ERGODIC THEORETIC PROOF OF EQUIDISTRIBUTION OF HECKE POINTS

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1. Introduction

Let G be a connected non-compact $\mathbb{Q}$-simple real algebraic group defined over $\mathbb{Q}$, that is, the identity component of the group of the real points of a connected $\mathbb{Q}$-simple algebraic group which is $\mathbb{R}$-isotropic. Let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup of G. As is well known, Γ has finite co-volume in G [BH]. Denote by μ_G the G-invariant Borel probability measure on $\Gamma\backslash G$. Two subgroups Γ_1 and Γ_2 of G are said to be *commensurable* with each other if $\Gamma_1 \cap \Gamma_2$ has a finite index both in Γ_1 and Γ_2. The commensurator group $\mathrm{Comm}(\Gamma)$ of Γ is defined as follows:

$$\mathrm{Comm}(\Gamma) = \{g \in G : \Gamma \text{ and } g\Gamma g^{-1} \text{ are commensurable with each other}\}.$$

Since Γ is an arithmetic subgroup, $\mathrm{Comm}(\Gamma)$ contains $G(\mathbb{Q})$ and in particular, Γ has an infinite index in $\mathrm{Comm}(\Gamma)$. Indeed, by the well known theorem of Margulis, this condition precisely characterizes arithmetic subgroups: an irreducible lattice is arithmetic if and only if Γ has an infinite index in $\mathrm{Comm}(\Gamma)$ [M, Ch IX, Thm 6.5].

For an element $a \in \mathrm{Comm}(\Gamma)$, the Γ-orbit $\Gamma\backslash\Gamma a\Gamma$ in $\Gamma\backslash G$ has finitely many points called *Hecke points* associated with a. We set

$$\deg(a) = \#\Gamma\backslash\Gamma a\Gamma.$$

It is easy to see that $\deg(a) = [\Gamma : \Gamma \cap a^{-1}\Gamma a]$.

In this paper, we are interested in the equidistribution problem of the Hecke points $\Gamma\backslash\Gamma a\Gamma$ as $\deg(a) \to \infty$. Namely, for any continuous function f in $\Gamma\backslash G$ with compact support, any $x \in \Gamma\backslash G$, and for any sequence $a_i \in \mathrm{Comm}(\Gamma)$ with $\deg(a_i) \to \infty$,

$$\text{(1.1)} \qquad \text{Does } \mathrm{T}_{a_i}(f)(x) := \frac{1}{\deg(a_i)} \sum_{\gamma \in \Gamma\backslash\Gamma a_i\Gamma} f(\gamma x) \text{ converge to } \int_{\Gamma\backslash G} f\, d\mu_G?$$

We remark that $\mathrm{T}_{a_i}(f)$ is well defined as a function on $\Gamma\backslash G$. When G is simple, $a_i \in G(\mathbb{Q})$ and Γ is a congruence subgroup, this was answered in the affirmative way in [COU], based on the adelic interpretation of Hecke operator T_{a_i} and an information on the local harmonic analysis of the p-adic groups $G(\mathbb{Q}_p)$. For smooth functions, the methods in [COU] also give rate of the convergence. The classical case of $G = \mathrm{SL}_2(\mathbb{R})$

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and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ were known for a long time and several partial results in this direction were known (see [C], [CU], [Sa], [GM], etc.) We remark that the result in [COU] was extended to reductive split groups as well in [GO].

It was pointed out by Burger and Sarnak [BS, "Theorem 5.2"] in 1991 that the equidistribution of Hecke points may follow from Ratner's measure classification theorem [Ra] provided a_i's converge to an element not belonging to $\mathrm{Comm}(\Gamma)$. However Burger and Sarnak did not give a detailed proof of this claim: this was done by Dani and Margulis [DM, Corollary 6.2] in 1993, where they deduced the assertion from their ergodic results built up on Ratner's fore-mentioned theorem.

Unlike the method used in [COU], this ergodic theoretic method does not provide a rate information of the equidistribution, which is of fundamental importance. However it works for much more general cases. Most importantly, the elements a_i's in $\mathrm{Comm}(\Gamma)$ need not be necessarily elements of $G(\mathbb{Q})$. It is not clear at all how to deal with these cases if one is using the adelic interpretation of Hecke operators as in [COU].

Our main purpose of this paper is to present an ergodic theoretic proof of the following result:

Theorem 1.2. *Let G be a connected non-compact $\mathbb{Q}$-simple real algebraic group defined over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{Q})$ an arithmetic subgroup of G. Let $\{a_i \in \mathrm{Comm}(\Gamma)\}$ be a sequence such that $\lim_{i\to\infty} \deg(a_i) = \infty$. Then for any bounded continuous function f on $\Gamma\backslash G$ and for any $x \in \Gamma\backslash G$,*

$$\lim_{i\to\infty} \mathrm{T}_{a_i}(f)(x) = \int_{\Gamma\backslash G} f(g)\, d\mu_G(g).$$

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2. Limits of H-invariant measures

Let G be a connected real semisimple algebraic group defined over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{Q})$ an arithmetic subgroup of G. Let H be a connected real non-compact semisimple $\mathbb{Q}$-simple subgroup of G. Then $\Gamma \cap H$ is an irreducible (H being $\mathbb{Q}$-simple) Zariski dense lattice in H.

Let $\{g_m \in G\}$ be a sequence such that $g_m^{-1}\Gamma g_m \cap H$ is commensurable with $\Gamma \cap H$. It follows that each $\Gamma\backslash\Gamma g_m H$ is closed [Rag] and there exists the unique H-invariant probability measure, say ν_m, in $\Gamma\backslash G$ supported on $\Gamma\backslash\Gamma g_m H$.

Let Y denote $\Gamma\backslash G$ if $\Gamma\backslash G$ is compact; and otherwise the one point compactification $\Gamma\backslash G \cup \{\infty\}$. The space $\mathcal{P}(Y)$ of the probability measures on Y equipped with the weak*-topology is weak* compact.

Our basic tools for the proof of Theorem 1.2 are the following proposition.

Proposition 2.1. *Suppose that ν_m weakly converges to a measure ν in $\mathcal{P}(\Gamma\backslash G)$ as $m \to \infty$. Then there exists a closed connected subgroup L of G containing H such that*

(1) *ν is an L-invariant measure supported on $\Gamma\backslash\Gamma c_0 L$ for some $c_0 \in G$;*

(2) $\Gamma \cap c_0 L c_0^{-1}$ *is a Zariski dense lattice in* $c_0 L c_0^{-1}$ *and hence in particular* $c_0 L c_0^{-1}$ *is defined over* $\mathbb{Q}$*;*
(3) *there exist* $m_0 \in \mathbb{N}$ *and a sequence* $\{x_m \in \Gamma g_m H\}$ *converging to* c_0 *as* $m \to \infty$ *such that* $c_0 L c_0^{-1}$ *contains the subgroup generated by* $\{x_m H x_m^{-1} : m \geq m_0\}$*.*

This proposition is deduced from the following theorem of Mozes and Shah:

Theorem 2.2 (MS, Theorem 1.1)**.** *Let* $\{u_i(t)\}_{t\in\mathbb{R}}$*,* $i \in \mathbb{N}$ *be a sequence of unipotent one-parameter subgroups of* G *and let* $\{\nu_m : m \in \mathbb{N}\}$ *be a sequence in* $\mathcal{P}(\Gamma\backslash G)$ *such that each* ν_i *is an ergodic* $\{u_i(t)\}$*-invariant measure. Suppose that* $\nu_m \to \nu$ *in* $\mathcal{P}(\Gamma\backslash G)$ *and let* $x \in \operatorname{supp}(\nu)$*. Then the following holds:*

(1) $\operatorname{supp}(\nu) = x\Lambda(\nu)$ *where* $\Lambda(\nu) = \{g \in G : \nu g = \nu\}$*.*
(2) *Let* $g_i' \to e$ *be a sequence in* G *such that for every* $i \in \mathbb{N}$*,* $xg_i' \in \operatorname{supp}(\nu_i)$ *and the trajectory* $\{xg_i' u_i(t)\}$ *is uniformly distributed with respect to* ν_i*. Then there exists an* $i_0 \in \mathbb{N}$ *such that for all* $i \geq i_0$*,*
$$\operatorname{supp}(\nu_i) \subset \operatorname{supp}(\nu) g_i'.$$
(3) ν *is invariant and ergodic for the action of the subgroup generated by the set* $\{g_i' u_i(t) g_i'^{-1} : i \geq i_0\}$*.*

Proof of Proposition 2.1 Since $g_m^{-1}\Gamma g_m \cap H$ is an irreducible lattice in H, every non-compact simple factor of H acts ergodically on each $\Gamma\backslash\Gamma g_m H$ with respect to ν_m. Denote by H_N the unique maximal connected normal subgroup of H with no compact factors. Then there exists a unipotent one-parameter subgroup $U := \{u(t)\}$ in H_N not contained in any proper closed normal subgroup of H_N (cf. Lemma 2.3 [MS]). Then by Moore's ergodicity theorem (cf. Theorem 2.1 in [BM]), U acts ergodically with respect to each ν_m. Moreover by the Birkhoff ergodic theorem, the following subset R has the zero co-measure in H:

$$\{h \in H : \Gamma\backslash\Gamma g_m h u(t) \text{ is uniformly distributed in } \Gamma\backslash\Gamma g_m H \text{ w. r. t. } \nu_m \text{ for each } m \in \mathbb{N}\}.$$

Hence for any $h \in R$ and for any continuous bounded function f on X with compact support, we have
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(g_m h u(t))\, dt = \int_X f\, d\nu_m.$$
If we set $L = \Lambda(\nu)$, we have that ν is supported on $\Gamma\backslash\Gamma c_0 L$ for some $c_0 \in \operatorname{supp}(\nu)$ by Theorem 2.2(1). There exist $\gamma_m \in \Gamma$ and $h_m \in R$ such that $\gamma_m g_m h_m \to c_0$ as $m \to \infty$. If we set
$$x_m := \gamma_m g_m h_m \quad \text{and} \quad g_m' := c_0^{-1} x_m,$$
then
$$\lim_{m\to\infty} g_m' = e$$
and $\Gamma\backslash\Gamma c_0 g_m' u(t) = \Gamma\backslash\Gamma g_m h_m u(t)$ is uniformly distributed with respect to ν_m. By Theorem 2.2(2), there exists an $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$
$$\operatorname{supp}(\nu_m) \subset \operatorname{supp}(\nu) g_m', \quad \text{or equivalently} \quad \Gamma\backslash\Gamma g_m H \subset \Gamma\backslash\Gamma c_0 L g_m'.$$

Hence $\Gamma x_m H x_m^{-1} \subset \Gamma(c_0 L c_0^{-1})$. By the connectedness of H, we may assume that $c_0 L c_0^{-1}$ is connected and

$$\{x_m H x_m^{-1} : m \geq m_0\} \subset c_0 L c_0^{-1}.$$

By Theorem 2.2(3), the subgroup generated by the set $\{x_m H x_m^{-1} : m \geq m_0\}$ acts ergodically on $\Gamma c_0 L c_0^{-1}$. Hence $c_0 L c_0^{-1}$ is the smallest closed subgroup containing the subgroup generated by the set $\{x_m H x_m^{-1} : m \geq m_0\}$ such that the orbit $\Gamma c_0 L c_0^{-1}$ is closed. This proves (3). The second claim (2) follows from [MS, Proposition 2.1].

3. Proof of Theorem 1.2

For a G-space X and a subgroup M of G, $\mathcal{P}(X)^M$ denotes the space of M-invariant Borel probability measures on X. We recall the ergodic theoretic approach suggested in [BS]. Let $\Delta(G)$ be the diagonal embedding of G into $G \times G$, that is, $\Delta(G) = \{(g, g) : g \in G\}$. For each $\nu \in \mathcal{P}(\Gamma\backslash G)^\Gamma$, the measure $\tilde{\nu}$ defined by

$$\tilde{\nu}(f) := \int_{\Gamma\backslash G} \int_{\Gamma\backslash G} f(g, hg)\, d\mu_G(g) d\nu(h)$$

for any bounded continuous function f on $\Gamma\backslash G \times \Gamma\backslash G$ is a $\Delta(G)$-invariant probability measure on $\Gamma\backslash G \times \Gamma\backslash G$. Moreover the map $\nu \mapsto \tilde{\nu}$ is a homeomorphism from $\mathcal{P}(\Gamma\backslash G)^\Gamma$ to $\mathcal{P}(\Gamma\backslash G \times \Gamma\backslash G)^{\Delta(G)}$. We make the following simple observation:

- For $a \in \mathrm{Comm}(\Gamma)$, if we set $\nu_a = \frac{1}{\deg(a)} \sum_{y \in \Gamma\backslash\Gamma a \Gamma} \delta_y$ where δ_y denotes the standard delta measure with support y, then $\tilde{\nu}_a$ is the (unique) $\Delta(G)$-invariant probability measure supported on $[(e, a)]\Delta(G) \subset (\Gamma \times \Gamma)\backslash(G \times G)$.
- The element $a \in G$ is contained in $\mathrm{Comm}(\Gamma)$ if and only if the orbit $[(e, a)]\Delta(G)$ is closed and supports a finite $\Delta(G)$-invariant measure.

Set $X = (\Gamma \times \Gamma)\backslash(G \times G)$ and let $\overline{X}$ be X if X is compact and its one point compactification $X \cup \{\infty\}$ otherwise. By the fact that $\mathcal{P}(\overline{X})$ is compact with respect to weak *-topology and the above observation, it suffices to show that, assuming the sequence $\{\tilde{\nu}_{a_i}\}$ weakly converging to $\tilde{\nu}$ in $\mathcal{P}(\overline{X})$, the limit $\tilde{\nu}$ is $G \times G$-invariant and supported on X.

First observe that the projections of $\tilde{\nu}_{a_i}$ to each factor $\Gamma\backslash G$ coincide with the G-invariant probability measure μ_G; hence the projections of the limit $\tilde{\nu}$ are μ_G as well. It follows that $\tilde{\nu}$ is supported on X.

Note that for each i, $(e, a_i)\Delta(G)(e, a_i^{-1}) \cap (\Gamma \times \Gamma)$ is commensurable with $(e, a_i)(\Delta(G) \cap (\Gamma \times \Gamma))(e, a_i^{-1})$. It follows that $(e, a_i)\Delta(G)(e, a_i^{-1})$ contains a Zariski dense subset contained in $(e, a_i)\Delta(G)(e, a_i^{-1})(\mathbb{Q})$. This implies that $(e, a_i)\Delta(G)(e, a_i^{-1})$ is a $\mathbb{Q}$-subgroup of $G \times G$ (cf. [Zi, Prop. 3.18]). It follows from the assumption G being $\mathbb{Q}$-simple that the subgroup $\Delta(G)$ as well as the subgroup $(e, a_i)\Delta(G)(e, a_i^{-1})$ are maximal closed connected subgroups of $G \times G$ defined over $\mathbb{Q}$. Therefore by applying Proposition 2.1, we obtain that $\tilde{\nu}$ is either $G \times G$-invariant or $\Delta(G)$-invariant supported on $x\Delta(G)$ for some $x \in X$. Suppose that the latter case happens. Then Proposition 2.1 also says that there exist i and $y_j \in \Gamma \times \Gamma$ such that $y_j(e, a_j)\Delta(G)(e, a_j^{-1})y_j^{-1} = (e, a_i)\Delta(G)(e, a_i^{-1})$

for all $j \geq i$. That is, $(e, a_i^{-1})y_j(e, a_j)$ belongs to the normalizer of $\Delta(G)$ in $G \times G$ for all $j \geq i$. Since $\Delta(G)$ has a finite index in its normalizer in $G \times G$, we have $(e, a_i^{-1})y_j(e, a_j) \in \Delta(G)$ and hence $[(e, a_i)]\Delta(G) = [(e, a_j)]\Delta(G)$ for infinitely many j. This implies that $\Gamma a_i = \Gamma a_j$ and hence $\deg(a_j)$ is constant for infinitely many j. This is a contradiction, since $\deg(a_i)$ tends to ∞ as $i \to \infty$. Therefore $\tilde{\nu}$ is the $G \times G$-invariant probability measure supported on X, as desired.

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