RIGIDITY OF KLEINIAN GROUPS VIA SELF-JOININGS: MEASURE THEORETIC CRITERION

DONGRYUL M. KIM AND HEE OH

Abstract. Let $n, m \geq 2$. Let $\Gamma < \mathrm{SO}^\circ(n+1,1)$ be a Zariski dense convex cocompact subgroup and $\Lambda \subset \mathbb{S}^n$ be its limit set. Let $\rho : \Gamma \to \mathrm{SO}^\circ(m+1,1)$ be a Zariski dense convex cocompact faithful representation and $f : \Lambda \to \mathbb{S}^m$ the ρ-boundary map. Let

$$\Lambda_f := \bigcup \left\{ C \cap \Lambda : \begin{array}{c} C \subset \mathbb{S}^n \text{ is a circle such that} \\ f(C \cap \Lambda) \text{ is contained in a proper sphere in } \mathbb{S}^m \end{array} \right\}.$$

When there exists at least one Λ-doubly stable circle in $\mathbb{S}^n$ (e.g., $\Omega = \mathbb{S}^n - \Lambda$ is disconnected), we prove the following dichotomy:

$$\text{either} \quad \Lambda_f = \Lambda \quad \text{or} \quad \mathcal{H}^\delta(\Lambda_f) = 0,$$

where $\mathcal{H}^\delta$ is the Hausdorff measure of dimension $\delta = \dim_H \Lambda$. Moreover, in the former case, we have $n = m$ and ρ is a conjugation by a Möbius transformation on $\mathbb{S}^n$. Our proof uses ergodic theory for directional diagonal flows and conformal measure theory of discrete subgroups of higher rank semisimple Lie groups, applied to the self-joining subgroup $\Gamma_\rho = (\mathrm{id} \times \rho)(\Gamma) < \mathrm{SO}^\circ(n+1,1) \times \mathrm{SO}^\circ(m+1,1)$.

1. Introduction

Let $\mathbb{H}^{n+1}$ denote the $(n+1)$-dimensional real hyperbolic space for $n \geq 2$. The group of its orientation-preserving isometries is given by the identity component $\mathrm{SO}^\circ(n+1,1)$ of the special orthogonal group. A discrete subgroup $\Gamma < \mathrm{SO}^\circ(n+1,1)$ is called *convex cocompact* if the convex core[1] of the associated hyperbolic manifold $\Gamma\backslash\mathbb{H}^{n+1}$ is compact. Let $\Gamma < \mathrm{SO}^\circ(n+1,1)$ be a Zariski dense convex cocompact subgroup for $n \geq 2$, and

$$\rho : \Gamma \to \mathrm{SO}^\circ(m+1,1)$$

be a faithful representation such that $\rho(\Gamma)$ is a Zariski dense convex cocompact cocompact subgroup of $\mathrm{SO}^\circ(m+1,1)$ where $m \geq 2$. For simplicity, we will call a discrete faithful representation $\rho : \Gamma \to \mathrm{SO}^\circ(m+1,1)$ a deformation of Γ into $\mathrm{SO}^\circ(m+1,1)$ and a (resp. Zariski dense) convex cocompact deformation of Γ if the image of ρ is a (resp. Zariski dense) convex cocompact subgroup. If $\Gamma < \mathrm{SO}^\circ(n+1,1)$ is cocompact and $n = m$, the Mostow strong rigidity theorem [19] says that ρ is always algebraic, more precisely, it is given by a conjugation by a Möbius transformation on $\mathbb{S}^n$. However in

Oh is partially supported by the NSF grant No. DMS-1900101.

[1]The convex core of $\Gamma\backslash\mathbb{H}^{n+1}$ is the smallest convex submanifold containing all closed geodesics.

other cases, Marden's isomorphism theorem and the Teichmüller theory imply that there exists a continuous family of convex cocompact deformations, modulo the conjugations by Möbius transformation on $\mathbb{S}^m$ (cf. [17, section 5]).

Let $\Lambda \subset \mathbb{S}^n$ denote the limit set of Γ, which is the set of all accumulation points of $\Gamma(o)$ in $\mathbb{S}^n$, $o \in \mathbb{H}^{n+1}$. Let $\mathcal{H}^\delta$ be the δ-dimensional Hausdorff measure on $\mathbb{S}^n$, where δ is the Hausdorff dimension of Λ with respect to the spherical metric on $\mathbb{S}^n$. Sullivan [20, Theorem 7] showed that for Γ convex cocompact, we have

$$0 < \mathcal{H}^\delta(\Lambda) < \infty.$$

The main aim of this paper is to present a criterion on when ρ is algebraic, in terms of the Hausdorff measure of the union of all circular slices of Λ that are mapped into circles, or more generally into some proper spheres in $\mathbb{S}^m$ by the ρ-boundary map. More precisely, by Tukia [23], there is a unique ρ-equivariant continuous embedding

$$f : \Lambda \to \mathbb{S}^m,$$

called the ρ-boundary map. We consider all circular slices of Λ which are mapped into some proper spheres in $\mathbb{S}^m$ by f:

$$\Lambda_f := \bigcup \left\{ C \cap \Lambda : \begin{array}{c} C \subset \mathbb{S}^n \text{ is a circle such that} \\ f(C \cap \Lambda) \text{ is contained in a proper sphere in } \mathbb{S}^m \end{array} \right\}.$$

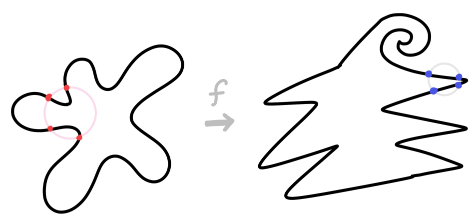

FIGURE 1. $f(C \cap \Lambda)$ is contained in a circle

We emphasize that the boundary map f is defined only on Λ and therefore our definition of Λ_f involves the image of the intersection $C \cap \Lambda$ under f, but not the whole circle C (see Figure 1). If $n = m$ and f is a Möbius transformation of $\mathbb{S}^n$, then f clearly maps all circles to circles and hence $\Lambda_f = \Lambda$. The following main theorem of this paper says that in all other cases, Λ_f has zero $\mathcal{H}^\delta$-measure. In other words, if $\mathcal{H}^\delta(\Lambda_f) > 0$, then f is the restriction of a Möbius transformation of $\mathbb{S}^n$ and ρ is algebraic.

Theorem 1.1. *Let $n, m \geq 2$. Let $\Gamma < \mathrm{SO}^\circ(n+1,1)$ be a Zariski dense convex cocompact subgroup such that the ordinary set $\Omega = \mathbb{S}^n - \Lambda$ has at least two components. Let $\rho : \Gamma \to \mathrm{SO}^\circ(m+1,1)$ be a Zariski dense convex cocompact deformation and $f : \Lambda \to \mathbb{S}^m$ the ρ-boundary map. Then*

$$\text{either} \quad \Lambda_f = \Lambda \quad \text{or} \quad \mathcal{H}^\delta(\Lambda_f) = 0.$$

In the former case, we have $n = m$, f extends to some $g \in \text{Möb}(\mathbb{S}^n)$ and ρ is a conjugation by g.

When $n = m = 2$, the topological version of the above theorem that either $\Lambda_f = \Lambda$ or Λ_f has empty interior was obtained in our earlier paper [12] for all finitely generated discrete subgroups. Theorem 1.1 provides its measure theoretic version. See Theorem 5.3 for the topological version for general $n, m \geq 2$.

Remark 1.2. If $\Gamma < \text{SO}^\circ(3,1)$ is convex cocompact with Λ connected, then Ω is disconnected [16, Chapter IX]; hence Theorem 1.1 applies.

Indeed, we prove Theorem 1.1 under a weaker condition that there exists a Λ-doubly stable circle (Theorem 5.1).

Definition 1.3. We say that a circle $C \subset \mathbb{S}^n$ is Λ-*doubly stable* if for any sequence of circles C_k converging to C,

$$\# \limsup (C_k \cap \Lambda) \geq 2,$$

where $\limsup E_k$ is defined as $\bigcap_{n\in\mathbb{N}} \overline{\bigcup_{k\geq n} E_k}$ for a sequence $E_k \subset \mathbb{S}^n$.

If Ω is disconnected, there exists a Λ-doubly stable circle (Lemma 4.2). If $\Omega = \emptyset$, i.e., $\Lambda = \mathbb{S}^n$, then every circle is $\mathbb{S}^n$-doubly stable. In particular, Theorem 5.1 applies to any uniform lattice Γ of $\text{SO}^\circ(n+1,1)$: either $f : \mathbb{S}^n \to \mathbb{S}^m$ preserves Lebesgue-almost none of the circles, or $n = m$ and f is induced by a Möbius transformation on $\mathbb{S}^n$.

Remark 1.4. It is an interesting question whether there exists a Zariski dense convex cocompact subgroup of $\text{SO}^\circ(3,1)$ whose limit set Λ is totally disconnected and there is no Λ-doubly stable circle.

In terms of the quasiconformal deformation indicated in Figure 2, our theorem implies that the union of circular slices of the left limit set which are mapped into circles has zero $\mathcal{H}^\delta$-measure.

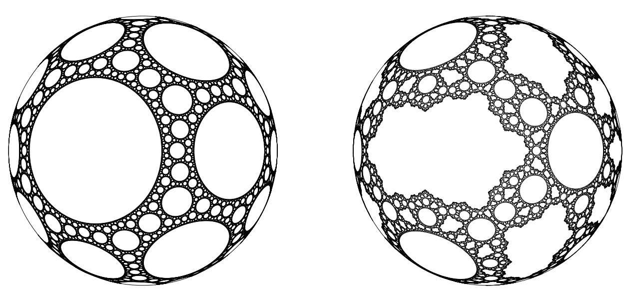

FIGURE 2. Non-trivial quasiconformal deformation[2]

[2]Image credit: Curtis McMullen and Yongquan Zhang [24]

Note that $(n+2)$-distinct points on $\mathbb{S}^n$ form the set of vertices of a unique ideal hyperbolic $(n+1)$-simplex of $\mathbb{H}^{n+1}$. Gromov-Thurston's proof of Mostow rigidity theorem ([8], [22]) uses the fact that a homeomorphism of $\mathbb{S}^n$ mapping vertices of every maximal volume $(n+1)$-simplex of $\mathbb{H}^{n+1}$ to vertices of a maximal volume $(n+1)$-simplex is a Möbius transformation.

Any $(n+2)$-distinct points on $\mathbb{S}^n$ form vertices of a zero-volume $(n+1)$-simplex of $\mathbb{H}^{n+1}$ if and only if they lie in some codimension one sphere in $\mathbb{S}^n$. We also prove the following higher dimensional version of [12, Theorem 1.3], which answered McMullen's question for $n = 2$:

Theorem 1.5. *Let $\Gamma < \mathrm{SO}^\circ(n+1,1)$ be a Zariski dense discrete subgroup. Suppose that there exists a Λ-doubly stable circle in $\mathbb{S}^n$. If the ρ-boundary map $f : \Lambda \to \mathbb{S}^n$ maps vertices of every $(n+1)$-simplex of zero-volume to vertices of an $(n+1)$-simplex of zero-volume, then f extends to a Möbius transformation of $\mathbb{S}^n$.*

We obtain a stronger statement that unless f extends to a Möbius transformation, the union of all vertices of $(n+1)$-simplexes of zero-volume whose images under f form vertices of zero-volume $(n+1)$-simplexes has empty interior in Λ.

On the proof of Theorem 1.1. We use the theory of Anosov representations. Consider the following self-joining subgroup of $G = \mathrm{SO}^\circ(n+1,1) \times \mathrm{SO}^\circ(m+1,1)$:

$$\Gamma_\rho := (\mathrm{id} \times \rho)(\Gamma) = \{(\gamma, \rho(\gamma)) : \gamma \in \Gamma\}.$$

The crucial point is that, under the assumption that both Γ and $\rho(\Gamma)$ are Zariski dense and convex cocompact and not conjugate to each other, we have that

$$\Gamma_\rho \text{ is a Zariski dense } \textit{Anosov} \text{ subgroup of } G \\ \text{with respect to a minimal parabolic subgroup}$$

(see the discussion around (2.2)). Hence the recent classification theorem on higher rank conformal measures by Lee-Oh [15] (Theorem 2.3) and the ergodicity theorem of Burger-Landesberg-Lee-Oh [4] (Theorem 2.4) apply to our setting, yielding that for any Γ_ρ-conformal measure on the limit set of Γ_ρ, the associated Bowen-Margulis-Sullivan measure on $\Gamma_\rho \backslash G$ is ergodic and conservative for a unique one-parameter diagonal flow $A_u = \{\exp tu : t \in \mathbb{R}\}$ where u is a vector in the interior of the positive Weyl chamber.

A general higher rank conformal measure seems mysterious. However, the graph structure of our self-joining group Γ_ρ allows us to pin down a very explicit Γ_ρ-conformal measure, which we call the graph-conformal measure [13]. Indeed, under the convex cocompactness hypothesis on Γ, the graph-conformal measure is given by the pushforward measure $(\mathrm{id} \times f)_*(\mathcal{H}^\delta|_\Lambda)$, and this is the reason why we can relate the Hausdorff measure $\mathcal{H}^\delta|_\Lambda$ with dynamics on the Anosov homogeneous space $\Gamma_\rho\backslash G$ in the proof of Theorem 1.1.

The conclusion of Theorem 1.1 follows if we show that Γ_ρ cannot be Zariski dense in G (Lemma 2.2). We give a proof by contradiction. Suppose that Γ_ρ is Zariski dense. Considering the action of Γ_ρ on the space Υ_ρ of all ordered pairs $Y = (C, S)$ of a circle $C \subset \mathbb{S}^n$ and a codimension one sphere $S \subset \mathbb{S}^m$ intersecting the limit set $\Lambda_\rho \subset \mathbb{S}^n \times \mathbb{S}^m$ of Γ_ρ, we are then able to prove, together with the work of Guivarch-Raugi [10] and the aforementioned ergodicity and conservativity result for the directional diagonal flows, that for $\mathcal{H}^\delta|_\Lambda$-almost all $\xi \in \Lambda$, the Γ_ρ-orbit of $Y \in \Upsilon_\rho$ containing $(\xi, f(\xi))$ is dense in the space Υ_ρ.

On the other hand, we show that the existence of a Λ-doubly stable circle in $\mathbb{S}^n$ implies that for any $Y_0 = (C_0, S_0) \in \Upsilon_\rho$ with $f(C_0 \cap \Lambda) \subset S_0$, the orbit $\Gamma_\rho Y_0$ cannot be dense in Υ_ρ (Theorem 4.1). This shows that Γ_ρ cannot be Zariski dense when $\mathcal{H}^\delta(\Lambda_f) > 0$. We also show that when Ω is disconnected, a Λ-doubly stable circle exists (Lemma 4.2).

Analogous question for rational maps. We close the introduction by the following question which seems natural in view of Sullivan's dictionary between Kleinian groups and rational maps ([21], [18]).

Question 1.6. Let $h_1, h_2 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be rational maps of degree at least 2 whose Julia sets are not contained in circles. Suppose that $h_2 = F \circ h_1 \circ F^{-1}$ for some quasiconformal homeomorphism $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Suppose that for the Julia set $J = J_{h_1}$ of h_1, there exists a J-doubly stable circle in $\hat{\mathbb{C}}$. Let

$$J_F := \bigcup \left\{ C \cap J : \begin{array}{c} C \subset \hat{\mathbb{C}} \text{ is a circle such that} \\ F(C \cap J) \text{ is contained in a circle} \end{array} \right\}.$$

(1) If $J_F = J$, is $F \in \text{Möb}(\hat{\mathbb{C}})$?
(2) Suppose that h_1, h_2 are hyperbolic. Let $\delta = \dim_H J$. Is it true that

$$\text{either} \quad J_F = J \quad \text{or} \quad \mathcal{H}^\delta(J_F) = 0?$$

Organization. The main goal of section 2 is to prove Theorem 2.6, which we deduce from the classification of conformal measures in [15] and the ergodicity and conservativity of directional diagonal flows in [4] with respect to the Bowen-Margulis-Sullivan measure associated to the Γ_ρ-conformal measure constructed from the δ-dimensional Hausdorff measure on Λ. The main theorem of section 3 is Theorem 3.3 which we deduce from Theorem 2.6 and a theorem of Guivarch-Raugi (Theorem 3.2). In section 4, we discuss an obstruction to dense Γ_ρ-orbits in the space Υ_ρ when a Λ-doubly stable circle exists. In section 5, we give a proof of Theorem 1.1. We also discuss a topological version of Theorem 1.1 without convex cocompactness assumption (Theorem 5.3).

Acknowledgement. We would like to thank Curt McMullen for useful comments on the preliminary version. We are also grateful to him and Yongquan Zhang for allowing us to use the beautiful image of Figure 2.

2. Ergodicity and graph-conformal measure

Let (X_1, d_1) and (X_2, d_2) be rank one Riemannian symmetric spaces. Let G be the product $G_1 \times G_2$ where $G_1 = \mathrm{Isom}^\circ(X_1)$ and $G_2 = \mathrm{Isom}^\circ(X_2)$ are connected simple real algebraic groups of rank one. Then $G = \mathrm{Isom}^\circ X$ where $X = X_1 \times X_2$ is the Riemannian product. We fix a Cartan involution θ of the Lie algebra $\mathfrak{g}$ of G, and decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ, respectively. We denote by K the maximal compact subgroup of G and choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Choosing a closed positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$, let $A := \exp \mathfrak{a}$ and $A^+ = \exp \mathfrak{a}^+$. The centralizer of A in K is denoted by M, and we let N^+ and $N = N^-$ be the horospherical subgroups so that $\log N^+$ and $\log N^-$ are the sum of all negative and positive root subspaces for our choice of A^+ respectively. We set

$$P^+ = MAN^+, \quad \text{and} \quad P = P^- = MAN;$$

they are minimal parabolic subgroups of G that are opposite to each other. The quotient $\mathcal{F} = G/P$ is known as the Furstenberg boundary of G, and is isomorphic to K/M. Let $\mathrm{N}_K(\mathfrak{a})$ be the normalizer of $\mathfrak{a}$ in K and let $\mathcal{W} := \mathrm{N}_K(\mathfrak{a})/M$ denote the Weyl group. Let $w_0 \in \mathrm{N}_K(\mathfrak{a})$ be the unique element in $\mathcal{W}$ such that $w_0 P w_0^{-1} = P^+$. For each $g \in G$, we define

$$g^+ := gP \in \mathcal{F} \quad \text{and} \quad g^- := g w_0 P \in \mathcal{F}.$$

An element $g \in G$ is loxodromic if $g = hamh^{-1}$ for some $a \in \operatorname{int} A^+$, $m \in M$ and $h \in G$. The Jordan projection of g is defined to be $\lambda(g) := \log a \in \operatorname{int} \mathfrak{a}^+$.

In the rest of the section, let Δ be a Zariski dense discrete subgroup of G. The *limit cone* $\mathcal{L}_\Delta \subset \mathfrak{a}^+$ is defined as the smallest closed cone containing all Jordan projections of loxodromic elements of Δ. It is a convex subset of $\mathfrak{a}^+$ with non-empty interior [1, Section 1.2]. Benoist showed that there exists a unique Δ-minimal subset of $\mathcal{F}$, which is called the limit set of Δ. We denote it by Λ_Δ.

Bowen-Margulis-Sullivan measures. Let $\mathcal{F}_i$ be the Furstenberg boundary of G_i, which is equal to the geometric boundary ∂X_i. For each $i = 1, 2$, the Busemann function $\beta_{\xi_i}(x_i, y_i)$ is defined as

$$\beta_{\xi_i}(x_i, y_i) = \lim_{t \to \infty} d_i(\xi_{i,t}, x_i) - d_i(\xi_{i,t}, y_i)$$

where $\xi_{i,t}$ is a geodesic ray toward to ξ_i. For $\xi = (\xi_1, \xi_2) \in \mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ and $x = (x_1, x_2), y = (y_1, y_2) \in X$, the $\mathfrak{a}$-valued Busemann function is defined componentwise:

$$\beta_\xi(x, y) = (\beta_{\xi_1}(x_1, y_1), \beta_{\xi_2}(x_2, y_2)) \in \mathfrak{a}$$

where we have identified $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ with $\mathbb{R}^2$.

In the following we fix $o = (o_1, o_2) \in X$ so that the stabilizer of o is K.

Definition 2.1. For a linear form $\psi \in \mathfrak{a}^*$, a Borel probability measure ν on $\mathcal{F}$ is called a (Δ, ψ)-*conformal measure* (with respect to o) if for any $g \in \Delta$ and $\xi \in \mathcal{F}$,

$$\frac{dg_*\nu}{d\nu}(\xi) = e^{\psi(\beta_\xi(o, go))}$$

where $g_*\nu(B) = \nu(g^{-1}B)$ for any Borel subset $B \subset \mathcal{F}$. By a Δ-conformal measure, we mean a (Δ, ψ)-conformal measure for some $\psi \in \mathfrak{a}^*$.

Two points $\xi = (\xi_1, \xi_2)$ and $\eta = (\eta_1, \eta_2)$ are in general position if $\xi_i \neq \eta_i$ for each $i = 1, 2$. Let $\mathcal{F}^{(2)}$ be the set of all pairs $(\xi, \eta) \in \mathcal{F} \times \mathcal{F}$ which are in general position. The map $G \to \mathcal{F}^{(2)} \times \mathfrak{a}$, $g \mapsto (g^+, g^-, \beta_{g^+}(o, go))$ induces a G-equivariant homeomorphism $G/M \simeq \mathcal{F}^{(2)} \times \mathfrak{a}$, called the Hopf-parametrization.

For a (Δ, ψ)-conformal measure ν supported on the limit set Λ_Δ for some $\psi \in \mathfrak{a}^*$, we can define the following Borel measure on G/M using the Hopf-parametrization:

$$d\tilde{m}_\nu^{\mathrm{BMS}}(gM) = e^{\psi(\beta_{g^+}(o,go) + \beta_{g^-}(o,go))} d\nu(g^+) d\nu(g^-) db \tag{2.1}$$

where db is the Haar measure on $\mathfrak{a}$. By integrating over the fiber of $G \to G/M$ with respect to the Haar measure of M, we will consider $\tilde{m}_\nu^{\mathrm{BMS}}$ as a Radon measure on G, which is then a left Δ-invariant and right AM-invariant measure. We denote by m_ν^{BMS} the Radon measure on $\Delta \backslash G$ induced by $\tilde{m}_\nu^{\mathrm{BMS}}$. This measure is called the Bowen-Margulis-Sullivan measure associated to ν. Its support is

$$\Omega_\Delta = \{[g] \in \Delta\backslash G : g^\pm \in \Lambda_\Delta\}.$$

We refer to [7] for a detailed discussion on the construction of this measure.

Self-joinings of convex cocompact groups. In the rest of the section, we will consider the following special type of discrete subgroups of G. Let $\Gamma < G_1$ be a Zariski dense convex cocompact subgroup and $\rho : \Gamma \to G_2$ be a Zariski dense convex cocompact faithful representation. Define the self-joining of Γ via ρ:

$$\Gamma_\rho := (\mathrm{id} \times \rho)(\Gamma) = \{(\gamma, \rho(\gamma)) : \gamma \in \Gamma\}$$

which is a discrete subgroup of G.

It follows from the convex cocompactness assumption for Γ and $\rho(\Gamma)$ that if we fix a word metric $|\cdot|$ on Γ for some finite generating set and fix $o_1 \in X_1$ and $o_2 \in X_2$, then there exist constants $C, C' > 0$ such that for all $\gamma \in \Gamma$,

$$\min\{d_1(\gamma o_1, o_1), d_2(\rho(\gamma) o_2, o_2)\} \geq C|\gamma| - C'. \tag{2.2}$$

In other words, Γ_ρ is an Anosov subgroup of G with respect to a minimal parabolic subgroup ([14], [9], [11]). This enables us to use the general theory developed for Anosov subgroups. We remark that ergodic theory for self-joining groups of convex cocompact groups was first studied in [3].

Since both G_1 and G_2 are simple, we have the following equivalence between Zariski density of the self-joining and the rigidity of ρ, first observed by Dal'Bo-Kim [6]:

Lemma 2.2. *The subgroup Γ_ρ is Zariski dense in G if and only if ρ does not extend to a Lie group isomorphism $G_1 \to G_2$.*

Since Γ and $\rho(\Gamma)$ are convex cocompact, there exists a unique ρ-equivariant continuous embedding $f : \Lambda \to \mathcal{F}_2$; this is a special case of a theorem of Tukia [23], which can also be seen directly as follows. Since $\Gamma < G_1$ is convex cocompact, Γ is a hyperbolic group and an orbit map $\Gamma \to X_1$ is a quasi-isometric embedding, where Γ is equipped with a word metric. Hence, it follows from a standard result for Gromov hyperbolic spaces (e.g. [2, Chapter III.H, Theorem 3.9]) that there exists a unique Γ-equivariant homeomorphism $f_1 : \partial\Gamma \to \Lambda_\Gamma = \Lambda$ where $\partial\Gamma$ is the Gromov boundary of Γ. Similarly, we obtain a unique $\rho(\Gamma)$-equivariant homeomorphism $f_2 : \partial\rho(\Gamma) \to \Lambda_{\rho(\Gamma)}$. Since $\rho : \Gamma \to \rho(\Gamma)$ is an isomorphism, there exists a unique ρ-equivariant homeomorphism $f_0 : \partial\Gamma \to \partial\rho(\Gamma)$. Therefore, $f := f_2 \circ f_0 \circ f_1^{-1} : \Lambda \to \Lambda_{\rho(\Gamma)}$ is the unique ρ-equivariant homeomorphism into $\mathcal{F}_2$.

Hence, for Γ_ρ Zariski dense, its limit set $\Lambda_\rho \subset \mathcal{F}$ is of the form

$$\Lambda_\rho = (\mathrm{id} \times f)(\Lambda)$$

where $\mathrm{id} \times f : \Lambda \to \Lambda_\rho$ is the diagonal embedding. We denote by $\mathcal{L}_\rho \subset \mathfrak{a}^+$ the limit cone of Γ_ρ:

$$\mathcal{L}_\rho = \mathcal{L}_{\Gamma_\rho}.$$

Since Γ_ρ is Anosov, the following Theorems 2.3 and 2.4 are special cases of theorems proved in those respective papers. Let $\mathfrak{a}^*$ denote the set of all $\mathbb{R}$-linear forms on $\mathfrak{a}$.

Theorem 2.3 (Classification of conformal measures, [15, Theorem 1.3, Proposition 4.4])**.** *Suppose that Γ_ρ is Zariski dense in G. The space of unit vectors in $\operatorname{int} \mathcal{L}_\rho$ is in bijection with the space of all Γ_ρ-conformal measures on Λ_ρ. Moreover, each Γ_ρ-conformal measure on Λ_ρ is a (Γ_ρ, ψ)-conformal measure for a unique linear form $\psi \in \mathfrak{a}^*$.*

We will denote this bijection by

$$u \mapsto \nu_u. \tag{2.3}$$

For each unit vector $u \in \operatorname{int} \mathcal{L}_\rho$, we also denote by $\psi_u \in \mathfrak{a}^*$ the (unique) linear form associated to ν_u, that is, ν_u is (Γ_ρ, ψ_u)-conformal.

Ergodicity. For simplicity, we set

$$\tilde{m}_u^{\mathrm{BMS}} := \tilde{m}_{\nu_u}^{\mathrm{BMS}} \quad \text{and} \quad m_u^{\mathrm{BMS}} := m_{\nu_u}^{\mathrm{BMS}}$$

For any non-zero vector $u \in \mathfrak{a}$, we consider the following one-parameter semigroup/subgroup:

$$A_u^+ := \{a_{tu} : t \geq 0\} \text{ and } A_u := \{a_{tu} : t \in \mathbb{R}\}.$$

where $a_{tu} = \exp tu$. The following ergodicity result due to Burger-Landesberg-Lee-Oh [4] is the main ingredient of our proof of Theorem 1.1:

Theorem 2.4 (Ergodicity of directional flows, [4])**.** *Suppose that Γ_ρ is Zariski dense in G. For any unit vector $u \in \operatorname{int} \mathcal{L}_\rho$, $(m_u^{\mathrm{BMS}}, \Gamma_\rho \backslash G)$ is ergodic and conservative for the A_u-action. In particular, for m_u^{BMS}-almost all x, xA_u^+ is dense in Ω_{Γ_ρ}.*

Graph-conformal measure. Let ν_Γ be the Γ-conformal measure supported on the limit set Λ of Γ; since Γ is convex cocompact, it exists uniquely [20]. It turns out that the measure $(\mathrm{id} \times f)_* \nu_\Gamma$ is a Γ_ρ-conformal measure, where $\mathrm{id} \times f : \Lambda \to \Lambda_\rho$ is the diagonal embedding. We called this measure the graph-conformal measure in [13]. More precisely, we have the following lemma, thanks to which we were able to apply Theorem 2.4 in the proof of Theorem 1.1: we denote by δ_Γ the critical exponent of Γ.[3]

Lemma 2.5. [13, Proposition 4.9] *The measure*

$$(\mathrm{id} \times f)_* \nu_\Gamma$$

is a (Γ_ρ, σ_1)-conformal measure supported on Λ_ρ, where $\sigma_1 \in \mathfrak{a}^*$ is the linear form given by $\sigma_1(t_1, t_2) = \delta_\Gamma t_1$ for $(t_1, t_2) \in \mathfrak{a}$.

We now deduce Theorem 2.6 from Theorems 2.3 and 2.4: first, there exists a unique unit vector

$$u_\rho \in \operatorname{int} \mathcal{L}_\rho \text{ such that } (\mathrm{id} \times f)_* \nu_\Gamma = \nu_{u_\rho}. \tag{2.4}$$

Hence if we write $\Omega_\rho := \Omega_{\Gamma_\rho}$, we get the following main theorem of this section:

Theorem 2.6. *Suppose that Γ_ρ is Zariski dense. Then there exists an $(\mathrm{id} \times f)_* \nu_\Gamma$-conull subset*

$$\Lambda'_\rho \subset \Lambda_\rho$$

such that for any $g \in G$ with $g^+ \in \Lambda'_\rho$, the closure $\overline{[g]A_{u_\rho}^+}$ contains Ω_ρ.

Proof. Since $\tilde{m}_{u_\rho}^{\mathrm{BMS}}$ is equivalent to the product measure $d\nu_{u_\rho} \times d\nu_{u_\rho} \times da \times dm$ where da and dm denote Haar measures on A and M respectively, it follows from Theorem 2.4 that there exists a ν_{u_ρ}-conull subset $\Lambda'_\rho \subset \Lambda_\rho$ such that for all $\xi \in \Lambda'_\rho$, there exists $g_0 \in G$ with $g_0^+ = \xi$ and $g_0^- \in \Lambda_\rho$ such that $[g_0]A_{u_\rho}^+$ is dense in Ω_ρ. Hence the claim follows by the following Lemma 2.7. $\square$

[3] The critical exponent δ_Γ is the abscissa of convergence of the Poincaré series $\sum_{\gamma \in \Gamma} e^{-s d_1(o_1, \gamma o_1)}$ for $o_1 \in X_1$.

Lemma 2.7. *Let $u \in \operatorname{int} \mathfrak{a}^+$ and $\Delta < G$ be a Zariski dense discrete subgroup. If $[g_0]A_u^+$ is dense in Ω_Δ, then for any $g \in G$ with $g^+ = g_0^+$, the closure $\overline{[g]A_u^+}$ contains Ω_Δ.*

Proof. This can be deduced from the proof of [12, Corollary 2.3], which we recall for readers' convenience. Since $g^+ = g_0^+$, $g = g_0 p$ for some $p \in P$. Writing $p = nam \in NAM$, we claim that

$$(\Omega_\Delta - [g_0]A_u^+)ma \subset \overline{[g]A_u^+}.$$

Let $x \in \Omega_\Delta - [g_0]A_u^+$. Since $\overline{[g_0]A_u^+} \supset \Omega_\Delta$, there exists a sequence $t_i \to +\infty$ such that $x = \lim_{i\to\infty}[g_0]a_{t_i u}$. Since $u \in \operatorname{int} \mathfrak{a}^+$, we have $a_{-t_i u} n a_{t_i u} \to e$ as $i \to \infty$. Therefore

$$\lim_{i\to\infty}[g]a_{t_i u} = \lim_{i\to\infty}[g_0]nama_{t_i u} = \lim_{i\to\infty}[g_0]a_{t_i u}(a_{-t_i u} n a_{t_i u})am = xam;$$

so $xam \in \overline{[g]A_u^+}$. This proves the claim.

Since Ω_Δ is AM-invariant and $\Omega_\Delta - [g_0]AM$ is dense in Ω_Δ, it follows that

$$\Omega_\Delta \subset \overline{[g]A_u^+}.$$

$\square$

3. Orbits in the space of circle-sphere pairs

Let $G_1 = \mathrm{SO}^\circ(n+1,1)$, $n \geq 2$ and $G_2 = \mathrm{SO}^\circ(m+1,1)$, $m \geq 2$. We set

$$\Upsilon = \{Y = (C, S) : C \subset \mathbb{S}^n \text{ a circle, } S \subset \mathbb{S}^m \text{ a codimension one sphere}\}.$$

Let $G = G_1 \times G_2$. The group G acts on Υ componentwise:

$$(g_1, g_2)(C, S) = (g_1 C, g_2 S)$$

for $(g_1, g_2) \in G_1 \times G_2$ and $(C, S) \in \Upsilon$. Let $\Delta < G$ be a Zariski dense discrete subgroup. Then Δ acts on the space

$$\Upsilon_\Delta = \{Y \in \Upsilon : Y \cap \Lambda_\Delta \neq \emptyset\},$$

which is a closed subset of Υ.

Denseness of Υ_Δ^*. Let

$$\Upsilon_\Delta^* := \{Y \in \Upsilon_\Delta : \# Y \cap \Lambda_\Delta \geq 2\}.$$

Theorem 3.1. *The subset Υ_Δ^* is dense in Υ_Δ.*

Recalling that $P = MAN$ and $\mathcal{F} = G/P \simeq K/M$, we have $G/AN \simeq K$. Consider the projection $\pi : G/AN = K \to G/P = K/M$, and set

$$\tilde{\Lambda}_\Delta = \pi^{-1}(\Lambda_\Delta) \subset G/AN = K.$$

Since $M \simeq \mathrm{SO}(n) \times \mathrm{SO}(m)$ is connected, the following is a special case of a theorem of Guivarch and Raugi [10]:

Theorem 3.2 ([10, Theorem 2])**.** *The action of Δ on $\tilde{\Lambda}_\Delta$ is minimal.*

Indeed, this theorem is a key ingredient of the proof of Theorem 3.1, which we now begin.

Proof of Theorem 3.1. For simplicity, we write Λ for Λ_Δ in this proof. Write $K = K_1 \times K_2$ where $K_1 = K \cap (G_1 \times \{e\}) = \mathrm{SO}(n+1)$ and $K_2 = K \cap (\{e\} \times G_2) = \mathrm{SO}(m+1)$, and similarly, we write $M = M_1 \times M_2 = \mathrm{SO}(n) \times \mathrm{SO}(m)$. Via the projection $K_i \to K_i/M_i = \mathcal{F}_i$, we can think of a point of K_i as an orthonormal frame f_ξ based at $\xi \in \mathcal{F}_i$. Hence an element of K is a pair of orthonormal frames $(\mathsf{f}_{\xi_1}, \mathsf{f}_{\xi_2}) \in K_1 \times K_2$. For an infinite sequence $(\xi_{1,j}, \xi_{2,j}) \in \mathcal{F}_1 \times \mathcal{F}_2$ converging to (ξ_1, ξ_2), we say that the convergence is $(1,1)$-tangential to the frame $(\mathsf{f}_{\xi_1}, \mathsf{f}_{\xi_2})$ if, for each $i = 1, 2$, the sequence of unit vectors $\frac{\overrightarrow{\xi_i \xi_{i,j}}}{\|\overrightarrow{\xi_i \xi_{i,j}}\|}$ at ξ_i converges to the first vector of the frame f_{ξ_i} as $j \to \infty$.

Let

$$\mathcal{E} = \left\{ (\mathsf{f}_{\xi_1}, \mathsf{f}_{\xi_2}) \in \tilde{\Lambda} : \begin{array}{c} \text{there exists a sequence } (\xi_{1,j}, \xi_{2,j}) \in \Lambda \\ \text{converging to } (\mathsf{f}_{\xi_1}, \mathsf{f}_{\xi_2}) \ (1,1)\text{-tangentially} \end{array} \right\}.$$

We first note that $\mathcal{E}$ is non-empty. Since Δ is Zariski dense in G , Δ contains a loxodromic element, say, $g \in \Delta$. Denote by $y_g \in \mathcal{F}$ the attracting fixed point of g. Choose $\zeta \in \Lambda$ which is in general position with $y_{g^{\pm 1}}$. Then the sequence $g^\ell \zeta$ converges to y_g as $\ell \to +\infty$. The claim follows from the compactness of the unit sphere in the tangent space of $\mathcal{F}$ at y_g.

On the other hand, since the action of G on $\mathcal{F}$ is conformal and Λ is Δ-invariant, $\mathcal{E}$ is a Δ-invariant subset of $\tilde{\Lambda}$. Hence by Theorem 3.2,

$$\overline{\mathcal{E}} = \tilde{\Lambda}.$$

Let $Y = (C, S) \in \Upsilon_\Delta$. We will construct a sequence $Y_k \in \Upsilon_\Delta^*$ converging to Y as $k \to \infty$. Choose $\xi = (\xi_1, \xi_2) \in Y \cap \Lambda$. Choose a unit vector v_1 at ξ_1 tangent to C and a unit vector v_2 at ξ_2 tangent to S. For each $i = 1, 2$, choose an orthonormal frame f_{ξ_i} in $\mathcal{F}_i$ based at ξ_i whose first vector is v_i. Since $(\mathsf{f}_{\xi_1}, \mathsf{f}_{\xi_2}) \in \tilde{\Lambda}$ and $\mathcal{E}$ is dense in $\tilde{\Lambda}$, we can find a sequence $(\mathsf{f}_{\eta_{1,k}}, \mathsf{f}_{\eta_{2,k}}) \in \mathcal{E}$ converging to $(\mathsf{f}_{\xi_1}, \mathsf{f}_{\xi_2})$ as $k \to \infty$. Hence, for each k, there exists a sequence $\{(\eta_{1,j}^{(k)}, \eta_{2,j}^{(k)}) \in \Lambda : j = 1, 2, \cdots\}$ converging $(1,1)$-tangentially to $(\mathsf{f}_{\eta_{1,k}}, \mathsf{f}_{\eta_{2,k}})$ as $j \to \infty$. Since $(\mathsf{f}_{\eta_{1,k}}, \mathsf{f}_{\eta_{2,k}}) \to (\mathsf{f}_{\xi_1}, \mathsf{f}_{\xi_2})$ as $k \to \infty$, we can choose large enough j_k for each k so that the following holds for each $i = 1, 2$:

(1) $\eta_{i,j_k}^{(k)} \to \xi_i$ as $k \to \infty$; and
(2) the unit tangent vector $\frac{\overrightarrow{\eta_{i,k} \eta_{i,j_k}^{(k)}}}{\|\overrightarrow{\eta_{i,k} \eta_{i,j_k}^{(k)}}\|}$ at $\eta_{i,k}$ converges to v_i as $k \to \infty$.

Now we are ready to construct a sequence $Y_k = (C_k, S_k) \in \Upsilon_\Delta^*$:

(1) Fix $z_1 \in C - \{\xi_1\}$ and let C_k be the circle passing through z_1, $\eta_{1,k}$ and $\eta_{1,j_k}^{(k)}$.
(2) Fix $z_2 \in S - \{\xi_2\}$. The tangent space $\mathsf{T}_{\xi_2} S$ of S at ξ_2 is a codimension one subspace of the tangent space $\mathsf{T}_{\xi_2} \mathcal{F}_2$. Noting that $\mathsf{v}_2 \in \mathsf{T}_{\xi_2} S$,

we can choose unit tangent vectors $\mathsf{w}_1, \cdots, \mathsf{w}_{m-2} \in \mathsf{T}_{\xi_2} S$ so that $\mathsf{v}_2, \mathsf{w}_1, \cdots, \mathsf{w}_{m-2}$ form a basis of $\mathsf{T}_{\xi_2} S$. For each $\ell = 1, \cdots, m-2$, we choose a sequence $\zeta_{\ell,k} \in \mathcal{F}_2$ converging to ξ_2 such that the unit vectors $\frac{\overrightarrow{\eta_{2,k}\zeta_{\ell,k}}}{\|\overrightarrow{\eta_{2,k}\zeta_{\ell,k}}\|}$ converges to w_ℓ as $k \to \infty$. Then for each $k \geq 1$ large enough, the set

$$\{z_2, \eta_{2,k}, \eta_{2,j_k}^{(k)}, \zeta_{1,k}, \cdots, \zeta_{m-2,k}\}$$

has cardinality $(m+1)$ and hence uniquely determines an $(m-1)$-dimensional sphere in $\mathcal{F}_2 = \mathbb{S}^m$, which we set to be S_k.

Since $(C_k, S_k) \cap \Lambda$ contains two distinct points $(\eta_{1,k}, \eta_{2,k})$ and $(\eta_{1,j_k}^{(k)}, \eta_{2,j_k}^{(k)})$, we have

$$(C_k, S_k) \in \Upsilon_\Delta^*.$$

Moreover, as $k \to \infty$, C_k converges to the unique circle passing through z_1 and tangent to v_1 which must be C, and S_k converges to the unique sphere passing through z_2 and whose tangent space at ξ_2 is same as $\mathsf{T}_{\xi_2} S$, which must be S. Therefore $(C_k, S_k) \in \Upsilon_\Delta^*$ converges to $Y = (C, S)$. This finishes the proof of Theorem 3.1.

Dense orbits. Let $\Gamma < \mathrm{SO}^\circ(n+1,1)$ be a convex cocompact subgroup where $n \geq 2$. Then ν_Γ is equal to δ_Γ-dimensional Hausdorff measure $\mathcal{H}^{\delta_\Gamma}|_\Lambda$ and $\delta := \delta_\Gamma$ is equal to the Hausdorff dimension of Λ by [20]. Let $\rho : \Gamma \to \mathrm{SO}^\circ(m+1,1)$ be a Zariski dense convex cocompact faithful representation. Let $\Gamma_\rho := (\mathrm{id} \times \rho)(\Gamma) < G$ and

$$\Upsilon_\rho := \Upsilon_{\Gamma_\rho} = \{Y = (C,S) \in \Upsilon : Y \cap \Lambda_\rho \neq \emptyset\}. \tag{3.1}$$

Theorem 3.3. *Suppose that Γ_ρ is Zariski dense. Then there exists a $\mathcal{H}^\delta|_\Lambda$-conull $\Lambda' \subset \Lambda$ such that for any $Y \in \Upsilon_\rho$ intersecting $(\mathrm{id} \times f)(\Lambda')$ non-trivially,*

$$\overline{\Gamma_\rho Y} = \Upsilon_\rho.$$

Proof. Since G acts transitively on Υ as homeomorphisms, we have the homeomorphism

$$\Upsilon \simeq G/H$$

where $H = \mathrm{Stab}(Y_0)$ is the stabilizer of some $Y_0 = (C_0, S_0) \in \Upsilon$. Noting that H° is a semisimple real algebraic subgroup conjugate to $(\mathrm{SO}^\circ(2,1) \times \mathrm{SO}(n-1)) \times \mathrm{SO}^\circ(m,1)$, we may choose Y_0 so that $H \supset A$ and that $H \cap P$ is a minimal parabolic subgroup of H.

Recall the subset $\Upsilon_\rho^* = \{Y \in \Upsilon_\rho : \# Y \cap \Lambda_\rho \geq 2\}$. Note that $\Upsilon_\rho^* = \Omega_\rho Y_0$. Suppose that there exists $g \in G$ such that the closure of $[g]A_u^+$ contains Ω_ρ for some $u \in \operatorname{int} \mathfrak{a}^+$. Since $A_u^+ \subset H$, the closure of $\Gamma_\rho g H$ contains $\Omega_\rho H$, in other words, the closure of $\Gamma_\rho g Y_0$ contains $\Omega_\rho Y_0 = \Upsilon_\rho^*$. Hence by Theorem 3.1,

$$\overline{\Gamma_\rho g Y_0} = \Upsilon_\rho.$$

Since $\Gamma < \mathrm{SO}^\circ(n+1,1)$ is convex cocompact, we have that $\mathcal{H}^\delta|_\Lambda$ is the unique Γ-conformal measure on Λ, up to a constant multiple [20]. Therefore Theorem 3.3 follows from Theorem 2.6 and Lemma 2.5. $\square$

4. Doubly stable condition

In this section, let $\Gamma < \mathrm{SO}^\circ(n+1,1)$ be a discrete group, $n \geq 2$, which is not necessarily convex cocompact. Let $\Lambda \subset \mathbb{S}^n$ denote its limit set.

We say that a circle $C \subset \mathbb{S}^n$ is Λ-*doubly stable* if for any sequence of circles C_k converging to C,

$$\# \limsup (C_k \cap \Lambda) \geq 2.$$

If Ω is disconnected, there exists a Λ-doubly stable circle (Lemma 4.2). Recall from (3.1) that $\Upsilon_\rho = \{Y \in \Upsilon : Y \cap \Lambda_\rho \neq \emptyset\}$.

Theorem 4.1. *Let $\Gamma < \mathrm{SO}^\circ(n+1,1)$ be a discrete subgroup and $\rho : \Gamma \to \mathrm{SO}^\circ(m+1,1)$, $m \geq 2$, be a discrete faithful representation with a boundary map $f : \Lambda \to \mathbb{S}^m$. Assume that there exists at least one Λ-doubly stable circle. If $(C_0, S_0) \in \Upsilon_\rho$ such that $f(C_0 \cap \Lambda) \subset S_0$, then*

$$\overline{\Gamma_\rho (C_0, S_0)} \neq \Upsilon_\rho.$$

Proof. Let $C \subset \mathbb{S}^n$ be a Λ-doubly stable circle. Then for any sequence of circles $C_k \subset \mathbb{S}^n$ converging to C as $k \to \infty$, we have

$$\# \limsup (C_k \cap \Lambda) \geq 2. \tag{4.1}$$

It follows that $\# C \cap \Lambda \geq 2$.

We first claim that there exists a codimension one sphere $S \subset \mathbb{S}^m$ such that

$$\# S \cap f(C \cap \Lambda) = 1. \tag{4.2}$$

Since $C \cap \Lambda$ is not homemorphic to $\mathbb{S}^m$, $m \geq 2$, the image $f(C \cap \Lambda)$ is a proper compact subset of $\mathbb{S}^m$. Therefore we can find a minimal closed m-ball $B \subset \mathbb{S}^m$ containing $f(C \cap \Lambda)$. By the minimality of B, there exists $\xi_0 \in C \cap \Lambda$ such that $f(\xi_0)$ lies in the boundary of B. Now any codimension one sphere S in $\mathbb{S}^m$ such that $S \cap B = \{f(\xi_0)\}$ satisfies (4.2).

Set $Y = (C, S)$. Since $(\xi_0, f(\xi_0)) \in (C, S)$, we have $Y \in \Upsilon_\rho$. We claim that for any $(C_0, S_0) \in \Upsilon_\rho$ such that $f(C_0 \cap \Lambda) \subset S_0$, we have $Y \notin \overline{\Gamma_\rho (C_0, S_0)}$; this implies the theorem. Suppose not. Then there exists a sequence $\gamma_k \in \Gamma$ such that $\gamma_k C_0 \to C$ and $\rho(\gamma_k) S_0 \to S$ as $k \to \infty$. By (4.1), we have

$$\# \limsup (\gamma_k C_0 \cap \Lambda) \geq 2. \tag{4.3}$$

By the ρ-equivariance of f, we have

$$f(\gamma_k C_0 \cap \Lambda) = f(\gamma_k (C_0 \cap \Lambda)) = \rho(\gamma_k) f(C_0 \cap \Lambda) \subset \rho(\gamma_k) S_0.$$

Hence

$$\limsup f(\gamma_k C_0 \cap \Lambda) \subset \limsup \rho(\gamma_k) S_0 = S.$$

Since $\limsup f(\gamma_k C_0 \cap \Lambda) \subset f(C \cap \Lambda)$ and f is injective, it follows from (4.3) that $\# S \cap f(C \cap \Lambda) \geq 2$. This contradicts (4.2), proving the claim. $\square$

We say that Λ is *doubly stable* if for any $\xi \in \Lambda$, there exists a Λ-doubly stable circle containing ξ.

Lemma 4.2. *Let $\Gamma < \mathrm{SO}^\circ(n+1,1)$ be a discrete subgroup. If Ω is disconnected, then Λ is doubly stable.*

Proof. Let Ω_1, Ω_2 be distinct connected components of Ω and fix any $\xi \in \Lambda$. Let C be a circle containing ξ and intersecting Ω_1 and Ω_2.

Let C_k be a sequence of circles converging to C as $k \to \infty$. We claim that $\# \limsup(C_k \cap \Lambda) \geq 2$. Suppose that $\# \limsup(C_k \cap \Lambda) \leq 1$. We will show that $C \cap \Omega_1$ is a singleton, which is a contradiction since $C \cap \Omega_1$ is an open subset of C.

For each k, let $I_k \subset C_k$ be a compact interval containing $C_k \cap \Lambda$ with minimal diameter. Since $C_k - I_k$ is a connected subset of Ω, $C_k - I_k \subset W_k$ for some connected component W_k of Ω. After passing to a subsequence and relabeling Ω_1 and Ω_2 if necessary, we may assume that $\Omega_1 \neq W_k$ and hence $\Omega_1 \cap W_k = \emptyset$ for all k.

Let $x, y \in C \cap \Omega_1$. Since the sequence C_k converges to C, $x = \lim_{k\to\infty} x_k$ and $y = \lim_{k\to\infty} y_k$ for some $x_k, y_k \in C_k$. Since Ω_1 is open, we may assume that $x_k, y_k \in C_k \cap \Omega_1$ for all $k \geq 1$. Hence $x_k, y_k \notin W_k$; so $x_k, y_k \in I_k$.

Since $\# \limsup(C_k \cap \Lambda) \leq 1$, the diameter of I_k tends to 0 as $k \to \infty$. Therefore the distance between x_k and y_k must go to 0 and hence $x = y$. This proves the claim, finishing the proof. $\square$

5. Rigidity via circular slices

Let $n, m \geq 2$. Let $\Gamma < \mathrm{SO}^\circ(n+1,1)$ be a Zariski dense convex cocompact subgroup. Let $\rho : \Gamma \to \mathrm{SO}^\circ(m+1,1)$ be a Zariski dense convex cocompact deformation and $f : \Lambda \to \mathbb{S}^m$ be its boundary map. Recall

$$\Lambda_f = \bigcup \left\{ C \cap \Lambda : \begin{array}{c} C \subset \mathbb{S}^n \text{ is a circle such that} \\ f(C \cap \Lambda) \text{ is contained in a } (m-1)\text{-sphere of } \mathbb{S}^m \end{array} \right\}.$$

Theorem 1.1 is a special case of the following:

Theorem 5.1. *Suppose that there exists a Λ-doubly stable circle. Then*

$$\textit{either} \quad \Lambda_f = \Lambda \quad \textit{or} \quad \mathcal{H}^\delta(\Lambda_f) = 0.$$

In the former case, we have $n = m$, f extends to some $g \in \mathrm{M\ddot{o}b}(\mathbb{S}^n)$ and ρ is a conjugation by g.

Remark 5.2. By Lemma 4.2, when Ω has at least two components, there exists a Λ-doubly stable circle. Hence Theorem 5.1 applies to this case.

Proof. Suppose that $\mathcal{H}^\delta(\Lambda_f) > 0$. We need to show that $\Lambda_f = \Lambda$. We claim that Γ_ρ cannot be Zariski dense in G. Suppose that Γ_ρ is Zariski dense. Let $\Lambda' \subset \Lambda$ be the $\mathcal{H}^\delta|_\Lambda$-conull subset given by Theorem 3.3. Since $\mathcal{H}^\delta(\Lambda_f) > 0$, there exists $\xi_0 \in \Lambda_f \cap \Lambda'$. By the definition of Λ_f, we can find

$Y_0 = (C_0, S_0) \in \Upsilon_\Delta$ so that $Y_0 \ni (\xi_0, f(\xi_0))$ and $f(C_0 \cap \Lambda) \subset S_0$. By the definition of Λ' as in Theorem 3.3, we have

$$\overline{\Gamma_\rho Y_0} = \Upsilon_\rho.$$

On the other hand, since there exists a Λ-doubly stable circle, Theorem 4.1 implies that $\overline{\Gamma_\rho Y_0} \neq \Upsilon_\rho$. This yields a contradiction, proving that Γ_ρ is not Zariski dense. Hence by Theorem 2.2, ρ extends to a Lie group isomorphism $\mathrm{SO}^\circ(n+1,1) \to \mathrm{SO}^\circ(m+1,1)$ and in particular $n = m$. Since the Lie group automorphism of $\mathrm{SO}^\circ(n+1,1)$ is a conjugation by some $g \in \text{Möb}(\mathbb{S}^n)$, it follows that ρ is a conjugation by g and by the uniqueness of the ρ-boundary map, f is the restriction of g to Λ. Therefore $\Lambda_f = \Lambda$. $\square$

Topological version without convex cocompactness. The assumption that Γ and $\rho(\Gamma)$ are convex cocompact was used to apply the ergodicity as in Theorem 2.4. The approach of our paper proves the following theorem without the convex cocompact hypothesis, which was shown in [12] for $n = m = 2$:

Theorem 5.3. *Let $\Gamma < \mathrm{SO}^\circ(n+1,1)$ be a Zariski dense discrete subgroup. Suppose that there exists a Λ-doubly stable circle. Let $\rho : \Gamma \to \mathrm{SO}^\circ(m+1,1)$ be a Zariski dense deformation with a ρ-boundary map $f : \Lambda \to \mathbb{S}^m$. Then*

$$\text{either} \quad \Lambda_f = \Lambda \quad \text{or} \quad \Lambda_f \text{ has empty interior in } \Lambda.$$

In the former case, we have $n = m$, f extends to some $g \in \text{Möb}(\mathbb{S}^n)$ and ρ is a conjugation by g.

For this, we need to replace the ergodicity theorem (Theorem 2.4) by the following theorem of Chow-Sarkar for $\Delta = \Gamma_\rho$:

Theorem 5.4 ([5, Theorem 8.1])**.** *Let $\Delta < G$ be a Zariski dense discrete subgroup. For any $u \in \operatorname{int} \mathcal{L}_\Delta$, there exists a dense A_u^+-orbit in*

$$\Omega_\Delta := \{[g] \in \Delta\backslash G : g^\pm \in \Lambda_\Delta\}.$$

This theorem provides a dense subset $\Lambda' \subset \Lambda$ such that for any $Y \subset \Upsilon_\rho$ intersecting $(\mathrm{id} \times f)(\Lambda')$ non-trivially, $\Gamma_\rho Y$ is dense in Υ_ρ, which is a topological version of Theorem 3.3. With this replacement, the rest of the proof can be repeated in verbatim. Theorem 1.5 is a direct consequence of Theorem 5.3 and Lemma 4.2.

References

[1] Y. Benoist. Propriétés asymptotiques des groupes linéaires. *Geom. Funct. Anal.*, 7(1):1–47, 1997.

[2] M. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[3] M. Burger. Intersection, the Manhattan curve, and Patterson-Sullivan theory in rank 2. *Internat. Math. Res. Notices*, (7):217–225, 1993.

[4] M. Burger, O. Landesberg, M. Lee, and H. Oh. The Hopf–Tsuji–Sullivan dichotomy in higher rank and applications to Anosov subgroups. *J. Mod. Dyn.*, 19:301–330, 2023.

[5] M. Chow and P. Sarkar. Local mixing of one-parameter diagonal flows on Anosov homogeneous spaces. *Int. Math. Res. Not. IMRN*, (18):15834–15895, 2023.
[6] F. Dal'Bo and I. Kim. A criterion of conjugacy for Zariski dense subgroups. *C. R. Acad. Sci. Paris Sér. I Math.*, 330(8):647–650, 2000.
[7] S. Edwards, M. Lee, and H. Oh. Anosov groups: local mixing, counting and equidistribution. *Geom. Topol.*, 27(2):513–573, 2023.
[8] M. Gromov. Hyperbolic manifolds. In *Bourbaki Seminar, Vol. 1979/80*, volume 842 of *Lecture Notes in Math.*, pages 40–53. Springer, 1981.
[9] O. Guichard and A. Wienhard. Anosov representations: domains of discontinuity and applications. *Invent. Math.*, 190(2):357–438, 2012.
[10] Y. Guivarc'h and A. Raugi. Actions of large semigroups and random walks on isometric extensions of boundaries. *Ann. Sci. École Norm. Sup. (4)*, 40(2):209–249, 2007.
[11] M. Kapovich, B. Leeb, and J. Porti. Anosov subgroups: dynamical and geometric characterizations. *Eur. J. Math.*, 3(4):808–898, 2017.
[12] D. M. Kim and H. Oh. Rigidity of Kleinian groups via self-joinings. *Invent. Math.*, 234(3):937–948, 2023.
[13] D. M. Kim and H. Oh. Conformal measure rigidity for representations via self-joinings. *Adv. Math.*, 458:Paper No. 109992, 40, 2024.
[14] F. Labourie. Anosov flows, surface groups and curves in projective space. *Invent. Math.*, 165(1):51–114, 2006.
[15] M. Lee and H. Oh. Invariant measures for horospherical actions and Anosov groups. *Int. Math. Res. Not. IMRN*, (19):16226–16295, 2023.
[16] B. Maskit. *Kleinian groups*, volume 287 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1988.
[17] K. Matsuzaki and M. Taniguchi. *Hyperbolic manifolds and Kleinian groups*. Oxford Mathematical Monographs. Oxford University Press, 1998.
[18] C. McMullen. The classification of conformal dynamical systems. In *Current developments in mathematics, 1995,* 323–360. Int. Press, Cambridge, MA, 1994.
[19] G. Mostow. *Strong rigidity of locally symmetric spaces*. Annals of Mathematics Studies, No. 78. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1973.
[20] D. Sullivan. The density at infinity of a discrete group of hyperbolic motions. *Inst. Hautes Études Sci. Publ. Math.*, (50):171–202, 1979.
[21] D. Sullivan. Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains. *Ann. of Math. (2)*, 122(3):401–418, 1985.
[22] W. Thurston. *The Geometry and Topology of Three-Manifolds*, volume 27. American Mathematical Society, 2022.
[23] P. Tukia. On isomorphisms of geometrically finite Möbius groups. *Inst. Hautes Études Sci. Publ. Math.*, (61):171–214, 1985.
[24] Y. Zhang. Construction of acylindrical hyperbolic 3-manifolds with quasifuchsian boundary. *Exp. Math.*, 31(3):883–896, 2022.

Department of Mathematics, Yale University, New Haven, CT 06511, USA
Email address: dongryul.kim@yale.edu

Department of Mathematics, Yale University, New Haven, CT 06511, USA
Email address: hee.oh@yale.edu