FINITENESS OF COMPACT MAXIMAL FLATS OF BOUNDED VOLUME

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Abstract. Let M be a complete Riemannian locally symmetric space of non-positive curvature and of finite volume. We show that there are only finitely many compact maximal flats in M of volume bounded by a given number. As a corollary in the case $M = \mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n$, we give a different proof of a theorem of Remak that for any $n \in \mathbb{N}$, there are only finitely many totally real number fields of degree n whose regulator is less than a given number.

1. Introduction

Let M be a complete Riemannian locally symmetric space of non-positive curvature and of finite volume. A flat in M is a complete totally geodesic submanifold of sectional curvature 0. A maximal flat means a flat of maximal dimension, i.e., of dimension equal to the rank of M. On each flat of M, we have an induced volume form.

The main aim of this note is to prove:

Theorem 1.1. *For any given $c > 0$, the number of compact maximal flats in M of volume less than c is finite.*

In the case when M is compact, Theorem 1.1 was proved earlier by Spatzier [Sp].

In proving the above theorem, by applying the theorem of Eberlein [Eb, Theorem 7.3.3], we may assume that M has no local Euclidean de Rham factor. We may further assume without loss of generality that M is irreducible, in the sense that there exists no finite covering of M which is a direct product (as Riemannian manifolds) of locally symmetric spaces of positive dimension [He].

Then M is of the form $\Gamma\backslash G/K$ where G is a connected center free semisimple real algebraic group with no compact factors, K a maximal compact subgroup of G and Γ an irreducible torsion free lattice in G, and the metric on M is induced from a left invariant Riemannian metric on G/K [He].

In the case when the real rank of G is one, that is, when M has negative curvature, a compact maximal flat in M is just a primitive closed geodesic in M. In this case,

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even the precise asymptotic for the number of (primitive) closed geodesics of length less than c, as $c \to \infty$, is known by Margulis [Ma1] for M compact and by Gangolli-Warner [GW] otherwise. The latter is deduced from Selberg trace formula.

Hence we may assume that the real rank of G is at least 2. By Margulis arithmeticity theorem [Ma2], Γ is then an arithmetic subgroup of G, that is, there exists a semisimple algebraic group H defined over $\mathbb{Q}$ and an epimorphism $f : H(\mathbb{R})^0 \to G$ with compact kernel such that Γ is commensurable with $f(H(\mathbb{Z}))$. Since any such Γ has a torsion free subgroup of finite index by Selberg's lemma [Se], there exist a maximal compact subgroup K_0 of $H(\mathbb{R})^0$ and a torsion free arithmetic subgroup Δ such that $\Delta\backslash H(\mathbb{R})^0/K_0$ is a finite covering space of M.

In what follows, a (resp. connected) real algebraic group defined over $\mathbb{Q}$ means (resp. the identity component of) the group of real points of a connected algebraic subgroup defined over $\mathbb{Q}$.

Summarizing above, to show Theorem 1.1, it suffices to show the following:

Theorem 1.2. *Let G be a connected semisimple real algebraic group defined over $\mathbb{Q}$ and K be a maximal compact subgroup of G. Let Γ be a torsion free arithmetic subgroup of G with respect to the given $\mathbb{Q}$-structure of G. For any given $c > 0$, the number of compact maximal flats in $\Gamma\backslash G/K$ of volume less than c is finite.*

In Section 4, we obtain a number theoretic application of our theorem.

Corollary 1.3. *Denote by Ω_n the set of all orders in totally real number fields of degree n. For any $c > 0$,*

$$\#\{\mathcal{D} \in \Omega_n : \mathrm{Reg}(\mathcal{D}) < c\} < \infty,$$

where $\mathrm{Reg}(\mathcal{D})$ denotes the regulator of the order $\mathcal{D}$.

In particular this implies that there are only finitely many totally real number fields of degree n with regulator less than a given number. This is a special case of a theorem of Remak who proved the same statement for any number fields which are not CM. He obtained this as a consequence of a lower bound for the regulator in terms of the discriminant of the field [Re] (see also [Si]).

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2. Compact maximal flats in $\Gamma\backslash G/K$

Let G be a connected center free semisimple linear real algebraic group. Let K be a maximal compact subgroup in G and Γ a torsion free lattice in G. Let

$$G = (\exp \mathfrak{p})K$$

be the Cartan decomposition, and consider the Riemannian symmetric space G/K with a fixed left invariant Riemannian metric.

Lemma 2.1. *There exists a maximal real split torus of G, whose identity component will be denoted by S, such that*

$$C_G(S)^0 = S(C_G(S)^0 \cap K) \qquad \textit{(as a direct product)}$$

where $C_G(S)$ denotes the centralizer of S in G.

Proof. Let S be the identity component of any maximal real split torus of G. Then $C_G(S)^0$ is a reductive algebraic subgroup, and hence for some $g \in G$, $gC_G(S)^0g^{-1}$ has a Cartan decomposition compatible with the one fixed for G [Mo]. On the other hand, since S is a maximal real split torus, $gC_G(S)^0g^{-1}$ is a direct product of gSg^{-1} with $gC_G(S)^0g^{-1} \cap K$. Hence gSg^{-1} satisfies the desired properties. □

Fix S as in the above lemma and set

$$A = C_G(S)^0 \quad \text{and} \quad M = A \cap K.$$

We refer to [Mo] for some known facts about maximal flats in the discussion below. Any maximal flat in the Riemannian symmetric space G/K is of the form gSK/K for $g \in G$. We say that a maximal flat $\mathcal{F}$ in G/K is Γ-compact if its image $\Gamma\backslash\Gamma\mathcal{F}$ under the projection map $G/K \to \Gamma\backslash G/K$ is compact. A compact maximal flat in $\Gamma\backslash G/K$ is then of the form $\Gamma\backslash\Gamma\mathcal{F}$ where $\mathcal{F}$ is a Γ-compact maximal flat in G/K.

Since $SK = AK$, we have $gSK/K = gAK/K$. Moreover by [Mo, Lemma 5.1],

$$gSK/K = hSK/K \quad \text{if and only if} \quad h^{-1}g \in N_G(S) \tag{2.2}$$

where $N_G(S)$ denotes the normalizer of S in G.

For a subgroup H of G, the notation $\bar{g}H$ denotes the orbit of $\bar{g} \in \Gamma\backslash G$ under the right translation action of H and the notation $[H]$ denotes the Γ-conjugacy class of subgroups of G containing H.

Now set

$$X := \{[gSg^{-1}] : gSK/K \text{ is } \Gamma\text{-compact}\}; \quad \text{and}$$

$$Y := \{[gAg^{-1} \cap \Gamma] : \bar{g}A \text{ is compact }\}.$$

By (2.2), the set X is in bijection with the space of compact maximal flats in $\Gamma\backslash G/K$.

Moreover we have:

Theorem 2.3. *The map*

$$[gSg^{-1}] \mapsto [gAg^{-1} \cap \Gamma]$$

defines a bijection from X to Y.

Proof. It can be easily checked that $N_G(S) \subset N_G(A)$. Hence $[gSg^{-1}] = [hSh^{-1}]$ implies $[gAg^{-1} \cap \Gamma] = [hAh^{-1} \cap \Gamma]$. Now supposing gSK/K is Γ-compact, we show that $A \cap g^{-1}\Gamma g$ is co-compact in A, or equivalently $\bar{g}A$ is compact in $\Gamma\backslash G$. Set $\mathcal{F} = gSK/K$. By (2.2), the stabilizer of the flat $\mathcal{F}$ in Γ is equal to $gN_G(S)g^{-1} \cap \Gamma$ and hence

$$\Gamma\backslash\Gamma\mathcal{F} = (gN_G(S)g^{-1} \cap \Gamma)\backslash\mathcal{F}.$$

Since $gAg^{-1} \cap \Gamma$ has finite index in $gN_G(S)g^{-1} \cap \Gamma$ and $\mathcal{F}$ is Γ-compact, we have $(gAg^{-1} \cap \Gamma)\backslash\mathcal{F}$ is compact as well. Since $gSg^{-1} = gAg^{-1}/gMg^{-1}$ and it acts simply transitively on $\mathcal{F}$,

$$(gAg^{-1} \cap \Gamma)\backslash\mathcal{F} = (gAg^{-1} \cap \Gamma)\backslash gAg^{-1}/gMg^{-1}.$$

Since M is compact, the canonical projection map

$$(gAg^{-1} \cap \Gamma)\backslash G \to (gAg^{-1} \cap \Gamma)\backslash G/gMg^{-1}$$

is proper, and hence it follows that $(gAg^{-1}\cap\Gamma)\backslash gAg^{-1}$ is compact, being the preimage of $(gAg^{-1} \cap \Gamma)\backslash gAg^{-1}/gMg^{-1}$. This proves the map is well defined.

To show that the map is injective, first note that if $gAg^{-1} \cap \Gamma$ is co-compact in gAg^{-1}, then the image of $gAg^{-1}\cap\Gamma$, under the natural projection of gAg^{-1} onto gSg^{-1}, is co-compact in gSg^{-1} as well, since $A = S/M$ with M compact. Since S is $\mathbb{R}$-split, it follows that the Zariski closure of $gAg^{-1}\cap\Gamma$ contains gSg^{-1} for any $gSg^{-1} \in X$. Since S is the unique maximal real split torus of A, it follows that $gAg^{-1} \cap \Gamma = hAh^{-1} \cap \Gamma$ implies $gSg^{-1} = hSh^{-1}$, proving the claim. Since $gSK/K = gAK/K$, the surjectivity is clear. □

Note that for $G = \mathrm{PSL}_2(\mathbb{R})$, $K = \mathrm{PSO}_2$ and $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, the above theorem is precisely the well known one to one correspondence between the set of primitive closed geodesics and the conjugacy classes of primitive hyperbolic elements in $\mathrm{PSL}_2(\mathbb{Z})$.

On the group G, we fix a left G-invariant and right K-invariant Riemannian metric which is compatible with the left invariant metric on G/K. For each closed subgroup H of G, this induces volume forms on H as well as on closed orbits $\bar{g}H$ in $\Gamma\backslash G$. If $\bar{g}H$ is compact, the volume of $\bar{g}H$ is then equal to $\mathrm{vol}((H \cap g^{-1}\Gamma g)\backslash H)$.

Lemma 2.4. *If $g \in G$ is such that $\mathcal{F} = gSK/K$ is Γ-compact, then*

$$\mathrm{vol}(\Gamma\backslash\Gamma\mathcal{F}) \geq r \cdot \mathrm{vol}(\bar{g}A)$$

where $r > 0$ is a constant independent of $\mathcal{F}$.

Proof. If Ω is a fundamental domain for Γ in $\mathcal{F}$, then $g^{-1}\Omega$ is a fundamental domain for $g^{-1}\Gamma g$ in $g^{-1}\mathcal{F} = SK/K$. Since the metric on G/K is left invariant,

$$\mathrm{vol}(\Gamma\backslash\Gamma\mathcal{F}) = \mathrm{vol}(N_G(S) \cap g^{-1}\Gamma g)\backslash g^{-1}\mathcal{F}).$$

Since $[N_G(S) \cap \Gamma : A \cap \Gamma] \leq [N_G(S) : A]$ and

$$(A \cap g^{-1}\Gamma g)\backslash g^{-1}\mathcal{F} = (A \cap g^{-1}\Gamma g)M\backslash A,$$

we have

$$\mathrm{vol}(\Gamma\backslash\Gamma\mathcal{F}) \geq \frac{1}{[N_G(S):A]}\mathrm{vol}((A\cap g^{-1}\Gamma g)M\backslash A).$$

If Ω is a subset of A which bijectively maps to a fundamental domain in A/M for the action of $A\cap g^{-1}\Gamma g$, then the product $\Omega\times M$ is a fundamental domain in A for the action of $A\cap g^{-1}\Gamma g$, which can be shown using the torsion free property of Γ. Therefore

$$\mathrm{vol}(\bar{g}A) = \mathrm{vol}((A\cap g^{-1}\Gamma g)\backslash A) = \mathrm{vol}((A\cap g^{-1}\Gamma g)M\backslash A)\cdot \mathrm{vol}(M)$$

Hence we deduce

$$\mathrm{vol}(\Gamma\backslash\Gamma\mathcal{F}) \geq \frac{\mathrm{vol}(\bar{g}A)}{\mathrm{vol}(M)\cdot[N_G(S):A]}.$$

□

3. Proof of Theorem 1.2

Let G be a connected linear semisimple real algebraic group defined over $\mathbb{Q}$ and Γ be an arithmetic subgroup with respect to this $\mathbb{Q}$-structure. Fix a left invariant Riemannian metric on G, and hence induced volume forms on closed subgroups of G as well as on closed orbits of the form $\Gamma\backslash\Gamma H$ for H closed subgroup.

By Theorem 2.3 and Lemma 2.4, Theorem 1.2 follows from:

Theorem 3.1. *Let S be the identity component of a maximal $\mathbb{R}$-split torus of G. Denote by A the identity component of the centralizer of S in G. For any $c>0$,*

$$\#\{[gAg^{-1}\cap\Gamma] : \bar{g}A \text{ is compact}, \mathrm{vol}(\bar{g}A)<c\}<\infty. \tag{3.2}$$

Moreover, if G is $\mathbb{R}$-split, then

$$\#\{\bar{g}S : \bar{g}S \text{ is compact}, \mathrm{vol}(\bar{g}S)<c\}<\infty.$$

Our main tools in proving the above theorem are the following:

Theorem 3.3 (Dani-Margulis, [DM, Theorem 5.1)**.** *Let G be a connected linear Lie group and Γ a discrete subgroup of G. For any $c>0$, there are only finitely many subgroups of the form $H\cap\Gamma$ such that H is a closed subgroup of G, $\Gamma\backslash\Gamma H$ is closed and the co-volume of $H\cap\Gamma$ in H is less than c.*

Theorem 3.4 (Tomanov- Weiss, [TW, Theorem 1.2])**.** *Let G be a real algebraic group defined over $\mathbb{Q}$, and S a maximal $\mathbb{R}$-split torus of G containing a maximal $\mathbb{Q}$-split torus of G. Let Γ be an arithmetic subgroup of G. Then there exists a compact subset $K\subset\Gamma\backslash G$ such that for any $g\in G$,*

$$\bar{g}S\cap K\neq\emptyset$$

Proof of Theorem 3.1 The subgroup A is a direct product of S and its unique maximal compact subgroup. First note that in G there exists a maximal real split torus containing a maximal $\mathbb{Q}$-split torus, e.g., take a maximal real split torus in the centralizer of a maximal $\mathbb{Q}$-split torus. Since all maximal real split tori of G are conjugate with each other, we may apply Theorem 3.4 to S to conclude that there is a compact subset $K_0 \subset G$ such that any A-orbit is of the form $\bar{g}A$ where $g \in K_0$.

Clearly the left hand side in (3.2) is bounded above by

$$\#\{gAg^{-1} \cap \Gamma : g \in K_0,\ \bar{g}A \text{ is compact, } \mathrm{vol}(\bar{g}A) < c\}. \tag{3.5}$$

Note that for any compact A-orbit $\bar{g}A$, the volume of $\bar{g}A$ is given by $\mathrm{vol}((g^{-1}\Gamma g \cap A)\backslash A)$ and

$$\mathrm{vol}((\Gamma \cap gAg^{-1})\backslash gAg^{-1}) = \mathrm{vol}((g^{-1}\Gamma g \cap A)\backslash A) \cdot \delta_g$$

where δ_g denotes the factor which volumes of subsets get multiplied under the transformation $a \to gag^{-1}$ for all $a \in A$ (here volumes are computed with respect to the induced metric on the submanifolds A and gAg^{-1} of G).

Since δ_g is a continuous function on G, we have

$$d := \max_{g \in K_0} \delta_g < \infty.$$

We now have for any $g \in K_0$ such that $\bar{g}A$ is compact,

$$\mathrm{vol}((\Gamma \cap gAg^{-1})\backslash gAg^{-1}) \le d \cdot \mathrm{vol}(\bar{g}A). \tag{3.6}$$

On the other hand, by Theorem 3.3, there are only finitely many subgroups of the form $gAg^{-1} \cap \Gamma$ such that $\bar{g}A$ is compact and

$$\mathrm{vol}((\Gamma \cap gAg^{-1})\backslash gAg^{-1}) < c \cdot d.$$

Hence (3.5) is finite, proving the first claim. To see the second claim, if G is $\mathbb{R}$-split, we have $A = S$, and $gSg^{-1} \cap \Gamma$ is Zariski dense in gSg^{-1} for any compact $\bar{g}S$. Hence the map from $\{[gSg^{-1}] : \bar{g}S \text{ is compact}\}$ to $\{[gSg^{-1} \cap \Gamma] : \bar{g}S \text{ is compact}\}$ induced by $gSg^{-1} \to gSg^{-1} \cap \Gamma$ is a bijection. Therefore the first part of the theorem implies that

$$\#\{[gSg^{-1}] : \bar{g}S \text{ is compact, } \mathrm{vol}(\bar{g}S) < c\} < \infty.$$

On the other hand, the cardinality of the fiber of the map $\bar{g}S \mapsto gSg^{-1}$ is $[N_G(S) : S]$, which is finite, since $S = A$. Therefore the number of compact $\bar{g}S$ with volume less than c is bounded above by

$$[N_G(S) : S] \cdot \#\{[gSg^{-1}] : \bar{g}S \text{ is compact, } \mathrm{vol}(\bar{g}S) < c\} < \infty.$$

This proves the claim.

4. Finiteness of orders of bounded regulators

We set up some notations as well as recall some basic definitions in number theory (cf. [BS]).

Notation: Let k be a number field of degree n, which is totally real, that is, any field embedding of k into $\mathbb{C}$ takes values in $\mathbb{R}$.

(1). We set k^* to be the set of totally positive elements in k, i.e.,

$$k^* = \{x \in k : \sigma(x) > 0 \quad \text{for any embedding } \sigma : k \to \mathbb{R}\}.$$

(2). Denote by $\mathcal{O}_k$ the ring of algebraic integers in k. An order of k is a subring of $\mathcal{O}_k$ containing 1 which has rank n as a $\mathbb{Z}$-submodule. Note that $\mathcal{O}_k$ is the unique maximal order in k.

(3). For an order $\mathcal{D}$ of k, the notation $\mathcal{D}^*$ denotes the group of units in $\mathcal{D}$. Then

$$\mathcal{D}^* = \{u \in k : u\mathcal{D} = \mathcal{D}\} = \{u \in \mathcal{D} : \mathrm{N}_{k/\mathbb{Q}}(u) = \pm 1\},$$

where $\mathrm{N}_{k/\mathbb{Q}}(u)$ denotes the norm of $u \in k$.

(4). The regulator $\mathrm{Reg}(\mathcal{D})$ of an order $\mathcal{D}$ of k is defined as the covolume in $\{(x_1, ..., x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i = 0\}$ of the discrete subgroup

$$\{(\log|\sigma_1(u)|, \cdots, \log|\sigma_n(u)|) : u \in \mathcal{D}^*\}$$

where $\sigma_1, \cdots, \sigma_n$ are n different embeddings of k into $\mathbb{R}$.

By Dirichlet unit theorem, $\mathrm{Reg}(\mathcal{D}) < \infty$ for any order $\mathcal{D}$ of k.

(5). The regulator of a field k means the regulator of its maximal order $\mathcal{O}_k$.

We now begin the proof of Theorem 1.3. Let S denote the identity component of the diagonal subgroup of $\mathrm{SL}_n(\mathbb{R})$, i.e.,

$$S = \{\mathrm{diag}(a_1, \cdots, a_n) \in \mathrm{SL}_n(\mathbb{R}) : a_i > 0\}.$$

Consider the space $Z\,\mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})$ where Z denotes the center of $\mathrm{GL}_n(\mathbb{R})$. Note that an S-orbit $\bar{g}S$, $g \in \mathrm{GL}_n(\mathbb{R})$, is compact in $Z\,\mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})$ if and only if the discrete subgroup $S \cap g^{-1}\,\mathrm{SL}_n(\mathbb{Z})g$ is co-compact in S.

We may identify $Z\,\mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})$ with $\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R})$ by the canonical isomorphism. Consider a left invariant Riemannian metric on $\mathrm{SL}_n(\mathbb{R})$. Since each compact S-orbit $\bar{g}S$ is a submanifold in $\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R})$, we have an induced volume form such that $\mathrm{vol}(\bar{g}S)$ is given by $\mathrm{vol}(S \cap g^{-1}\,\mathrm{SL}_n(\mathbb{Z})g\backslash S)$.

Setting $\mathcal{C}$ to be the set of all compact S-orbits in $Z\,\mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})$, we first establish an injective map from Ω_n to $\mathcal{C}$ (recall that Ω_n denotes the set of all orders in totally real number fields of degree n). To do so, for each totally real number field k of degree n, fix an n-tuple $\sigma_k := (\sigma_1, \cdots, \sigma_n)$ of different embeddings of k into $\mathbb{R}$

with σ_1 being the identity. For an order $\mathcal{D}$ of k, we set

$$\phi(\mathcal{D}) = \bar{g}_\xi S$$

where $g_\xi := (\sigma_j(\xi_i))_{ij}$ for a $\mathbb{Z}$-basis $\xi = (\xi_1, \cdots, \xi_n)$ of $\mathcal{D}$. Clearly $\phi(\mathcal{D})$ does not depend on the choice of $\mathbb{Z}$-basis of $\mathcal{D}$, and hence ϕ is a well defined map from Ω_n to the set of S-orbits in $Z\,\mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})$.

To show that ϕ is injective, suppose that $\phi(\mathcal{D}) = \phi(\mathcal{D}')$ for some $\mathcal{D}, \mathcal{D}' \in \Omega_n$. Then

$$g_{\xi'} \in Z\,\mathrm{GL}_n(\mathbb{Z}) g_\xi S$$

where $\xi = (\xi_1, \cdots, \xi_n)$ and $\xi' = (\xi'_1 \cdots, \xi'_n)$ are $\mathbb{Z}$-bases for $\mathcal{D}$ and $\mathcal{D}'$ respectively.

By comparing the first columns, we obtain that

$$\mathcal{D}' = a\mathcal{D} \quad \text{for some non-zero } a \in \mathbb{R}.$$

Since $1 \in \mathcal{D}$, we have $a \in \mathcal{D}'$. Therefore

$$a\mathcal{D}' \subset \mathcal{D}' = a\mathcal{D}; \quad \text{hence } \mathcal{D}' \subset \mathcal{D}.$$

On the other hand, since $1 \in \mathcal{D}'$, $a^{-1} \in \mathcal{D}$. Hence

$$\mathcal{D} = aa^{-1}\mathcal{D} \subset a\mathcal{D} = \mathcal{D}'.$$

Therefore $\mathcal{D} = \mathcal{D}'$, showing that ϕ is injective.

Lemma 4.1. *We have*

$$S \cap g_\xi^{-1} \mathrm{SL}_n(\mathbb{Z}) g_\xi = \{\mathrm{diag}(\sigma_1(u), \cdots, \sigma_n(u)) : u \in \mathcal{D}^* \cap k^*\}$$

where ξ is a $\mathbb{Z}$-basis of $\mathcal{D}$.

Proof. Note that $\mathrm{diag}(a_1, \cdots, a_n) \in S \cap g_\xi^{-1} \mathrm{SL}_n(\mathbb{Z}) g_\xi$ if and only if

$$\mathbb{Z}^n\, g_\xi\, \mathrm{diag}(a_1, \cdots, a_n) = \mathbb{Z}^n g_\xi \tag{4.2}$$

where $\mathbb{Z}^n$ are integral row vectors. This is again same to say that for any non-zero $x \in \mathcal{D}$, there exists $y \in \mathcal{D}$ such that

$$a_j \sigma_j(x) = \sigma_j(y) \quad \text{for all } 1 \le j \le n.$$

Setting $u = yx^{-1}$, we have $u \in k$ and $a_j = \sigma_j(u)$. Since $a_j\sigma_j(\mathcal{D}) = \sigma_j(\mathcal{D})$ from (4.2), $u\mathcal{D} = \mathcal{D}$ and hence $u \in \mathcal{D}^*$. Moreover $\sigma_j(u) = a_j > 0$ for each $1 \le j \le n$, and hence $u \in k^*$. This proves $\subset$. The other inclusion is clear as well. □

Since $\{\mathrm{diag}(\sigma_1(u), \cdots, \sigma_n(u)) : u \in \mathcal{D}^* \cap k^*\}$ has index at most 2^n in the subgroup

$$\{\mathrm{diag}(\sigma_1(u), \cdots, \sigma_n(u)) : u \in \mathcal{D}^*\}$$

by Dirichlet unit theorem and the above lemma, we have $\phi(\Omega_n) \subset \mathcal{C}$.

Summarizing the above,

Lemma 4.3. *The map $\phi : \Omega_n \to \mathcal{C}$ is a well defined injective map and for any $\mathcal{D} \in \Omega_n$,*

$$c \cdot \mathrm{Reg}(\mathcal{D}) \leq \mathrm{vol}(\phi(\mathcal{D})) \leq c \cdot 2^n \cdot \mathrm{Reg}(\mathcal{D})$$

where $c > 0$ is a constant depending only on the volume form on S.

Now Theorem 1.3 follows from Theorem 3.1.

Remark

- In fact, any compact S-orbit in $\mathcal{C}$ can be constructed from a rank n free $\mathbb{Z}$-submodule (not necessarily order) in a totally real number field of degree n. We refer to [Oh] for a precise description of the bijection between $\mathcal{C}$ and *module classes*.
- To get an injective map into the maximal flats in $\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n$ rather than into $\mathcal{C}$, we need to put an equivalence relation on the set Ω_n so that $\mathcal{D} \sim \mathcal{D}'$ if and only if there exists an isomorphism between the fields containing $\mathcal{D}$ and $\mathcal{D}'$ as orders which maps $\mathcal{D}$ to $\mathcal{D}'$.

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