FINITENESS OF COMPACT MAXIMAL FLATS OF BOUNDED VOLUME

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Abstract. Let $M$ be a complete Riemannian locally symmetric space of non-positive curvature and of finite volume. We show that there are only finitely many compact maximal flats in $M$ of volume bounded by a given number. As a corollary in the case $M = \text{SL}_n(\mathbb{Z}) \backslash \text{SL}_n(\mathbb{R}) / \text{SO}_n$, we give a different proof of a theorem of Remak that for any $n \in \mathbb{N}$, there are only finitely many totally real number fields of degree $n$ whose regulator is less than a given number.

1. Introduction

Let $M$ be a complete Riemannian locally symmetric space of non-positive curvature and of finite volume. A flat in $M$ is a complete totally geodesic submanifold of sectional curvature 0. A maximal flat means a flat of maximal dimension, i.e., of dimension equal to the rank of $M$. On each flat of $M$, we have an induced volume form.

The main aim of this note is to prove:

**Theorem 1.1.** For any given $c > 0$, the number of compact maximal flats in $M$ of volume less than $c$ is finite.

In the case when $M$ is compact, Theorem 1.1 was proved earlier by Spatzier [Sp].

In proving the above theorem, by applying the theorem of Eberlein [Eb, Theorem 7.3.3], we may assume that $M$ has no local Euclidean de Rham factor. We may further assume without loss of generality that $M$ is irreducible, in the sense that there exists no finite covering of $M$ which is a direct product (as Riemannian manifolds) of locally symmetric spaces of positive dimension [He].

Then $M$ is of the form $\Gamma \backslash G / K$ where $G$ is a connected center free semisimple real algebraic group with no compact factors, $K$ a maximal compact subgroup of $G$ and $\Gamma$ an irreducible torsion free lattice in $G$, and the metric on $M$ is induced from a left invariant Riemannian metric on $G / K$ [He].

In the case when the real rank of $G$ is one, that is, when $M$ has negative curvature, a compact maximal flat in $M$ is just a primitive closed geodesic in $M$. In this case,

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Partially supported by DMS-0070544.
even the precise asymptotic for the number of (primitive) closed geodesics of length less than \(c\), as \(c \to \infty\), is known by Margulis [Ma1] for \(M\) compact and by Gangolli-Warner [GW] otherwise. The latter is deduced from Selberg trace formula.

Hence we may assume that the real rank of \(G\) is at least 2. By Margulis arithmeticity theorem [Ma2], \(\Gamma\) is then an arithmetic subgroup of \(G\), that is, there exists a semisimple algebraic group \(H\) defined over \(\mathbb{Q}\) and an epimorphism \(f : H(\mathbb{R})^0 \to G\) with compact kernel such that \(\Gamma\) is commensurable with \(f(H(\mathbb{Z}))\). Since any such \(\Gamma\) has a torsion free subgroup of finite index by Selberg’s lemma [Se], there exist a maximal compact subgroup \(K_0\) of \(H(\mathbb{R})^0\) and a torsion free arithmetic subgroup \(\Delta\) such that \(\Delta\backslash H(\mathbb{R})^0/K_0\) is a finite covering space of \(M\).

In what follows, a (resp. connected) real algebraic group defined over \(\mathbb{Q}\) means (resp. the identity component of) the group of real points of a connected algebraic subgroup defined over \(\mathbb{Q}\).

Summarizing above, to show Theorem 1.1, it suffices to show the following:

**Theorem 1.2.** Let \(G\) be a connected semisimple real algebraic group defined over \(\mathbb{Q}\) and \(K\) be a maximal compact subgroup of \(G\). Let \(\Gamma\) be a torsion free arithmetic subgroup of \(G\) with respect to the given \(\mathbb{Q}\)-structure of \(G\). For any given \(c > 0\), the number of compact maximal flats in \(\Gamma \backslash G/K\) of volume less than \(c\) is finite.

In Section 4, we obtain a number theoretic application of our theorem.

**Corollary 1.3.** Denote by \(\Omega_n\) the set of all orders in totally real number fields of degree \(n\). For any \(c > 0\),

\[
\#\{D \in \Omega_n : \text{Reg}(D) < c\} < \infty,
\]

where \(\text{Reg}(D)\) denotes the regulator of the order \(D\).

In particular this implies that there are only finitely many totally real number fields of degree \(n\) with regulator less than a given number. This is a special case of a theorem of Remak who proved the same statement for any number fields which are not CM. He obtained this as a consequence of a lower bound for the regulator in terms of the discriminant of the field [Re] (see also [Si]).

**Acknowledgment** I would like to thank Dima Dolgopyat and Barak Weiss for helpful discussion. I am also grateful to the referee for useful remarks.

2. **Compact maximal flats in \(\Gamma \backslash G/K\)**

Let \(G\) be a connected center free semisimple linear real algebraic group. Let \(K\) be a maximal compact subgroup in \(G\) and \(\Gamma\) a torsion free lattice in \(G\). Let

\[
G = (\exp \mathfrak{p})K
\]
be the Cartan decomposition, and consider the Riemannian symmetric space $G/K$ with a fixed left invariant Riemannian metric.

**Lemma 2.1.** There exists a maximal real split torus of $G$, whose identity component will be denoted by $S$, such that

$$C_G(S)^0 = S(C_G(S)^0 \cap K)$$

(as a direct product)

where $C_G(S)$ denotes the centralizer of $S$ in $G$.

**Proof.** Let $S$ be the identity component of any maximal real split torus of $G$. Then $C_G(S)^0$ is a reductive algebraic subgroup, and hence for some $g \in G$, $gC_G(S)^0g^{-1}$ has a Cartan decomposition compatible with the one fixed for $G$ [Mo]. On the other hand, since $S$ is a maximal real split torus, $gC_G(S)^0g^{-1}$ is a direct product of $gSg^{-1}$ with $gC_G(S)^0g^{-1} \cap K$. Hence $gSg^{-1}$ satisfies the desired properties. \[\square\]

Fix $S$ as in the above lemma and set

$$A = C_G(S)^0 \quad \text{and} \quad M = A \cap K.$$

We refer to [Mo] for some known facts about maximal flats in the discussion below. Any maximal flat in the Riemannian symmetric space $G/K$ is of the form $gSK/K$ for $g \in G$. We say that a maximal flat $F$ in $G/K$ is $\Gamma$-compact if its image $\Gamma \backslash F$ under the projection map $G/K \rightarrow \Gamma \backslash G/K$ is compact. A compact maximal flat in $\Gamma \backslash G/K$ is then of the form $\Gamma \backslash F$ where $F$ is a $\Gamma$-compact maximal flat in $G/K$.

Since $SK = AK$, we have $gSK/K = gAK/K$. Moreover by [Mo, Lemma 5.1],

$$gSK/K = hSK/K \quad \text{if and only if} \quad h^{-1}g \in N_G(S)$$

where $N_G(S)$ denotes the normalizer of $S$ in $G$.

For a subgroup $H$ of $G$, the notation $gH$ denotes the orbit of $\bar{g} \in \Gamma \backslash G$ under the right translation action of $H$ and the notation $[H]$ denotes the $\Gamma$-conjugacy class of subgroups of $G$ containing $H$.

Now set

$$X := \{ [gSg^{-1}] : gSK/K \text{ is } \Gamma\text{-compact} \}; \quad \text{and}$$

$$Y := \{ [gAg^{-1} \cap \Gamma] : \bar{g}A \text{ is compact } \}.$$

By (2.2), the set $X$ is in bijection with the space of compact maximal flats in $\Gamma \backslash G/K$.

Moreover we have:

**Theorem 2.3.** The map

$$[gSg^{-1}] \mapsto [gAg^{-1} \cap \Gamma]$$

defines a bijection from $X$ to $Y$. 
Proof. It can be easily checked that $N_G(S) \subset N_G(A)$. Hence $[gSg^{-1}] = [hSh^{-1}]$ implies $[gAg^{-1} \cap \Gamma] = [hAh^{-1} \cap \Gamma]$. Now supposing $gSK/K$ is $\Gamma$-compact, we show that $A \cap g^{-1}\Gamma g$ is co-compact in $A$, or equivalently $\bar{g}A$ is compact in $\Gamma \backslash G$. Set $\mathcal{F} = gSK/K$. By (2.2), the stabilizer of the flat $\mathcal{F}$ in $\Gamma$ is equal to $gN_G(S)g^{-1} \cap \Gamma$ and hence

$$\Gamma \backslash \mathcal{F} = (gN_G(S)g^{-1} \cap \Gamma) \backslash \mathcal{F}.$$  

Since $gAg^{-1} \cap \Gamma$ has finite index in $gN_G(S)g^{-1} \cap \Gamma$ and $\mathcal{F}$ is $\Gamma$-compact, we have $(gAg^{-1} \cap \Gamma) \backslash \mathcal{F}$ is compact as well. Since $gSg^{-1} = gAg^{-1}/gMg^{-1}$ and it acts simply transitively on $\mathcal{F}$,

$$(gAg^{-1} \cap \Gamma) \backslash \mathcal{F} = (gAg^{-1} \cap \Gamma)gAg^{-1}/gMg^{-1}.$$  

Since $M$ is compact, the canonical projection map

$$(gAg^{-1} \cap \Gamma) \backslash G \to (gAg^{-1} \cap \Gamma) \backslash G/gMg^{-1}$$

is proper, and hence it follows that $(gAg^{-1} \cap \Gamma)gAg^{-1}$ is compact, being the preimage of $(gAg^{-1} \cap \Gamma)gAg^{-1}/gMg^{-1}$. This proves the map is well defined.

To show that the map is injective, first note that if $gAg^{-1} \cap \Gamma$ is co-compact in $gAg^{-1}$, then the image of $gAg^{-1} \cap \Gamma$, under the natural projection of $gAg^{-1}$ onto $gSg^{-1}$, is co-compact in $gSg^{-1}$ as well, since $A = S/M$ with $M$ compact. Since $S$ is $\mathbb{R}$-split, it follows that the Zariski closure of $gAg^{-1} \cap \Gamma$ contains $gSg^{-1}$ for any $gSg^{-1} \in X$. Since $S$ is the unique maximal real split torus of $A$, it follows that $gAg^{-1} \cap \Gamma = hAh^{-1} \cap \Gamma$ implies $gSg^{-1} = hSh^{-1}$, proving the claim. Since $gSK/K = gAK/K$, the surjectivity is clear. \hfill \Box

Note that for $G = PSL_2(\mathbb{R})$, $K = PSO_2$ and $\Gamma = PSL_2(\mathbb{Z})$, the above theorem is precisely the well known one to one correspondence between the set of primitive closed geodesics and the conjugacy classes of primitive hyperbolic elements in $PSL_2(\mathbb{Z})$.

On the group $G$, we fix a left $G$-invariant and right $K$-invariant Riemannian metric which is compatible with the left invariant metric on $G/K$. For each closed subgroup $H$ of $G$, this induces volume forms on $H$ as well as on closed orbits $\bar{g}H$ in $\Gamma \backslash G$. If $\bar{g}H$ is compact, the volume of $\bar{g}H$ is then equal to $\text{vol}((H \cap g^{-1}\Gamma g) \backslash H)$.

**Lemma 2.4.** If $g \in G$ is such that $\mathcal{F} = gSK/K$ is $\Gamma$-compact, then

$$\text{vol}(\Gamma \backslash \mathcal{F}) \geq r \cdot \text{vol}(\bar{g}A)$$

where $r > 0$ is a constant independent of $\mathcal{F}$.

**Proof.** If $\Omega$ is a fundamental domain for $\Gamma$ in $\mathcal{F}$, then $g^{-1}\Omega$ is a fundamental domain for $g^{-1}\Gamma g$ in $g^{-1}\mathcal{F} = SK/K$. Since the metric on $G/K$ is left invariant, 

$$\text{vol}(\Gamma \backslash \mathcal{F}) = \text{vol}(N_G(S) \cap g^{-1}\Gamma g \backslash g^{-1}\mathcal{F}).$$

Since $[N_G(S) \cap \Gamma : A \cap \Gamma] \leq [N_G(S) : A]$ and

$$(A \cap g^{-1}\Gamma g) \backslash g^{-1}\mathcal{F} = (A \cap g^{-1}\Gamma g)M \backslash A,$$
we have
\[ \text{vol}(\Gamma \backslash \Gamma \mathcal{F}) \geq \frac{1}{[N_G(S) : A]} \text{vol}((A \cap g^{-1} \Gamma g)M \backslash A). \]

If \( \Omega \) is a subset of \( A \) which bijectively maps to a fundamental domain in \( A/M \) for the action of \( A \cap g^{-1} \Gamma g \), then the product \( \Omega \times M \) is a fundamental domain in \( A \) for the action of \( A \cap g^{-1} \Gamma g \), which can be shown using the torsion free property of \( \Gamma \). Therefore
\[ \text{vol}(gA) = \text{vol}((A \cap g^{-1} \Gamma g) \backslash A) = \text{vol}((A \cap g^{-1} \Gamma g)M \backslash A) \cdot \text{vol}(M). \]
Hence we deduce
\[ \text{vol}(\Gamma \backslash \Gamma \mathcal{F}) \geq \frac{\text{vol}(gA)}{\text{vol}(M) \cdot [N_G(S) : A]}. \]

\[ \square \]

3. Proof of Theorem 1.2

Let \( G \) be a connected linear semisimple real algebraic group defined over \( \mathbb{Q} \) and \( \Gamma \) be an arithmetic subgroup with respect to this \( \mathbb{Q} \)-structure. Fix a left invariant Riemannian metric on \( G \), and hence induced volume forms on closed subgroups of \( G \) as well as on closed orbits of the form \( \Gamma \backslash \Gamma H \) for \( H \) closed subgroup.

By Theorem 2.3 and Lemma 2.4, Theorem 1.2 follows from:

**Theorem 3.1.** Let \( S \) be the identity component of a maximal \( \mathbb{R} \)-split torus of \( G \). Denote by \( A \) the identity component of the centralizer of \( S \) in \( G \). For any \( c > 0 \),
\[ (3.2) \quad \#\{[gAg^{-1} \cap \Gamma] : gA \text{ is compact, } \text{vol}(gA) < c\} < \infty. \]

Moreover, if \( G \) is \( \mathbb{R} \)-split, then
\[ \#\{\bar{g}S : \bar{g}S \text{ is compact, } \text{vol}(\bar{g}S) < c\} < \infty. \]

Our main tools in proving the above theorem are the following:

**Theorem 3.3** (Dani-Margulis, [DM, Theorem 5.1]). Let \( G \) be a connected linear Lie group and \( \Gamma \) a discrete subgroup of \( G \). For any \( c > 0 \), there are only finitely many subgroups of the form \( H \cap \Gamma \) such that \( H \) is a closed subgroup of \( G \), \( \Gamma \backslash \Gamma H \) is closed and the co-volume of \( H \cap \Gamma \) in \( H \) is less than \( c \).

**Theorem 3.4** (Tomanov- Weiss, [TW, Theorem 1.2]). Let \( G \) be a real algebraic group defined over \( \mathbb{Q} \), and \( S \) a maximal \( \mathbb{R} \)-split torus of \( G \) containing a maximal \( \mathbb{Q} \)-split torus of \( G \). Let \( \Gamma \) be an arithmetic subgroup of \( G \). Then there exists a compact subset \( K \subset \Gamma \backslash G \) such that for any \( g \in G \),
\[ \bar{g}S \cap K \neq \emptyset \]
Proof of Theorem 3.1 The subgroup $A$ is a direct product of $S$ and its unique maximal compact subgroup. First note that in $G$ there exists a maximal real split torus containing a maximal $\mathbb{Q}$-split torus, e.g., take a maximal real split torus in the centralizer of a maximal $\mathbb{Q}$-split torus. Since all maximal real split tori of $G$ are conjugate with each other, we may apply Theorem 3.4 to $S$ to conclude that there is a compact subset $K_0 \subset G$ such that any $A$-orbit is of the form $\bar{g}A$ where $g \in K_0$.

Clearly the left hand side in (3.2) is bounded above by
\[
\#\{gAg^{-1} \cap \Gamma : g \in K_0, \bar{g}A \text{ is compact}, \text{vol}(\bar{g}A) < c\}.
\]

Note that for any compact $A$-orbit $\bar{g}A$, the volume of $\bar{g}A$ is given by $\text{vol}((g^{-1} \Gamma g \cap A) \setminus A)$ and
\[
\text{vol}((\Gamma \cap gAg^{-1}) \setminus gAg^{-1}) = \text{vol}((g^{-1} \Gamma g \cap A) \setminus A) \cdot \delta_g
\]
where $\delta_g$ denotes the factor which volumes of subsets get multiplied under the transformation $a \mapsto gag^{-1}$ for all $a \in A$ (here volumes are computed with respect to the induced metric on the submanifolds $A$ and $gAg^{-1}$ of $G$).

Since $\delta_g$ is a continuous function on $G$, we have
\[
d := \max_{g \in K_0} \delta_g < \infty.
\]
We now have for any $g \in K_0$ such that $\bar{g}A$ is compact,
\[
\text{vol}((\Gamma \cap gAg^{-1}) \setminus gAg^{-1}) \leq d \cdot \text{vol}(gA).
\]

On the other hand, by Theorem 3.3, there are only finitely many subgroups of the form $gAg^{-1} \cap \Gamma$ such that $\bar{g}A$ is compact and
\[
\text{vol}((\Gamma \cap gAg^{-1}) \setminus gAg^{-1}) < c \cdot d.
\]
Hence (3.5) is finite, proving the first claim. To see the second claim, if $G$ is $\mathbb{R}$-split, we have $A = S$, and $gSg^{-1} \cap \Gamma$ is Zariski dense in $gSg^{-1}$ for any compact $\bar{g}S$. Hence the map from $\{[gSg^{-1}] : \bar{g}S \text{ is compact}\}$ to $\{[gSg^{-1} \cap \Gamma] : \bar{g}S \text{ is compact}\}$ induced by $gSg^{-1} \mapsto gSg^{-1} \cap \Gamma$ is a bijection. Therefore the first part of the theorem implies that
\[
\#\{[gSg^{-1}] : \bar{g}S \text{ is compact}, \text{vol}(\bar{g}S) < c\} < \infty.
\]
On the other hand, the cardinality of the fiber of the map $\bar{g}S \mapsto gSg^{-1}$ is $[N_G(S) : S]$, which is finite, since $S = A$. Therefore the number of compact $\bar{g}S$ with volume less than $c$ is bounded above by
\[
[N_G(S) : S] \cdot \#\{[gSg^{-1}] : \bar{g}S \text{ is compact}, \text{vol}(\bar{g}S) < c\} < \infty.
\]
This proves the claim.
4. Finiteness of orders of bounded regulators

We set up some notations as well as recall some basic definitions in number theory (cf. [BS]).

**Notation:** Let $k$ be a number field of degree $n$, which is totally real, that is, any field embedding of $k$ into $\mathbb{C}$ takes values in $\mathbb{R}$.

1. We set $k^*$ to be the set of totally positive elements in $k$, i.e.,
   
   $$ k^* = \{ x \in k : \sigma(x) > 0 \text{ for any embedding } \sigma : k \to \mathbb{R} \}. $$

2. Denote by $\mathcal{O}_k$ the ring of algebraic integers in $k$. An order of $k$ is a subring of $\mathcal{O}_k$ containing 1 which has rank $n$ as a $\mathbb{Z}$-submodule. Note that $\mathcal{O}_k$ is the unique maximal order in $k$.

3. For an order $\mathcal{D}$ of $k$, the notation $\mathcal{D}^*$ denotes the group of units in $\mathcal{D}$. Then
   
   $$ \mathcal{D}^* = \{ u \in k : u\mathcal{D} = \mathcal{D} \} = \{ u \in \mathcal{D} : N_{k/\mathbb{Q}}(u) = \pm 1 \}, $$

   where $N_{k/\mathbb{Q}}(u)$ denotes the norm of $u \in k$.

4. The regulator $\text{Reg}(\mathcal{D})$ of an order $\mathcal{D}$ of $k$ is defined as the covolume in
   
   $$ \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i = 0 \} $$

   of the discrete subgroup
   
   $$ \{ (\log |\sigma_1(u)|, \ldots, \log |\sigma_n(u)|) : u \in \mathcal{D}^* \} $$

   where $\sigma_1, \ldots, \sigma_n$ are $n$ different embeddings of $k$ into $\mathbb{R}$.

   By Dirichlet unit theorem, $\text{Reg}(\mathcal{D}) < \infty$ for any order $\mathcal{D}$ of $k$.

5. The regulator of a field $k$ means the regulator of its maximal order $\mathcal{O}_k$.

We now begin the proof of Theorem 1.3. Let $S$ denote the identity component of the diagonal subgroup of $\text{SL}_n(\mathbb{R})$, i.e.,

$$ S = \{ \text{diag}(a_1, \ldots, a_n) \in \text{SL}_n(\mathbb{R}) : a_i > 0 \}. $$

Consider the space $Z \text{GL}_n(\mathbb{Z}) \backslash \text{GL}_n(\mathbb{R})$ where $Z$ denotes the center of $\text{GL}_n(\mathbb{R})$. Note that an $S$-orbit $gS$, $g \in \text{GL}_n(\mathbb{R})$, is compact in $Z \text{GL}_n(\mathbb{Z}) \backslash \text{GL}_n(\mathbb{R})$ if and only if the discrete subgroup $S \cap g^{-1} \text{SL}_n(\mathbb{Z})g$ is co-compact in $S$.

We may identify $Z \text{GL}_n(\mathbb{Z}) \backslash \text{GL}_n(\mathbb{R})$ with $\text{SL}_n(\mathbb{Z}) \backslash \text{SL}_n(\mathbb{R})$ by the canonical isomorphism. Consider a left invariant Riemannian metric on $\text{SL}_n(\mathbb{R})$. Since each compact $S$-orbit $gS$ is a submanifold in $\text{SL}_n(\mathbb{Z}) \backslash \text{SL}_n(\mathbb{R})$, we have an induced volume form such that $\text{vol}(gS)$ is given by $\text{vol}(S \cap g^{-1} \text{SL}_n(\mathbb{Z})g \backslash S)$.

Setting $\mathcal{C}$ to be the set of all compact $S$-orbits in $Z \text{GL}_n(\mathbb{Z}) \backslash \text{GL}_n(\mathbb{R})$, we first establish an injective map from $\Omega_n$ to $\mathcal{C}$ (recall that $\Omega_n$ denotes the set of all orders in totally real number fields of degree $n$). To do so, for each totally real number field $k$ of degree $n$, fix an $n$-tuple $\sigma_k := (\sigma_1, \ldots, \sigma_n)$ of different embeddings of $k$ into $\mathbb{R}$.
with \( \sigma_1 \) being the identity. For an order \( \mathcal{D} \) of \( k \), we set

\[
\phi(\mathcal{D}) = \bar{g}_\xi S
\]

where \( g_\xi := (\sigma_j(\xi_i))_{ij} \) for a \( \mathbb{Z} \)-basis \( \xi = (\xi_1, \cdots, \xi_n) \) of \( \mathcal{D} \). Clearly \( \phi(\mathcal{D}) \) does not depend on the choice of \( \mathbb{Z} \)-basis of \( \mathcal{D} \), and hence \( \phi \) is a well defined map from \( \Omega_n \) to the set of \( S \)-orbits in \( \mathbb{Z} \text{GL}_n(\mathbb{Z}) \backslash \text{GL}_n(\mathbb{R}) \).

To show that \( \phi \) is injective, suppose that \( \phi(\mathcal{D}) = \phi(\mathcal{D}') \) for some \( \mathcal{D}, \mathcal{D}' \in \Omega_n \). Then

\[
g_{\xi'} \in \mathbb{Z} \text{GL}_n(\mathbb{Z})g_\xi S
\]

where \( \xi = (\xi_1, \cdots, \xi_n) \) and \( \xi' = (\xi'_1, \cdots, \xi'_n) \) are \( \mathbb{Z} \)-bases for \( \mathcal{D} \) and \( \mathcal{D}' \) respectively.

By comparing the first columns, we obtain that

\[
\mathcal{D}' = a\mathcal{D} \quad \text{for some non-zero } a \in \mathbb{R}.
\]

Since \( 1 \in \mathcal{D} \), we have \( a \in \mathcal{D}' \). Therefore

\[
a\mathcal{D}' \subset \mathcal{D}' = a\mathcal{D}; \quad \text{hence } \mathcal{D}' \subset \mathcal{D}.
\]

On the other hand, since \( 1 \in \mathcal{D}'', a^{-1} \in \mathcal{D} \). Hence

\[
\mathcal{D} = a a^{-1} \mathcal{D} \subset a \mathcal{D} = \mathcal{D}'.
\]

Therefore \( \mathcal{D} = \mathcal{D}' \), showing that \( \phi \) is injective.

**Lemma 4.1.** We have

\[
S \cap g_\xi^{-1} \text{SL}_n(\mathbb{Z})g_\xi = \{ \text{diag}(\sigma_1(u), \cdots, \sigma_n(u)) : u \in \mathcal{D}^* \cap k^* \}
\]

where \( \xi \) is a \( \mathbb{Z} \)-basis of \( \mathcal{D} \).

**Proof.** Note that \( \text{diag}(a_1, \cdots, a_n) \in S \cap g_\xi^{-1} \text{SL}_n(\mathbb{Z})g_\xi \) if and only if

\[
Z^n g_\xi \text{ diag}(a_1, \cdots, a_n) = Z^n g_\xi
\]

where \( Z^n \) are integral row vectors. This is again same to say that for any non-zero \( x \in \mathcal{D} \), there exists \( y \in \mathcal{D} \) such that

\[
a_j \sigma_j(x) = \sigma_j(y) \quad \text{for all } 1 \leq j \leq n.
\]

Setting \( u = yx^{-1} \), we have \( u \in k \) and \( a_j = \sigma_j(u) \). Since \( a_j \sigma_j(\mathcal{D}) = \sigma_j(\mathcal{D}) \) from (4.2), \( u \mathcal{D} = \mathcal{D} \) and hence \( u \in \mathcal{D}^* \). Moreover \( \sigma_j(u) = a_j > 0 \) for each \( 1 \leq j \leq n \), and hence \( u \in k^* \). This proves \( \subset \). The other inclusion is clear as well. \( \square \)

Since \( \{ \text{diag}(\sigma_1(u), \cdots, \sigma_n(u)) : u \in \mathcal{D}^* \cap k^* \} \) has index at most \( 2^n \) in the subgroup

\[
\{ \text{diag}(\sigma_1(u), \cdots, \sigma_n(u)) : u \in \mathcal{D}^* \}
\]

by Dirichlet unit theorem and the above lemma, we have \( \phi(\Omega_n) \subset \mathcal{C} \).

Summarizing the above,
Lemma 4.3. The map $\phi : \Omega_n \to C$ is a well defined injective map and for any $D \in \Omega_n$,
\[ c \cdot \text{Reg}(D) \leq \text{vol}(\phi(D)) \leq c \cdot 2^n \cdot \text{Reg}(D) \]
where $c > 0$ is a constant depending only on the volume form on $S$.

Now Theorem 1.3 follows from Theorem 3.1.

Remark

- In fact, any compact $S$-orbit in $C$ can be constructed from a rank $n$ free $\mathbb{Z}$-submodule (not necessarily order) in a totally real number field of degree $n$. We refer to [Oh] for a precise description of the bijection between $C$ and module classes.
- To get an injective map into the maximal flats in $\text{SL}_n(\mathbb{Z}) \backslash \text{SL}_n(\mathbb{R}) / \text{SO}_n$ rather than into $C$, we need to put an equivalence relation on the set $\Omega_n$ so that $D \sim D'$ if and only if there exists an isomorphism between the fields containing $D$ and $D'$ as orders which maps $D$ to $D'$.

References


[Oh] H. Oh *Hecke orbits of compact maximal flats on $\mathbb{Z} \text{GL}_n(\mathbb{Z}) \backslash \text{GL}_n(\mathbb{R}) / \text{O}_n$*, Preprint


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