. Hence, since q_n normalizes U and since Z is U-minimal with respect to Ω, the set Zq_n is included in Z.

Now suppose that g_n is not in AU_1U_2. By Lemmas 2.3 and 3.1, there exist sequences $s_n \in I(z_1)$ and $t_n \in \mathbb{R}$ such that, after passing to a subsequence, the products $\widetilde{u}_{-s_n} g_n \widetilde{u}_{t_n}$ converge to a non-trivial element $q \in AU_2 \cap G_0$. Since the elements $z\widetilde{u}_{t_n}$ belong to Z^*, they have a limit point $z' \in Z^*$. Since we have

$$z'q = \lim_{n\to\infty} z\widetilde{u}_{s_n}(\widetilde{u}_{-s_n} g_n \widetilde{u}_{t_n})$$

the element $z'q$ belongs to Z. As q normalizes U, it follows that Zq is contained in Z. $\square$

Proposition 3.6. *There exist an element $z \in \overline{xH}$ and a one-parameter semi-group $U_2^+ \subset U_2$ such that $zU_2^+ \subset \overline{xH}$.*

Proof. By Proposition 3.5 there exists a one-parameter semigroup $L^+ \subset AU_2$ such that $ZL^+ \subset Z$. This semigroup L^+ is equal to one of the following: U_2^+, A^+ or $v_0^{-1}A^+v_0$ for some non-trivial element $v_0 \in U_2$, where U_2^+ and A^+ are one-parameter semigroups of U_2 and A respectively.

When $L^+ = U_2^+$, our claim is proved.

Suppose now $L^+ = A^+$. By Lemma 3.3 there exists a non-trivial element $v \in U_2$ such that $Zv \subset \overline{xH}$. Then one has the inclusions

$$ZA^+vA \subset ZvA \subset \overline{xH}A \subset \overline{xH}.$$

Choose a point $z' \in Z^*$ and a sequence $\widetilde{a}_{t_n} \in A^+$ going to ∞. Since $z'\widetilde{a}_{t_n}$ belong to Ω, after passing to a subsequence, the sequence $z'\widetilde{a}_{t_n}$ converges to a point $z \in \overline{xH} \cap \Omega$. Moreover, since the Hausdorff limit of the sets $\widetilde{a}_{-t_n}A^+$ is A, one has the inclusions

$$zAvA \subset \lim_{n\to\infty} z'\widetilde{a}_{t_n}(\widetilde{a}_{-t_n}A^+)vA = z'A^+vA \subset \overline{xH}.$$

Now by a simple computation, we can check that the set AvA contains a one-parameter semigroup U_2^+ of U_2, and hence the orbit zU_2^+ is included in $\overline{xH}$ as desired.

Suppose finally $L^+ = v_0^{-1}A^+v_0$ for some $v_0 \in U_2$. We can assume without loss of generality that $A^+ = \{\widetilde{a}_{\varepsilon t} : t \geq 0\}$ where $\varepsilon = \pm 1$ and that $v_0 = (e, u_1)$. A simple computation shows that the set $v_0^{-1}A^+v_0A$ contains the set $U_2' := \{(e, u_{\varepsilon t}) : 0 \leq t \leq 1\}$. Hence one has the inclusions

$$ZU_2' \subset Zv_0^{-1}A^+v_0A \subset ZA \subset \overline{xH}.$$

Choose a point $z' \in Z^*$ and let $z \in \overline{xH}$ be a limit of a sequence $z'\widetilde{a}_{-t_n}$ with t_n going to $+\infty$. Since the Hausdorff limit of the sets $\widetilde{a}_{t_n}U_2'\widetilde{a}_{-t_n}$ is the semigroup $U_2^+ := \{(e, u_{\varepsilon t}) : t \geq 0\}$, one has the inclusions

$$zU_2^+ \subset \lim_{n\to\infty} (z'\widetilde{a}_{-t_n})\widetilde{a}_{t_n}U_2'\widetilde{a}_{-t_n} \subset \overline{ZU_2'A} \subset \overline{xH}. \qquad \square$$

3.3. **Conclusion.**

Proof of Theorem 1.2. Suppose that the orbit xH is not closed. By Proposition 3.6, the orbit closure $\overline{xH}$ contains an orbit zU_2^+ of a one-parameter subsemigroup of U_2. Since Γ_2 is cocompact in H_2, by Lemma 2.5, this orbit zU_2^+ is dense in zH_2. Hence we have the inclusions

$$X = zG = zH_2H \subset H\overline{zU_2^+} \subset \overline{xH}.$$

This proves the claim. $\square$

Proof of Theorem 1.1. Let $x = [g]$ be a point of $X_2 = \Gamma_2 \backslash H_2$. By replacing Γ_1 by $g^{-1}\Gamma_1 g$, we may assume without loss of generality that $g = e$. One deduces Theorem 1.1 from Theorem 1.2 thanks to the following equivalences:
The orbit $[e]H$ is closed (resp. dense) in $\Gamma \backslash G \iff$
The orbit $\Gamma[e]$ is closed (resp. dense) in $G/H \iff$
The product $\Gamma_2\Gamma_1$ is closed (resp. dense) in $\mathrm{PSL}_2(\mathbb{R}) \iff$
The orbit $[e]\Gamma_1$ is closed (resp. dense) in $\Gamma_2 \backslash \mathrm{PSL}_2(\mathbb{R})$. □

References

[1] Y. Benoist and J. F. Quint *Stationary measures and invariant subsets of homogeneous spaces I.* Annals of Math, Vol 174 (2011), p. 1111-1162

[2] Y. Benoist and J. F. Quint *Stationary measures and invariant subsets of homogeneous spaces III.* Annals of Math, Vol 178 (2013), p. 1017-1059

[3] G. Hedlund *Fuchsian groups and transitive horocycles.* Duke Math. J. Vol 2, 1936, p. 530-542

[4] G. Margulis *Indefinite quadratic forms and unipotent flows on homogeneous spaces.* In Dynamical systems and ergodic theory (Warsaw, 1986), volume 23. Banach Center Publ., 1989.

[5] C. McMullen, A. Mohammadi and H. Oh *Geodesic planes in hyperbolic* 3*-manifolds.* Preprint, 2015

[6] M. Ratner *Raghunathan's topological conjecture and distributions of unipotent flows.* Duke Math. J. 63 (1991), p. 235 - 280.

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