Quantum affine Gelfand–Tsetlin bases and quantum toroidal algebra via K-theory of affine Laumon spaces

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Abstract   Laumon moduli spaces are certain smooth closures of the moduli spaces of maps from the projective line to the flag variety of GL_n. We construct the action of the quantum loop algebra $U_v(\mathbf{L}\mathfrak{sl}_n)$ in the K-theory of Laumon spaces by certain natural correspondences. Also we construct the action of the quantum toroidal algebra $\ddot{U}_v(\widehat{\mathfrak{sl}}_n)$ in the K-theory of the affine version of Laumon spaces.

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1 Introduction

This note is a sequel to [3,4]. The moduli spaces $\mathfrak{Q}_{\underline{d}}$ were introduced by Laumon [9,10]. They are certain partial compactifications of the moduli spaces of degree $\underline{d}$ based maps from $\mathbb{P}^1$ to the flag variety $\mathcal{B}_n$ of GL_n. The authors of [3,4] considered the localized equivariant cohomology $R = \bigoplus_{\underline{d}} H^\bullet_{\widetilde{T} \times \mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{H^\bullet_{\widetilde{T} \times \mathbb{C}^*}(pt)}$ $\mathrm{Frac}(H^\bullet_{\widetilde{T} \times \mathbb{C}^*}(pt))$ where $\widetilde{T}$ is a Cartan torus of GL_n acting naturally on the target $\mathcal{B}_n$, and $\mathbb{C}^*$ acts as "loop rotations" on the source $\mathbb{P}^1$. They constructed the action of the Yangian $Y(\mathfrak{sl}_n)$ on R, the new Drinfeld generators acting by natural correspondences.

In this note, we write (in style of [4]) the formulas for the action of "Drinfeld generators" of the quantum loop algebra in the localized equivariant K-theory $M = \bigoplus_{\underline{d}} K^{\widetilde{T} \times \mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{K^{\widetilde{T} \times \mathbb{C}^*}(pt)} \mathrm{Frac}(K^{\widetilde{T} \times \mathbb{C}^*}(pt))$. In fact, the correspondences defining this action are very similar to the correspondences used by Nakajima [12] to construct the action of loop algebra in the equivariant K-theory of quiver varieties.

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We prove the main theorem directly by checking all relations in the fixed points basis.

There is an affine version of the Laumon spaces, namely the moduli spaces $\mathcal{P}_{\underline{d}}$ of parabolic sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$, a certain partial compactification of the moduli spaces of degree d based maps from $\mathbb{P}^1$ to the "thick" flag variety of the loop group $\widehat{SL_n}$, see [5]. The similar correspondences give rise to the action of the quantum toroidal algebra $\ddot{U}_v(\widehat{\mathfrak{sl}}_n)$ on the sum of localized equivariant K-groups $V = \bigoplus_{\underline{d}} K^{\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*}(\mathcal{P}_{\underline{d}}) \otimes_{K^{\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*}(pt)} \mathrm{Frac}(K^{\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*}(pt))$ where the second copy of $\mathbb{C}^*$ acts by the loop rotation on the second copy of $\mathbb{P}^1$ (Theorem 4.13).

Since the fixed point basis of M corresponds to the Gelfand–Tsetlin basis of the universal Verma module over $U_v(\mathfrak{gl}_n)$ (Theorem 6.3 in [3]), we propose to call the fixed point basis of V the *affine Gelfand–Tsetlin basis*. We expect that the specialization of the affine Gelfand–Tsetlin basis gives rise to a basis in the integrable $U_v(\widehat{\mathfrak{gl}}_n)$-modules (which we also propose to call the affine Gelfand–Tsetlin basis). We expect (see 4.17) that the action of $\ddot{U}_v(\widehat{\mathfrak{sl}}_n)$ on the integrable $U_v(\widehat{\mathfrak{gl}}_n)$-modules coincides with the action of Uglov and Takemura [15]. It seems likely that these $\ddot{U}_v(\widehat{\mathfrak{sl}}_n)$-modules are obtained by the application of the *Schur* functor [7] to the irreducible $\mathcal{X}$-semisimple modules over the double affine Cherednik algebra $\ddot{H}_n(v)$ of type A_{n-1}, see [13].

2 Laumon spaces and quantum loop algebra $U_q(L\mathfrak{sl}_n)$

2.1 Laumon spaces

We recall the setup of [2–4]. Let $\mathbf{C}$ be a smooth projective curve of genus zero. We fix a coordinate z on $\mathbf{C}$, and consider the action of $\mathbb{C}^*$ on $\mathbf{C}$ such that $v(z) = v^{-2}z$. We have $\mathbf{C}^{\mathbb{C}^*} = \{0, \infty\}$.

We consider an n-dimensional vector space W with a basis $w_1, \ldots, w_n$. This defines a Cartan torus $T \subset G = GL_n \subset Aut(W)$. We also consider its 2^n-fold cover, the bigger torus $\widetilde{T}$, acting on W as follows: for $\widetilde{T} \ni \underline{t} = (t_1, \ldots, t_n)$ we have $\underline{t}(w_i) = t_i^2 w_i$. We denote by $\mathcal{B}$ the flag variety of G.

Given an $(n-1)$-tuple of nonnegative integers $\underline{d} = (d_1, \ldots, d_{n-1})$, we consider the Laumon's quasiflags' space $\mathcal{Q}_{\underline{d}}$, see [10], 4.2. It is the moduli space of flags of locally free subsheaves

$$0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_{n-1} \subset \mathcal{W} = W \otimes \mathcal{O}_{\mathbf{C}}$$

such that $\mathrm{rank}(\mathcal{W}_k) = k$, and $\deg(\mathcal{W}_k) = -d_k$.

It is known to be a smooth projective variety of dimension $2d_1 + \cdots + 2d_{n-1} + \dim \mathcal{B}$, see [9], 2.10.

$\mathfrak{Q}_{\underline{d}} \subset \mathcal{Q}_{\underline{d}}$ (quasiflags based at $\infty \in \mathbf{C}$) formed by the flags

$$0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_{n-1} \subset \mathcal{W} = W \otimes \mathcal{O}_{\mathbf{C}}$$

such that $\mathcal{W}_i \subset \mathcal{W}$ is a vector subbundle in a neighbourhood of $\infty \in \mathbf{C}$, and the fiber of $\mathcal{W}_i$ at ∞ equals the span $\langle w_1, \ldots, w_i \rangle \subset W$.

It is known to be a smooth quasiprojective variety of dimension $2d_1 + \cdots + 2d_{n-1}$.

2.2 Fixed points

The group $G \times \mathbb{C}^*$ acts naturally on $\mathcal{Q}_{\underline{d}}$, and the group $\widetilde{T} \times \mathbb{C}^*$ acts naturally on $\mathfrak{Q}_{\underline{d}}$. The set of fixed points of $\widetilde{T} \times \mathbb{C}^*$ on $\mathfrak{Q}_{\underline{d}}$ is finite; we recall its description from [6], 2.11.

Let $\underline{\widetilde{d}}$ be a collection of nonnegative integers (d_{ij}), $i \geq j$, such that $d_i = \sum_{j=1}^{i} d_{ij}$, and for $i \geq k \geq j$ we have $d_{kj} \geq d_{ij}$. Abusing notation, we denote by $\underline{\widetilde{d}}$ the corresponding $\widetilde{T} \times \mathbb{C}^*$-fixed point in $\mathfrak{Q}_{\underline{d}}$:

$$\mathcal{W}_1 = \mathcal{O}_{\mathbf{C}}(-d_{11} \cdot 0)w_1,$$
$$\mathcal{W}_2 = \mathcal{O}_{\mathbf{C}}(-d_{21} \cdot 0)w_1 \oplus \mathcal{O}_{\mathbf{C}}(-d_{22} \cdot 0)w_2,$$
$$\vdots$$
$$\mathcal{W}_{n-1} = \mathcal{O}_{\mathbf{C}}(-d_{n-1,1} \cdot 0)w_1 \oplus \mathcal{O}_{\mathbf{C}}(-d_{n-1,2} \cdot 0)w_2 \oplus \cdots \oplus \mathcal{O}_{\mathbf{C}}(-d_{n-1,n-1} \cdot 0)w_{n-1}.$$

2.3 Correspondences

For $i \in \{1, \ldots, n-1\}$, and $\underline{d} = (d_1, \ldots, d_{n-1})$, we set $\underline{d} + i := (d_1, \ldots, d_i + 1, \ldots, d_{n-1})$. We have a correspondence $\mathsf{E}_{\underline{d},i} \subset \mathcal{Q}_{\underline{d}} \times \mathcal{Q}_{\underline{d}+i}$ formed by the pairs $(\mathcal{W}_\bullet, \mathcal{W}'_\bullet)$ such that for $j \neq i$ we have $\mathcal{W}_j = \mathcal{W}'_j$, and $\mathcal{W}'_i \subset \mathcal{W}_i$, see [6], 3.1. In other words, $\mathsf{E}_{\underline{d},i}$ is the moduli space of flags of locally free sheaves

$$0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_{i-1} \subset \mathcal{W}'_i \subset \mathcal{W}_i \subset \mathcal{W}_{i+1} \subset \cdots \subset \mathcal{W}_{n-1} \subset \mathcal{W}$$

such that $\mathrm{rank}(\mathcal{W}_k) = k$, and $\deg(\mathcal{W}_k) = -d_k$, while $\mathrm{rank}(\mathcal{W}'_i) = i$, and $\deg(\mathcal{W}'_i) = -d_i - 1$.

According to [9], 2.10, $\mathsf{E}_{\underline{d},i}$ is a smooth projective algebraic variety of dimension $2d_1 + \cdots + 2d_{n-1} + \dim \mathcal{B} + 1$.

We denote by $\mathbf{p}$ (respectively, $\mathbf{q}$) the natural projection $\mathsf{E}_{\underline{d},i} \to \mathcal{Q}_{\underline{d}}$ (respectively, $\mathsf{E}_{\underline{d},i} \to \mathcal{Q}_{\underline{d}+i}$). We also have a map $\mathbf{s} : \mathsf{E}_{\underline{d},i} \to \mathbf{C}$,

$$(0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_{i-1} \subset \mathcal{W}'_i \subset \mathcal{W}_i \subset \mathcal{W}_{i+1} \subset \cdots \subset \mathcal{W}_{n-1} \subset \mathcal{W}) \mapsto \mathrm{supp}(\mathcal{W}_i/\mathcal{W}'_i).$$

The correspondence $\mathsf{E}_{\underline{d},i}$ comes equipped with a natural line bundle L_i whose fiber at a point

$$(0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_{i-1} \subset \mathcal{W}'_i \subset \mathcal{W}_i \subset \mathcal{W}_{i+1} \subset \cdots \subset \mathcal{W}_{n-1} \subset \mathcal{W})$$

equals $\Gamma(\mathbf{C}, \mathcal{W}_i/\mathcal{W}'_i)$.

Finally, we have a transposed correspondence ${}^{\mathsf{T}}\mathsf{E}_{\underline{d},i} \subset \mathcal{Q}_{\underline{d}+i} \times \mathcal{Q}_{\underline{d}}$.

Restricting to $\mathfrak{Q}_{\underline{d}} \subset \mathcal{Q}_{\underline{d}}$, we obtain the correspondence $\mathsf{E}_{\underline{d},i} \subset \mathfrak{Q}_{\underline{d}} \times \mathfrak{Q}_{\underline{d}+i}$ together with line bundle L_i and the natural maps $\mathbf{p} : \mathsf{E}_{\underline{d},i} \to \mathfrak{Q}_{\underline{d}}$, $\mathbf{q} : \mathsf{E}_{\underline{d},i} \to \mathfrak{Q}_{\underline{d}+i}$, $\mathbf{s} : \mathsf{E}_{\underline{d},i} \to \mathbf{C} - \infty$. We also have a transposed correspondence ${}^{\mathsf{T}}\mathsf{E}_{\underline{d},i} \subset \mathfrak{Q}_{\underline{d}+i} \times \mathfrak{Q}_{\underline{d}}$. It is a smooth quasiprojective variety of dimension $2d_1 + \cdots + 2d_{n-1} + 1$.

2.4 Equivariant K-groups

We denote by $'M$ the direct sum of equivariant (complexified) K-groups:

$$'M = \oplus_{\underline{d}} K^{\widetilde{T} \times \mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}).$$

It is a module over $K^{\widetilde{T} \times \mathbb{C}^*}(pt) = \mathbb{C}[T \times \mathbb{C}^*] = \mathbb{C}[x_1, \ldots, x_n, v]$. We define

$$M = {}'M \otimes_{K^{\widetilde{T} \times \mathbb{C}^*}(pt)} \operatorname{Frac}(K^{\widetilde{T} \times \mathbb{C}^*}(pt)).$$

We have an evident grading

$$M = \oplus_{\underline{d}} M_{\underline{d}}, \quad M_{\underline{d}} = K^{\widetilde{T} \times \mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{K^{\widetilde{T} \times \mathbb{C}^*}(pt)} \operatorname{Frac}(K^{\widetilde{T} \times \mathbb{C}^*}(pt)).$$

2.5 Quantum universal enveloping algebra $U_v(\mathfrak{gl}_n)$

For the quantum universal enveloping algebra $U_v(\mathfrak{gl}_n)$, we follow the notations of Sect. 2 of [11]. Namely, $U_v(\mathfrak{gl}_n)$ has generators $\mathfrak{t}_1^{\pm 1}, \ldots \mathfrak{t}_n^{\pm 1}, \mathfrak{e}_1, \ldots, \mathfrak{e}_{n-1}, \mathfrak{f}_1, \ldots, \mathfrak{f}_{n-1}$ with the following defining relations (formulas (2.1) of *loc. cit.*):

$$\mathfrak{t}_i \mathfrak{t}_j = \mathfrak{t}_j \mathfrak{t}_i, \ \mathfrak{t}_i \mathfrak{t}_i^{-1} = \mathfrak{t}_i^{-1} \mathfrak{t}_i = 1 \tag{1}$$

$$\mathfrak{t}_i \mathfrak{e}_j \mathfrak{t}_i^{-1} = \mathfrak{e}_j v^{\delta_{i,j} - \delta_{i,j+1}}, \ \mathfrak{t}_i \mathfrak{f}_j \mathfrak{t}_i^{-1} = \mathfrak{f}_j v^{-\delta_{i,j} + \delta_{i,j+1}} \tag{2}$$

$$[\mathfrak{e}_i, \mathfrak{f}_j] = \delta_{i,j} \frac{\mathfrak{k}_i - \mathfrak{k}_i^{-1}}{v - v^{-1}}, \ \mathfrak{k}_i = \mathfrak{t}_i \mathfrak{t}_{i+1}^{-1} \tag{3}$$

$$[e_i, e_j] = [f_i, f_j] = 0 \ (|i - j| > 1) \tag{4}$$

$$[\mathfrak{e}_i, [\mathfrak{e}_i, \mathfrak{e}_{i \pm 1}]_v]_v = [\mathfrak{f}_i, [\mathfrak{f}_i, \mathfrak{f}_{i \pm 1}]_v]_v = 0, \ [a, b]_v := ab - vba \tag{5}$$

The subalgebra generated by $\mathfrak{k}_i, \mathfrak{k}_i^{-1}, \mathfrak{e}_i, \mathfrak{f}_i (1 \leq i \leq n-1)$ is isomorphic to $U_v(\mathfrak{sl}_n)$. We denote by $U_v(\mathfrak{gl}_n)_{\leq 0}$ the subalgebra of $U_v(\mathfrak{gl}_n)$ generated by $\mathfrak{t}_i, \mathfrak{t}_i^{-1}, \mathfrak{f}_i$. It acts on the field $\mathbb{C}(\widetilde{T} \times \mathbb{C}^*)$ as follows: $\mathfrak{f}_i$ acts trivially for any $1 \leq i \leq n-1$, and $\mathfrak{t}_i$ acts by multiplication by $t_i v^{i-1}$. We define the *universal Verma module* $\mathfrak{M}$ over $U_v(\mathfrak{gl}_n)$ as $\mathfrak{M} := U_v(\mathfrak{gl}_n) \otimes_{U_v(\mathfrak{gl}_n)_{\leq 0}} \mathbb{C}(\widetilde{T} \times \mathbb{C}^*)$.

We define the following operators on M:

$$\mathfrak{t}_i = t_i v^{d_{i-1} - d_i + i - 1} : M_{\underline{d}} \to M_{\underline{d}} \tag{6}$$

$$\mathfrak{e}_i = t_{i+1}^{-1} v^{d_{i+1} - d_i - i + 1} \mathbf{p}_* \mathbf{q}^* : M_{\underline{d}} \to M_{\underline{d} - i} \tag{7}$$

$$\mathfrak{f}_i = -t_i^{-1} v^{d_i - d_{i-1} + i - 1} \mathbf{q}_* (L_i \otimes \mathbf{p}^*) : M_{\underline{d}} \to M_{\underline{d} + i} \tag{8}$$

The following theorem is Theorem 2.12 of [2].

Theorem 2.6 *These operators satisfy the relations in $U_v(\mathfrak{gl}_n)$, i.e. they give rise to the action of $U_v(\mathfrak{gl}_n)$ on M. Moreover, there is a unique isomorphism $\psi : M \to \mathfrak{M}$ carrying $[\mathcal{O}_{\mathfrak{Q}_0}] \in M$ to the lowest weight vector $1 \in \mathbb{C}(\widetilde{T} \times \mathbb{C}^*) \subset \mathfrak{M}$.*

Remark 2.7 These notations coincide with those from [2] (see Theorem 2.12 and Conjecture 3.7 of *loc. cit.*) after Chevalley involution.

2.8 Gelfand–Tsetlin basis of the universal Verma module

The construction of Gelfand–Tsetlin basis for the representations of quantum $\mathfrak{gl}_n$ goes back to Jimbo [8]. We will follow the approach of [11]. To a collection $\widetilde{d} = (d_{ij})$, $n - 1 \geq i \geq j$ we associate a *Gelfand–Tsetlin pattern* $\Lambda = \Lambda(\widetilde{d}) := (\lambda_{ij})$, $n \geq i \geq j$ as follows: $v^{\lambda_{nj}} := t_j v^{j-1}$, $n \geq j \geq 1$; $v^{\lambda_{ij}} := t_j v^{j-1-d_{ij}}$, $n - 1 \geq i \geq j \geq 1$. Now we define $\xi_{\widetilde{d}} = \xi_{\Lambda} \in \mathfrak{M}$ by the formula (5.12) of [11]. According to Proposition 5.1 of *loc. cit.*, the set $\{\xi_{\widetilde{d}}\}$ (over all collections $\widetilde{d}$) forms a basis of $\mathfrak{M}$.

According to the Thomason localization theorem, restriction to the $\widetilde{T} \times \mathbb{C}^*$-fixed point set induces an isomorphism

$$K^{\widetilde{T} \times \mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{K^{\widetilde{T} \times \mathbb{C}^*}(pt)} \mathrm{Frac}(K^{\widetilde{T} \times \mathbb{C}^*}(pt))$$

$$\to K^{\widetilde{T} \times \mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}^{\widetilde{T} \times \mathbb{C}^*}) \otimes_{K^{\widetilde{T} \times \mathbb{C}^*}(pt)} \mathrm{Frac}(K^{\widetilde{T} \times \mathbb{C}^*}(pt))$$

The structure sheaves $[\widetilde{d}]$ of the $\widetilde{T} \times \mathbb{C}^*$-fixed points $\widetilde{d}$ (see 2.2) form a basis in $\oplus_{\underline{d}} K^{\widetilde{T} \times \mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}^{\widetilde{T} \times \mathbb{C}^*}) \otimes_{K^{\widetilde{T} \times \mathbb{C}^*}(pt)} \mathrm{Frac}(K^{\widetilde{T} \times \mathbb{C}^*}(pt))$. The embedding of a point $\widetilde{d}$ into $\mathfrak{Q}_{\underline{d}}$ is a proper morphism, so the direct image in the equivariant K-theory is well defined, and we will denote by $\{[\widetilde{d}]\} \in M_{\underline{d}}$ the direct image of the structure sheaves of the point $\widetilde{d}$. The set $\{[\widetilde{d}]\}$ forms a basis of M.

The following theorem is Theorem 6.3 of [3] and Corollary 2.20 of [2].

Theorem 2.9 (a) *The isomorphism* $\Psi : M \xrightarrow{\sim} \mathfrak{M}$ *of Theorem* 2.6 *takes* $\{[\widetilde{d}]\}$ *to*

$$(v^2 - 1)^{-|\underline{d}|} \prod_j t_j^{\sum_{i \geq j} d_{i,j}} v^{\sum_i i d_i - \frac{|\underline{d}|}{2} - \frac{\sum_{i,j} d_{i,j}^2}{2}} \xi_{\widetilde{d}}.$$

(b) *The matrix coefficients of the operators* $\mathfrak{e}_i$, $\mathfrak{f}_i$ *in the fixed point basis* $\{[\widetilde{d}]\}$ *of* M *are as follows*:

$$\mathfrak{f}_{i[\widetilde{d},\widetilde{d}']} = -t_i^{-1} v^{d_i - d_{i-1} - 1 + i} t_j^2 v^{-2d_{i,j}}$$

$$\times (1 - v^2)^{-1} \prod_{j \neq k \leq i} (1 - t_j^2 t_k^{-2} v^{2d_{i,k} - 2d_{i,j}})^{-1} \prod_{k \leq i-1} (1 - t_j^2 t_k^{-2} v^{2d_{i-1,k} - 2d_{i,j}})$$

if $d_{i,j}' = d_{i,j} + 1$ *for certain* $j \leq i$;

$$\mathfrak{e}_{i[\underline{d},\underline{d}']} = t_{i+1}^{-1} v^{d_{i+1} - d_i + 1 - i}$$
$$\times (1 - v^2)^{-1} \prod_{j \neq k \leq i} (1 - t_k^2 t_j^{-2} v^{2d_{i,j} - 2d_{i,k}})^{-1} \prod_{k \leq i+1} (1 - t_k^2 t_j^{-2} v^{2d_{i,j} - 2d_{i+1,k}})$$

if $d'_{i,j} = d_{i,j} - 1$ for certain $j \leq i$;
  All the other matrix coefficients of $\mathfrak{e}_i$, $\mathfrak{f}_i$ vanish.

2.10 Quantum loop algebra $\mathbf{U}_v(\mathbf{L}\mathfrak{sl}_n)$

Let $(a_{kl})_{1 \leq k,l \leq n-1} = A_{n-1}$ stand for the Cartan matrix of $\mathfrak{sl}_n$. For the quantum loop algebra $\mathbf{U}_v(\mathbf{L}\mathfrak{sl}_n)$ we follow the notations of [12]. Namely, the quantum loop algebra $U_v(\mathbf{L}\mathfrak{sl}_n)$ is an associative algebra over $\mathbb{Q}(v)$ generated by $e_{k,r}$, $f_{k,r}$, $v^{\pm h_k}$, $h_{k,m}$ ($1 \leq k, l \leq n - 1, r \in \mathbb{Z}, m \in \mathbb{Z} \backslash \{0\}$) with the following defining relations:

$$\psi_k^s(z)\psi_l^{s'}(w) = \psi_l^{s'}(w)\psi_k^s(z), \tag{9}$$

$$(z - v^{\pm a_{kl}} w)\psi_l^s(z)x_k^{\pm}(w) = x_k^{\pm}(w)\psi_l^s(z)(v^{\pm a_{kl}} z - w), \tag{10}$$

$$[x_k^+(z), x_l^-(w)] = \frac{\delta_{kl}}{v - v^{-1}}\{\delta(w/z)\psi_k^+(w) - \delta(z/w)\psi_k^-(z)\}, \tag{11}$$

$$(z - v^{\pm 2} w)x_k^{\pm}(z)x_k^{\pm}(w) = x_k^{\pm}(w)x_k^{\pm}(z)(v^{\pm 2} z - w), \tag{12}$$

$$(z - v^{\pm a_{k,l}} w)x_k^{\pm}(z)x_l^{\pm}(w) = x_l^{\pm}(w)x_k^{\pm}(z)(v^{\pm a_{k,l}} z - w), k \neq l, \tag{13}$$

$$\{x_i^s(z_1)x_i^s(z_2)x_{i\pm1}^s(w) - (v + v^{-1})x_i^s(z_1)x_{i\pm1}^s(w)x_i^s(z_2)$$
$$+ x_{i\pm1}^s(w)x_i^s(z_1)x_i^s(z_2)\} + \{z_1 \longleftrightarrow z_2\} = 0, \tag{14}$$

where $s, s' = \pm$. Here, $\delta(z), x_k^{\pm}(z), \psi_k^{\pm}(z)$ are generating functions defined as following

$$\delta(z) := \sum_{r=-\infty}^{\infty} z^r, \quad x_k^+(z) := \sum_{r=-\infty}^{\infty} e_{k,r} z^{-r}, \quad x_k^-(z) := \sum_{r=-\infty}^{\infty} f_{k,r} z^{-r},$$

$$\psi_k^{\pm}(z) := v^{\pm h_k} \exp\left(\pm(v - v^{-1}) \sum_{m=1}^{\infty} h_{k,\pm m} z^{\mp m}\right).$$

2.11 Action of $\mathbf{U}_v(\mathbf{L}\mathfrak{sl}_n)$ on M

For any $0 \leq i \leq n$, we will denote by $\underline{\mathcal{W}}_i$ the tautological i-dimensional vector bundle on $\mathfrak{Q}_d \times \mathbf{C}$. By the Künneth formula we have $K^{\widetilde{T} \times \mathbb{C}^*}(\mathfrak{Q}_d \times \mathbf{C}) = K^{\widetilde{T} \times \mathbb{C}^*}(\mathfrak{Q}_d) \otimes 1 \oplus K^{\widetilde{T} \times \mathbb{C}^*}(\mathfrak{Q}_d) \otimes \tau$ where $\tau \in K^{\widetilde{T} \times \mathbb{C}^*}(\mathbf{C})$ is the class of $\mathcal{O}(1)$. Under this decomposition, for the exterior power $\Lambda^j(\underline{\mathcal{W}}_i)$ we have $\Lambda^j(\underline{\mathcal{W}}_i) =: \Lambda^j_{(j)}(\underline{\mathcal{W}}_i) \otimes 1 + \Lambda^j_{(j-1)}(\underline{\mathcal{W}}_i) \otimes \tau$ where $\Lambda^j_{(j)}(\underline{\mathcal{W}}_i), \Lambda^j_{(j-1)}(\underline{\mathcal{W}}_i) \in K^{\widetilde{T} \times \mathbb{C}^*}(\mathfrak{Q}_d)$.

For $0 \leq m \leq n$, we introduce the generating series $\mathbf{b}_m(z)$ with coefficients in the equivariant K-theory of $\mathfrak{Q}_{\underline{d}}$ as follows: $\mathbf{b}_m(z) := 1 + \sum_{1 \leq j \leq m} \left(\Lambda_{(j)}^j (\mathcal{W}_m) - v \Lambda_{(j-1)}^j (\mathcal{W}_m) \right) (-z)^{-j}$. In particular, $\mathbf{b}_0(z) := 1$.

Let v stand for the character of $\widetilde{T} \times \mathbb{C}^*$: $(\underline{t}, v) \mapsto v$. We define the line bundle $L_k' := v^k L_k$ on the correspondence $\mathsf{E}_{\underline{d},k}$, that is L_k' and L_k are isomorphic as line bundles but the equivariant structure of L_k' is obtained from the equivariant structure of L_k by the twist by the character v^k.

We also define the operators

$$e_{k,r} := t_{k+1}^{-1} v^{d_{k+1} - d_k + 1 - k} \mathbf{p}_* ((L_k')^{\otimes r} \otimes \mathbf{q}^*) : M_{\underline{d}} \to M_{\underline{d}-k} \tag{15}$$

$$f_{k,r} := -t_k^{-1} v^{d_k - d_{k-1} - 1 + k} \mathbf{q}_* (L_k \otimes (L_k')^{\otimes r} \otimes \mathbf{p}^*) : M_{\underline{d}} \to M_{\underline{d}+k} \tag{16}$$

We consider the following generating series of operators on M:

$$x_k^+(z) = \sum_{r=-\infty}^{\infty} e_{k,r} z^{-r} : M_{\underline{d}} \to M_{\underline{d}-k}[[z, z^{-1}]] \tag{17}$$

$$x_k^-(z) = \sum_{r=-\infty}^{\infty} f_{k,r} z^{-r} : M_{\underline{d}} \to M_{\underline{d}+k}[[z, z^{-1}]] \tag{18}$$

$$\psi_k^+(z) \,|_{M_{\underline{d}}} = \sum_{r=0}^{\infty} \psi_{k,r}^+ z^{-r} =:$$
$$= t_{k+1}^{-1} t_k v^{d_{k+1} - 2d_k + d_{k-1} - 1}$$
$$\times \left(\mathbf{b}_k(zv^{-k-2})^{-1} \mathbf{b}_k(zv^{-k})^{-1} \mathbf{b}_{k-1}(zv^{-k}) \mathbf{b}_{k+1}(zv^{-k-2}) \right)^+$$
$$\in M_{\underline{d}}[[z^{-1}]]; \tag{19}$$

$$\psi_k^-(z) \,|_{M_{\underline{d}}} = \sum_{r=0}^{-\infty} \psi_{k,r}^- z^{-r} :=$$
$$= t_{k+1}^{-1} t_k v^{d_{k+1} - 2d_k + d_{k-1} - 1}$$
$$\times \left(\mathbf{b}_k(zv^{-k-2})^{-1} \mathbf{b}_k(zv^{-k})^{-1} \mathbf{b}_{k-1}(zv^{-k}) \mathbf{b}_{k+1}(zv^{-k-2}) \right)^-$$
$$\in M_{\underline{d}}[[z]]; \tag{20}$$

where $()^{\pm}$ denotes the expansion at $z = \infty, 0$, respectively.

Theorem 2.12 *These generating series of operators $\psi_k^{\pm}(z), x_k^{\pm}(z)$ on M defined in 2.11 satisfy the relations in $\mathbf{U}_v(\mathbf{L}\mathfrak{sl}_n)$, i.e. they give rise to the action of $\mathbf{U}_v(\mathbf{L}\mathfrak{sl}_n)$ on M.*

Remark 2.13 For the quantum group $U_v(\mathfrak{sl}_n)$ (generated by $e_{k,0}, f_{k,0}, \psi_{k,0}^{\pm}$ in $U_v(\mathbf{L}\mathfrak{sl}_n)$) we get formulas (6–8). Formulas (17–20) are very similar to those for equivariant cohomology in [4].

Definition 2.14 We assign to a collection $[\widetilde{\underline{d}}]$ a collection of weights $s_{ij} := t_j^2 v^{-2d_{ij}}$.

Proposition 2.15 *a)* *The matrix coefficients of the operators* $f_{i,r}, e_{i,r}$ *in the fixed points basis* $\{[\widetilde{\underline{d}}]\}$ *of* M *are as follows*:

$$f_{i,r[\widetilde{\underline{d}},\widetilde{\underline{d}}']} = -t_i^{-1} v^{d_i - d_{i-1} - 1 + i} s_{i,j} (s_{i,j} v^i)^r (1 - v^2)^{-1}$$
$$\times \prod_{j \neq k \leq i} (1 - s_{i,j} s_{i,k}^{-1})^{-1} \prod_{k \leq i-1} (1 - s_{i,j} s_{i-1,k}^{-1})$$

if $d'_{i,j} = d_{i,j} + 1$ *for certain* $j \leq i$;

$$e_{i,r[\widetilde{\underline{d}},\widetilde{\underline{d}}']} = t_{i+1}^{-1} v^{d_{i+1} - d_i + 1 - i} (s_{i,j} v^{i+2})^r (1 - v^2)^{-1}$$
$$\times \prod_{j \neq k \leq i} (1 - s_{i,k} s_{i,j}^{-1})^{-1} \prod_{k \leq i+1} (1 - s_{i+1,k} s_{i,j}^{-1})$$

if $d'_{i,j} = d_{i,j} - 1$ *for certain* $j \leq i$;
All the other matrix coefficients of $e_{i,r}, f_{i,r}$ *vanish.*
b) *The eigenvalue of* $\psi_i^{\pm}(z)$ *on* $\{[\widetilde{\underline{d}}]\}$ *equals*

$$t_{i+1}^{-1} t_i v^{d_{i+1} - 2d_i + d_{i-1} - 1} \prod_{j \leq i} (1 - z^{-1} v^{i+2} s_{i,j})^{-1} (1 - z^{-1} v^i s_{i,j})^{-1}$$
$$\times \prod_{j \leq i+1} (1 - z^{-1} v^{i+2} s_{i+1,j}) \prod_{j \leq i-1} (1 - z^{-1} v^i s_{i-1,j}).$$

Proof (a)   Follows directly from Theorem 2.9b.
(b)   Follows from the exactness of functor $\Lambda_z(L) := \sum_{j=0}^{\text{rank}(L)} z^i \Lambda^i L$ on the category of coherent sheaves and the fact that $\{s_{i,j}\}_{j \leq i}$ is the set of characters of $\widetilde{T} \times \mathbb{C}^*$ in the stalk of $\underline{\mathcal{W}}_i$ at the fixed point $\{[\widetilde{\underline{d}}], 0\} \in \mathfrak{Q}_{\underline{d}} \times \mathbb{C}$. $\qquad\square$

Now we formulate a corollary which will be used in Sect. 4.
For any $0 \leq m < i \leq n$ we will denote by $\underline{\mathcal{W}}_{mi}$ the quotient $\underline{\mathcal{W}}_i / \underline{\mathcal{W}}_m$ of the tautological vector bundles on $\mathfrak{Q}_{\underline{d}} \times \mathbb{C}$. Under the Künneth decomposition, for the exterior power $\Lambda^j(\underline{\mathcal{W}}_{mi})$ we have $\Lambda^j(\underline{\mathcal{W}}_{mi}) =: \Lambda^j_{(j)}(\underline{\mathcal{W}}_{mi}) \otimes 1 + \Lambda^j_{(j-1)}(\underline{\mathcal{W}}_{mi}) \otimes \tau$ where $\Lambda^j_{(j)}(\underline{\mathcal{W}}_{mi}), \Lambda^j_{(j-1)}(\underline{\mathcal{W}}_{mi}) \in K^{\widetilde{T} \times \mathbb{C}^*}(\mathfrak{Q}_{\underline{d}})$. We introduce the generating series $\mathbf{b}_{mi}(z) := 1 + \sum_{j=1}^{\infty} \left(\Lambda^j_{(j)}(\underline{\mathcal{W}}_{mi}) - v \Lambda^j_{(j-1)}(\underline{\mathcal{W}}_{mi}) \right)(-z)^{-j}$.

Corollary 2.16 $\psi_i^{\pm}(z)|_{M_{\underline{d}}} = t_{i+1}^{-1} t_i v^{d_{i+1} - 2d_i + d_{i-1} - 1} \left(\mathbf{b}_{mi}(zv^{-i-2})^{-1} \mathbf{b}_{mi}(zv^{-i})^{-1} \right.$
$\mathbf{b}_{m,i-1}(zv^{-i}) \mathbf{b}_{m,i+1}(zv^{-i-2}) \big)^{\pm}$ *for any* $m < i$.

Proof Let $\Lambda_z(L) := \sum_{j=0}^{rank(L)} z^i \Lambda^i L$. Then on the one hand

$$\Lambda_z(\underline{\mathcal{W}}_j) = \Lambda_z(\underline{\mathcal{W}}_m) \Lambda_z(\underline{\mathcal{W}}_{mj}),$$

while on the other hand

$$\Lambda_{-1/z}(\mathcal{W}_j)\mid_{\tau:=-v}= \mathbf{b}_j(z), \quad \Lambda_{-1/z}(\mathcal{W}_{mj})\mid_{\tau:=-v}= \mathbf{b}_{mj}(z),$$

$$\Lambda_{-1/z}(\mathcal{W}_m)\mid_{\tau:=-v}= \mathbf{b}_m(z).$$

$\square$

3 Proof of Theorem 2.12

Let us check (12) first. We will prove it for x_k^- (case x_k^+ is entirely analogous).

Proof We need to check for any integers a, b: $f_{i,a+1}f_{i,b} - v^{-2}f_{i,a}f_{i,b+1} = v^{-2}f_{i,b}f_{i,a+1} - f_{i,b+1}f_{i,a}$. Let us compute both sides in the fixed points basis:

(a) $[\widetilde{\underline{d}}, \widetilde{\underline{d}}' = \widetilde{\underline{d}} + \delta_{i,j_1} + \delta_{i,j_2}]\,(j_1 \neq j_2)$.

$$\left(f_{i,a+1}f_{i,b} - v^{-2}f_{i,a}f_{i,b+1}\right)_{[\widetilde{\underline{d}},\widetilde{\underline{d}}']}$$

$$= Pv^{i(a+b+1)}\left[s_{i,j_1}^b s_{i,j_2}^{a+1}(1 - s_{i,j_1}s_{i,j_2}^{-1})^{-1}(1 - v^2 s_{i,j_2}s_{i,j_1}^{-1})^{-1}\right.$$

$$\left. -v^{-2}s_{i,j_1}^{b+1}s_{i,j_2}^a(1 - s_{i,j_1}s_{i,j_2}^{-1})^{-1}(1 - v^2 s_{i,j_2}s_{i,j_1}^{-1})^{-1} + \{j_1 \longleftrightarrow j_2\}\right]$$

$$= Pv^{i(a+b+1)}\left[(s_{i,j_1}^b s_{i,j_2}^{a+1} - v^{-2}s_{i,j_1}^{b+1}s_{i,j_2}^a)(1 - s_{i,j_1}s_{i,j_2}^{-1})^{-1}(1 - v^2 s_{i,j_2}s_{i,j_1}^{-1})^{-1}\right.$$

$$\left. + \{j_1 \longleftrightarrow j_2\}\right].$$

Similarly,

$$\left(v^{-2}f_{i,b}f_{i,a+1} - f_{i,b+1}f_{i,a}\right)_{[\widetilde{\underline{d}},\widetilde{\underline{d}}']}$$

$$= Pv^{i(a+b+1)}\left[(v^{-2}s_{i,j_1}^{a+1}s_{i,j_2}^b - s_{i,j_1}^a s_{i,j_2}^{b+1})(1 - s_{i,j_1}s_{i,j_2}^{-1})^{-1}\right.$$

$$\left. \times (1 - v^2 s_{i,j_2}s_{i,j_1}^{-1})^{-1} + \{j_1 \longleftrightarrow j_2\}\right].$$

Here,

$$P = (-t_i)^{-2}v^{2d_i - 2d_{i-1} + 2i - 1}s_{i,j_1}s_{i,j_2}(1 - v^2)^{-2}$$

$$\times \prod_{j_1, j_2 \neq k \leq i}(1 - s_{i,j_1}s_{i,k}^{-1})^{-1}(1 - s_{i,j_2}s_{i,k}^{-1})^{-1}$$

$$\times \prod_{k \leq i-1}(1 - s_{i,j_1}s_{i-1,k}^{-1})(1 - s_{i,j_2}s_{i-1,k}^{-1}).$$

So, we have to prove that

$$(s_{i,j_1}^b s_{i,j_2}^{a+1} - v^{-2} s_{i,j_1}^{b+1} s_{i,j_2}^a - v^{-2} s_{i,j_1}^{a+1} s_{i,j_2}^b + s_{i,j_1}^a s_{i,j_2}^{b+1})$$

$$\times (1 - s_{i,j_1} s_{i,j_2}^{-1})^{-1} (1 - v^2 s_{i,j_2} s_{i,j_1}^{-1})^{-1}$$

$$= (s_{i,j_1}^b s_{i,j_2}^a (s_{i,j_2} - v^{-2} s_{i,j_1}) + s_{i,j_1}^a s_{i,j_2}^b (s_{i,j_2} - v^{-2} s_{i,j_1})) s_{i,j_1} s_{i,j_2}$$

$$\times (s_{i,j_2} - s_{i,j_1})^{-1} (s_{i,j_1} - v^2 s_{i,j_2})^{-1}$$

$$= \frac{s_{i,j_1} s_{i,j_2} (s_{i,j_1}^a s_{i,j_2}^b + s_{i,j_1}^b s_{i,j_2}^a)}{v^2 (s_{i,j_1} - s_{i,j_2})}$$

is antisymmetric with respect to $\{j_1 \longleftrightarrow j_2\}$ which is obvious.

(b) $[\tilde{\underline{d}}, \tilde{\underline{d}}' = \tilde{\underline{d}} + 2\delta_{i,j_1}]$.

In this case, define

$$P' := (-t_i)^{-2} v^{2d_i - 2d_{i-1} + 2i - 3} s_{i,j_1} s_{i,j_2} \times (1 - v^2)^{-2}$$

$$\times \prod_{j_1 \neq k \leq i} (1 - s_{i,j_1} s_{i,k}^{-1})^{-1} (1 - s_{i,j_2} s_{i,k}^{-1})^{-1} \prod_{k \leq i-1} (1 - s_{i,j_1} s_{i-1,k}^{-1})(1 - s_{i,j_2} s_{i-1,k}^{-1}).$$

Then,

$$\left(f_{i,a+1} f_{i,b} - v^{-2} f_{i,a} f_{i,b+1} \right)_{[\tilde{\underline{d}}, \tilde{\underline{d}}']} = P' v^{i(a+b+1)} s_{i,j_1}^{a+b+1} (v^{-2(a+1)} - v^{-2} v^{-2a}) = 0$$

$$= \left(v^{-2} f_{i,b} f_{i,a+1} - f_{i,b+1} f_{i,a} \right)_{[\tilde{\underline{d}}, \tilde{\underline{d}}']}.$$

So the equality holds again. □

Let us check (13) now. We will prove it only for x_k^- again.

Proof If $|k - l| > 1$ than it is obvious that in the fixed points basis the formulas are the same. So let us check it for $l = i + 1, k = i$, i.e. for any integers a, b:
$f_{i,a+1} f_{i+1,b} - v f_{i,a} f_{i+1,b+1} = v f_{i+1,b} f_{i,a+1} - f_{i+1,b+1} f_{i,a}$.

Let us compute both sides in the fixed points basis:

(a) $[\tilde{\underline{d}}, \tilde{\underline{d}}' = \tilde{\underline{d}} + \delta_{i,j_1} + \delta_{i+1,j_2}] \, (j_1 \neq j_2)$.

$$(f_{i,a+1} f_{i+1,b} - v f_{i,a} f_{i+1,b+1})_{[\tilde{\underline{d}}, \tilde{\underline{d}}']} = P v \left(1 - s_{i+1,j_2} s_{i,j_1}^{-1} \right)$$

$$\times \left[v^{i(a+1)+(i+1)b} s_{i+1,j_2}^b s_{i,j_1}^{a+1} - v^{ia+(i+1)(b+1)+1} s_{i+1,j_2}^{b+1} s_{i,j_1}^a \right],$$

$$(v f_{i+1,b} f_{i,a+1} - f_{i+1,b+1} f_{i,a})_{[\tilde{\underline{d}}, \tilde{\underline{d}}']} P \left(1 - v^2 s_{i+1,j_2} s_{i,j_1}^{-1} \right)$$

$$= \left[v^{i(a+1)+(i+1)b+1} s_{i+1,j_2}^b s_{i,j_1}^{a+1} - v^{ia+(i+1)(b+1)} s_{i+1,j_2}^{b+1} s_{i,j_1}^a \right],$$

where

$$P = t_{i+1}^{-1} t_i^{-1} v^{d_{i+1}-d_{i-1}+2i-2} (1-v^2)^{-2}$$

$$\times \prod_{j_1 \neq k \leq i} (1 - s_{i,j_1} s_{i,k}^{-1})^{-1} \prod_{k \leq i-1} (1 - s_{i,j_1} s_{i-1,k}^{-1})$$

$$\times \prod_{j_2 \neq k \leq i+1} (1 - s_{i+1,j_2} s_{i+1,k}^{-1})^{-1} \prod_{j_1 \neq k \leq i} (1 - s_{i+1,j_2} s_{i,k}^{-1}).$$

After dividing both right hand sides by $P s_{i,j_1}^{a-1} s_{i+1,j_2}^{b} v^{ia+(i+1)b}$ we get an equality:

$$v(v^i s_{i,j_1} - v^{i+2} s_{i+1,j_2})(s_{i,j_1} - s_{i+1,j_2})$$
$$= (v^{i+1} s_{i,j_1} - v^{i+1} s_{i+1,j_2})(s_{i,j_1} - v^2 s_{i+1,j_2}).$$

(b) $[\widetilde{\underline{d}}, \widetilde{\underline{d}}' = \widetilde{\underline{d}} + \delta_{i,j} + \delta_{i+1,j}].$

$$(f_{i,a+1} f_{i+1,b} - v f_{i,a} f_{i+1,b+1})_{[\widetilde{\underline{d}},\widetilde{\underline{d}}']} = P' v t_j^{a+b+1} (1 - v^{2d_{i,j}-2d_{i+1,j}})$$

$$\times \left[v^{i(a+1)+(i+1)b} v^{-2bd_{i+1,j}-2(a+1)d_{i,j}} - v^{ia+(i+1)(b+1)+1} v^{-2(b+1)d_{i+1,j}-2ad_{i,j}} \right]$$

$$= P' t_j^{a+b+1} v^{i(a+1)+(i+1)b} v^{-2ad_{i,j}-2bd_{i+1,j}} (v^{-2d_{i,j}} - v^2 v^{-2d_{i+1,j}})$$
$$\times v (1 - v^{2d_{ij}-2d_{i+1,j}}).$$

Similarly, we get

$$(v f_{i+1,b} f_{i,a+1} - f_{i+1,b+1} f_{i,a})_{[\widetilde{\underline{d}},\widetilde{\underline{d}}']}$$
$$= P' t_j^{a+b+1} v^{i(a+1)+(i+1)b} v^{-2ad_{i,j}-2bd_{i+1,j}} (vv^{-2d_{i,j}} - vv^{-2d_{i+1,j}})(1-v^{2d_{ij}-2d_{i+1,j}+2}),$$

where P' is a specialization of P under $j_1 = j_2 = j$.
Again, we get an equality.　　　　　　　　　　　　　　　　　□

Let us check (11) for the case $k \neq l$.

Proof We have to show for any integers a, b the equality $e_{k,a} f_{l,b} = f_{l,b} e_{k,a}$ holds.
　　If $|k - l| > 1$ then this equation is obvious, since matrix elements in the fixed points basis are the same. So let us check the only nontrivial case: $k = i, l = i + 1$ (the pair $k = i + 1, l = i$ is analogous). We consider the pair of fixed points $[\widetilde{\underline{d}}, \widetilde{\underline{d}}' = \widetilde{\underline{d}} + \delta_{i,j_1} + \delta_{i+1,j_2}]$ (here j_1, j_2 might be equal).

$$e_{i,a} f_{i+1,b} \mid_{[\widetilde{\underline{d}},\widetilde{\underline{d}'}]} = P(1 - s_{i+1,j_2} s_{i,j_1}^{-1})(1 - v^{-2} s_{i+1,j_2} s_{i,j_1}^{-1}),$$

$$f_{i+1,b} e_{i,a} \mid_{[\widetilde{\underline{d}},\widetilde{\underline{d}'}]} = P(1 - v^{-2} s_{i+1,j_2} s_{i,j_1}^{-1})(1 - s_{i+1,j_2} s_{i,j_1}^{-1}),$$

where

$$P = -t_{i+1}^{-1} t_i^{-1} v^{2d_{i+1} - 2d_i} (1 - v^2)^{-1}$$

$$\times \prod_{j_2 \neq k \leq i+1} (1 - s_{i+1,j_2} s_{i+1,k}^{-1})^{-1} \prod_{j_1 \neq k \leq i} (1 - s_{i+1,j_2} s_{i,k}^{-1})$$

$$\times \prod_{j_1 \neq k \leq i} (1 - s_{i,j_1}^{-1} s_{i,k})^{-1} \prod_{j_2 \neq k \leq i+1} (1 - s_{i,j_1}^{-1} s_{i+1,k}).$$

This completes the proof of (11).   □

Let us check (14) fourthly. We will prove it only for x_k^- again.

Proof We have to prove for any integers a, b, c and $j = i \pm 1$ the equality holds:

$$\{f_{i,a} f_{i,b} f_{j,c} - (v + v^{-1}) f_{i,a} f_{j,c} f_{i,b} + f_{j,c} f_{i,a} f_{i,b}\} + \{a \longleftrightarrow b\} = 0.$$

Let us consider the case $j = i + 1$ (the second case is similar).

We will show that matrix elements in the fixed points basis of the first bracket is antisymmetric with respect to a change $\{a \longleftrightarrow b\}$.

(a) $[\widetilde{\underline{d}}, \widetilde{\underline{d}'} = \widetilde{\underline{d}} + \delta_{i,j_1} + \delta_{i,j_2} + \delta_{i+1,j_3}]$ $(j_1 \neq j_2)$.

$$f_{i,a} f_{i,b} f_{i+1,c} \mid_{[\widetilde{\underline{d}},\widetilde{\underline{d}'}]}$$

$$= P v^2 \left[s_{i,j_1}^a s_{i,j_2}^b (1 - s_{i+1,j_3} s_{i,j_2}^{-1})(1 - s_{i+1,j_3} s_{i,j_1}^{-1}) \right.$$

$$\left. \times (1 - s_{i,j_2} s_{i,j_1}^{-1})^{-1} (1 - v^2 s_{i,j_1} s_{i,j_2}^{-1})^{-1} + \{j_1 \longleftrightarrow j_2\} \right],$$

$$f_{i,a} f_{i+1,c} f_{i,b} \mid_{[\widetilde{\underline{d}},\widetilde{\underline{d}'}]}$$

$$= P v \left[s_{i,j_1}^a s_{i,j_2}^b (1 - v^2 s_{i+1,j_3} s_{i,j_2}^{-1})(1 - s_{i+1,j_3} s_{i,j_1}^{-1}) \right.$$

$$\left. \times (1 - s_{i,j_2} s_{i,j_1}^{-1})^{-1} (1 - v^2 s_{i,j_1} s_{i,j_2}^{-1})^{-1} + \{j_1 \longleftrightarrow j_2\} \right],$$

$$f_{i+1,c} f_{i,a} f_{i,b} \mid_{[\widetilde{\underline{d}},\widetilde{\underline{d}'}]}$$

$$= P \left[s_{i,j_1}^a s_{i,j_2}^b (1 - v^2 s_{i+1,j_3} s_{i,j_2}^{-1})(1 - v^2 s_{i+1,j_3} s_{i,j_1}^{-1}) \right.$$

$$\left. \times (1 - s_{i,j_2} s_{i,j_1}^{-1})^{-1} (1 - v^2 s_{i,j_1} s_{i,j_2}^{-1})^{-1} + \{j_1 \longleftrightarrow j_2\} \right].$$

Thus,

$$(f_{i,a} f_{i,b} f_{i+1,c} - (v + v^{-1}) f_{i,a} f_{i+1,c} f_{i,b} + f_{i+1,c} f_{i,a} f_{i,b})_{[\underline{\tilde{d}}, \underline{\tilde{d}}']}$$

$$= P s_{i,j_1}^a s_{i,j_2}^b \left(\frac{v^2 (s_{i,j_2} - s_{i+1,j_3})(s_{i,j_1} - s_{i+1,j_3})}{(s_{i,j_1} - s_{i,j_2})(s_{i,j_2} - v^2 s_{i,j_1})} \right.$$

$$- \frac{(1 + v^2)(s_{i,j_2} - v^2 s_{i+1,j_3})(s_{i,j_1} - s_{i+1,j_3})}{(s_{i,j_1} - s_{i,j_2})(s_{i,j_2} - v^2 s_{i,j_1})}$$

$$\left. + \frac{(s_{i,j_2} - v^2 s_{i+1,j_3})(s_{i,j_1} - v^2 s_{i+1,j_3})}{(s_{i,j_1} - s_{i,j_2})(s_{i,j_2} - v^2 s_{i,j_1})} \right)$$

$$+ \{j_1 \longleftrightarrow j_2\} = P(1 - v^2) \frac{s_{i+1,j_3} s_{i,j_1}^a s_{i,j_2}^b}{s_{i,j_1} - s_{i,j_2}} + \{j_1 \longleftrightarrow j_2\}$$

$$= P s_{i+1,j_3} (1 - v^2) \frac{s_{i,j_1}^a s_{i,j_2}^b - s_{i,j_1}^b s_{i,j_2}^a}{s_{i,j_1} - s_{i,j_2}},$$

where

$$P = - t_{i+1}^{-1} t_i^{-2} v^{d_{i+1} + d_i - 2d_{i-1} + 3i - 3} s_{i,j_1} s_{i,j_2} s_{i+1,j_3}^{c+1} v^{c(i+1) + i(a+b)}$$

$$\times \prod_{k \leq i-1} \left((1 - s_{i,j_2} s_{i-1,k}^{-1})(1 - s_{i,j_1} s_{i-1,k}^{-1}) \right)$$

$$\times \prod_{j_3 \neq k \leq i+1} (1 - s_{i+1,j_3} s_{i+1,k}^{-1})^{-1} \prod_{j_1,j_2 \neq k \leq i} (1 - s_{i+1,j_3} s_{i,k}^{-1})$$

$$\times \prod_{j_1,j_2 \neq k \leq i} (1 - s_{i,j_2} s_{i,k}^{-1})^{-1} \prod_{j_1,j_2 \neq k \leq i} (1 - s_{i,j_1} s_{i,k}^{-1})^{-1}.$$

We see that

$$f_{i,a} f_{i,b} f_{i+1,c} - (v + v^{-1}) f_{i,a} f_{i+1,c} f_{i,b} + f_{i+1,c} f_{i,a} f_{i,b} \big|_{[\underline{\tilde{d}}, \underline{\tilde{d}}']}$$

$$= P s_{i+1,j_3} (1 - v^2) \frac{s_{i,j_1}^a s_{i,j_2}^b - s_{i,j_1}^b s_{i,j_2}^a}{s_{i,j_1} - s_{i,j_2}}$$

is antisymmetric with respect to $a \longleftrightarrow b$.

(b) $[\underline{\tilde{d}}, \underline{\tilde{d}}' = \underline{\tilde{d}} + 2\delta_{i,j_1} + \delta_{i+1,j_3}]$.

By the same calculation one gets:

$$(f_{i,a} f_{i,b} f_{i+1,c} - (v + v^{-1}) f_{i,a} f_{i+1,c} f_{i,b} + f_{i+1,c} f_{i,a} f_{i,b})_{[\underline{\tilde{d}}, \underline{\tilde{d}}']}$$

$$= P' \left[v^2 (1 - s_{i+1,j_3} s_{i,j_1}^{-1}) - (1 + v^2)(1 - v^2 s_{i+1,j_3} s_{i,j_1}^{-1}) + (1 - v^4 s_{i+1,j_3} s_{i,j_1}^{-1}) \right] = 0,$$

where

$$P' = -t_{i+1}^{-1} t_i^{-2} v^{d_{i+1}+d_i-2d_{i-1}+3i-3} s_{i,j_1}^{a+b+2} s_{i+1,j_3}^{c+1} v^{c(i+1)+i(a+b)}$$

$$\times \prod_{k \leq i-1} \left((1 - s_{i,j_2} s_{i-1,k}^{-1})(1 - s_{i,j_1} s_{i-1,k}^{-1}) \right)$$

$$\times \prod_{j_3 \neq k \leq i+1} (1 - s_{i+1,j_3} s_{i+1,k}^{-1})^{-1}$$

$$\times \prod_{j_1 \neq k \leq i} \left((1 - s_{i+1,j_3} s_{i,k}^{-1})(1 - s_{i,j_2} s_{i,k}^{-1})^{-1}(1 - s_{i,j_1} s_{i,k}^{-1}) \right)^{-1}.$$

This completes the proof of (14). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now, we will introduce the operators $\varphi_k^+(z)_{|M_{\underline{d}}} = \sum_{r=0}^{\infty} \varphi_{k,r|M_{\underline{d}}}^+ z^{-r}$, $\varphi_k^-(z)_{|M_{\underline{d}}} = \sum_{r=0}^{-\infty} \varphi_{k,r|M_{\underline{d}}}^- z^{-r}$ (here $\varphi_{k,r|M_{\underline{d}}}^{\pm} \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, v^{\pm 1}]$) diagonalizable in the fixed points basis and satisfying the equation

$$[x_k^+(z), x_k^-(w)] = \frac{1}{v - v^{-1}} \{ \delta(w/z)\varphi_k^+(w) - \delta(z/w)\varphi_k^-(z) \} \tag{21}$$

We will show that $\varphi_k^{\pm}(z)$ are determined uniquely by the conditions $\varphi_{i,0|M_{\underline{d}}}^+ = t_i t_{i+1}^{-1} v^{d_{i+1}-2d_i+d_{i-1}-1}$, $\varphi_{i,0|M_{\underline{d}}}^- = t_i^{-1} t_{i+1} v^{-d_{i+1}+2d_i-d_{i-1}+1}$ (we get these formulas from the fact that the subalgebra generated by $\psi_{k,0}^{\pm}$, E_k, F_k is a quantum group $U_v(\mathfrak{sl}_n)$, for which the formulas are already known). Let us further omit $|_{M_{\underline{d}}}$ for brevity. Next we will check

$$\varphi_k^s(z)\varphi_l^{s'}(w) = \varphi_l^{s'}(w)\varphi_k^s(z), \tag{22}$$

$$(z - v^{\pm a_{kl}} w)\varphi_l^s(z)x_k^{\pm}(w) = x_k^{\pm}(w)\varphi_l^s(z)(v^{\pm a_{kl}} z - w). \tag{23}$$

Finally, by showing that $\varphi_k^{\pm}(z) = \psi_k^{\pm}(z)$ we will get (9–11) from (21–23). And so Theorem 2.12 will be proved.

From Proposition 2.15 one gets that $(v - v^{-1})[x_i^+(z), x_i^-(w)]$ is diagonalizable in the fixed points basis and moreover its eigenvalue at $\{\widetilde{\underline{d}}\}$ equals to

$$\sum_{a,b \in \mathbb{Z}} z^{-a} w^{-b} \chi_{i,a+b},$$

where

$$\chi_{i,a} = -t_{i+1}^{-1} t_i^{-1} v^{-1} (v^2 - 1)^{-1} v^{d_{i+1}-d_{i-1}}$$

$$\times \sum_{j \leq i} s_{ij} \left(\prod_{j \neq k \leq i} (1 - s_{i,j} s_{i,k}^{-1})^{-1} \prod_{j \neq k \leq i} (1 - v^2 s_{i,k} s_{i,j}^{-1})^{-1} \right.$$

$$\times \prod_{k \le i-1} (1 - s_{i,j}s_{i-1,k}^{-1}) \prod_{k \le i+1} (1 - v^2 s_{i+1,k}s_{i,j}^{-1})(s_{i,j}v^i)^a$$

$$- v^2 \prod_{j \ne k \le i} (1 - s_{i,j}^{-1}s_{i,k})^{-1} \prod_{j \ne k \le i} (1 - v^2 s_{i,k}^{-1}s_{i,j})^{-1}$$

$$\times \left. \prod_{k \le i-1} (1 - v^2 s_{i,j}s_{i-1,k}^{-1}) \prod_{k \le i+1} (1 - s_{i+1,k}s_{i,j}^{-1})(s_{i,j}v^{i+2})^a \right).$$

So as we want an equality $(v - v^{-1})[x_i^+(z), x_i^-(w)] = \delta\left(\frac{z}{w}\right)\varphi_i^+(w) - \delta\left(\frac{w}{z}\right)\varphi_i^-(z) =$

$$\sum_{a,b|a+b>0} z^{-a}w^{-b}\varphi_{i,a+b}^+ - \sum_{a,b|a+b<0} z^{-a}w^{-b}\varphi_{i,a+b}^- + \sum_{a,b|a+b=0} z^{-a}w^{-b}(\varphi_{i,0}^+ - \varphi_{i,0}^-)$$

to hold, we determine $\varphi_{i,s>0}^+$, $\varphi_{i,s<0}^-$, $\varphi_{i,s=0}^+ - \varphi_{i,s=0}^-$ uniquely as they are equal to the corresponding $\chi_{i,s}$. But we know from [2] that for $\varphi_{i,0}^+ = t_i t_{i+1}^{-1} v^{d_{i+1} - 2d_i + d_{i-1} - 1}$, $\varphi_{i,0}^- = t_i^{-1} t_{i+1} v^{-d_{i+1} + 2d_i - d_{i-1} + 1}$ the above equality for $\varphi_{i,0}^+ - \varphi_{i,0}^-$ holds. So we have determined all coefficients of the series $\varphi_i^\pm(z)$.

Let us note that as all operators $\varphi_{i,s}^\pm$ are diagonalizable in the fixed points basis the (22) holds automatically. So let us check the (23), i.e.

$$(z - v^{s'a_{kl}}w)\varphi_l^s(z)x_k^{s'}(w) = x_k^{s'}(w)\varphi_l^s(z)(v^{s'a_{kl}}z - w).$$

Proof We claim that for $k \ne l$ it follows directly from the (13) and the construction of $\varphi_i^\pm(z)$. So let us now check it for the case $k = l, s = +, s' = -$.

Now, we are computing the matrix elements of both sides in the fixed points basis on the pair $[\widetilde{d}, \widetilde{d}' = \widetilde{d} + \delta_{i,p \le i}]$.

Let us notice that $f_{i,b+1} |_{[\widetilde{d},\widetilde{d}']} = f_{i,b} |_{[\widetilde{d},\widetilde{d}']} \cdot s_{i,p} v^i$. And as $\varphi_{i,s \ge 0}^+$ are diagonalizable in the fixed points basis the only thing we have to check is the following equality for any $a \ge 0$:

$$(\varphi_{i,a+1}^+ - v^{-2}s_{i,p}v^i\varphi_{i,a}^+) |_{\widetilde{d}+\delta_{i,p}} = (v^{-2}\varphi_{i,a+1}^+ - s_{i,p}v^i\varphi_{i,a}^+) |_{\widetilde{d}}.$$

(a) $a > 0$. Here we use the notations of Proposition 2.21 ([2]).

Namely $q := v^2$, $s_j := s_{ij} = t_j^2 v^{-2d_{i,j}}$, $r_k := t_k^2 v^{-2d_{i-1,k}}$, $p_k := t_k^2 v^{-2d_{i+1,k}}$. Then

$$\varphi_{i,a}^+ |_{\widetilde{d}} = v^{ia} \prod_{j \le i} s_j \prod_{k \le i-1} p_k^{-1} \left(\sum_{j \le i} s_j^{-2} \prod_{k \le i+1} (s_j - qr_k) \prod_{k \le i-1} (p_k - s_j)\right.$$

$$\times \left. \prod_{j \ne k \le i} \left((s_j - qs_k)^{-1}(s_k - s_j)^{-1}\right) s_j^a\right)$$

$$- q \sum_{j \le i} s_j^{-2} \prod_{k \le i+1} (s_j - r_k) \prod_{k \le i-1} (p_k - q s_j)$$

$$\times \prod_{j \ne k \le i} \left((s_j - s_k)^{-1} (s_k - q s_j)^{-1} \right) (q s_j)^a \Bigg).$$

$$\varphi_{i,a}^+ \mid_{\underline{\widetilde{d}} + \delta_{i,p}} = v^{ia} \prod_{j \le i} s_j \prod_{k \le i-1} p_k^{-1} q^{-1} \left(\sum_{p \ne j \le i} s_j^{-2} \prod_{k \le i+1} (s_j - q r_k) \prod_{k \le i-1} (p_k - s_j) \right.$$

$$\times \prod_{j, p \ne k \le i} \left((s_j - q s_k)^{-1} (s_k - s_j)^{-1} \right) (s_j - s_p)^{-1} (q^{-1} s_p - s_j)^{-1} s_j^a$$

$$- q \sum_{p \ne j \le i} s_j^{-2} \prod_{k \le i+1} (s_j - r_k) \prod_{k \le i-1} (p_k - q s_j)$$

$$\times \prod_{j, p \ne k \le i} \left((s_j - s_k)^{-1} (s_k - q s_j)^{-1} \right) (s_j - q^{-1} s_p)^{-1}$$

$$\times (q^{-1} s_p - q s_j)^{-1} (q s_j)^a + s_p^{-2} q^2 \prod_{k \le i+1} (q^{-1} s_p - q r_k)$$

$$\times \prod_{k \le i-1} (p_k - q^{-1} s_p) \prod_{p \ne k \le i} \left((q^{-1} s_p - q s_k)^{-1} (s_k - q^{-1} s_p) \right) (q^{-1} s_p)^a$$

$$- q s_p^{-2} q^2 \prod_{k \le i+1} (q^{-1} s_p - r_k) \prod_{k \le i-1} (p_k - s_p)$$

$$\times \prod_{p \ne k \le i} \left((q^{-1} s_p - s_k)^{-1} (s_k - s_p)^{-1} \right) s_p^a \Bigg).$$

Hence,

$$(v^{-2} \varphi_{i,a+1}^+ - s_p v^i \varphi_{i,a}^+) \mid_{\underline{\widetilde{d}}} = v^{i(a+1)} \prod_{j \le i} s_j \prod_{k \le i-1} p_k^{-1} \left(\sum_{p \ne j \le i} s_j^{-2} \right.$$

$$\times \prod_{k \le i+1} (s_j - q r_k) \prod_{k \le i-1} (p_k - s_j)$$

$$\times \prod_{j \ne k \le i} \left((s_j - q s_k)^{-1} (s_k - s_j)^{-1} \right) (q^{-1} s_j - s_p) s_j^a$$

$$- q \sum_{p \ne j \le i} s_j^{-2} \prod_{k \le i+1} (s_j - r_k) \prod_{k \le i-1} (p_k - q s_j)$$

$$\times \prod_{j \ne k \le i} \left((s_j - s_k)^{-1} (s_k - q s_j)^{-1} \right) (s_j - s_p) (q s_j)^a$$

$$+ s_p^{-2} \prod_{k \leq i+1} (s_p - q r_k) \prod_{k \leq i-1} (p_k - s_p)$$

$$\times \prod_{p \neq k \leq i} \left((s_p - q s_k)^{-1} (s_k - s_p)^{-1} \right) (q^{-1} - 1) s_p^{a+1} \Bigg).$$

$$(\varphi_{i,a+1}^+ - v^{-2} s_p v^i \varphi_{i,a}^+) \,|_{\widetilde{\underline{d}} + \delta_{i,p}} = v^{i(a+1)} \prod_{j \leq i} s_j \prod_{k \leq i-1} p_k^{-1} q^{-1} \left(\sum_{p \neq j \leq i} s_j^{-2} \right.$$

$$\times \prod_{k \leq i+1} (s_j - q r_k) \prod_{k \leq i-1} (p_k - s_j)$$

$$\times \prod_{j, p \neq k \leq i} \left((s_j - q s_k)^{-1} (s_k - s_j)^{-1} \right) (s_j - s_p)^{-1}$$

$$\times (q^{-1} s_p - s_j)^{-1} (s_j - q^{-1} s_p) s_j^a$$

$$- q \sum_{p \neq j \leq i} s_j^{-2} \prod_{k \leq i+1} (s_j - r_k) \prod_{k \leq i-1} (p_k - q s_j)$$

$$\times \prod_{j, p \neq k \leq i} \left((s_j - s_k)^{-1} (s_k - q s_j)^{-1} \right) (s_j - q^{-1} s_p)^{-1}$$

$$\times (q^{-1} s_p - q s_j)^{-1} (q s_j - q^{-1} s_p) (q s_j)^a$$

$$- q s_p^{-2} q^2 \prod_{k \leq i+1} (q^{-1} s_p - r_k) \prod_{k \leq i-1} (p_k - s_p)$$

$$\times \prod_{p \neq k \leq i} \left((q^{-1} s_p - s_k)^{-1} (s_k - s_p)^{-1} \right) (1 - q^{-1}) s_p^{a+1} \Bigg).$$

It is straightforward to check that these two expressions coincide.

b) $a = 0$. In this case, we know that $\varphi_{i,0}^+ = \chi_{i,0} + \varphi_{i,0}^-$, $\varphi_{i,1}^+ = \chi_{i,1}$. The argument of (a) shows

$$(\chi_{i,1} - v^{-2} s_{i,p} v^i \chi_{i,0}) \,|_{\widetilde{\underline{d}} + \delta_{i,p}} = (v^{-2} \chi_{i,1} - s_{i,p} v^i \chi_{i,0}) \,|_{\widetilde{\underline{d}}} .$$

So in order to prove

$$(\varphi_{i,1}^+ - v^{-2} s_{i,p} v^i \varphi_{i,0}^+) \,|_{\widetilde{\underline{d}} + \delta_{i,p}} = (v^{-2} \varphi_{i,1}^+ - s_{i,p} v^i \varphi_{i,0}^+) \,|_{\widetilde{\underline{d}}}$$

it is enough to check $v^{-2} \varphi_{i,0}^- \,|_{\widetilde{\underline{d}} + \delta_{i,p}} = \varphi_{i,0}^- \,|_{\widetilde{\underline{d}}}$, which follows directly from the formula $\varphi_{i,0}^- = t_i^{-1} t_{i+1} v^{-d_{i+1} + 2 d_i - d_{i-1} + 1}$. $\qquad\square$

Now, we rewrite the formulas for $\varphi_i^{\pm}(z)$. From (23), we get for any $a > 0$:

$$(\varphi_{l,a+1}^+ - v^{-a_{k,l}} t_p^2 v^{-2 d_{k,p}} v^k \varphi_{l,a}^+) \,|_{\widetilde{\underline{d}} + \delta_{k,p}} = (v^{-a_{k,l}} \varphi_{l,a+1}^+ - t_p^2 v^{-2 d_{k,p}} v^k \varphi_{l,a}^+) \,|_{\widetilde{\underline{d}}},$$

i.e.

$$\varphi_l^+(z)(1 - t_p^2 v^{-a_{k,l}-2d_{k,p}+k} z^{-1})\,|_{\underline{\widetilde{d}}+\delta_{k,p}} = \varphi_l^+(z)(v^{-a_{k,l}} - t_p^2 v^{-2d_{k,p}+k} z^{-1})\,|_{\underline{\widetilde{d}}}\;.$$

This equation is especially interesting whenever $a_{k,l} \neq 0$ giving the following information:

$$\frac{\varphi_l^+(z)\,|_{\underline{\widetilde{d}}+\delta_{l+1,p}}}{\varphi_l^+(z)\,|_{\underline{\widetilde{d}}}} = v\frac{1 - z^{-1}v^l t_p^2 v^{-2d_{l+1,p}}}{1 - z^{-1}v^{l+2}t_p^2 v^{-2d_{l+1,p}}}, \tag{24}$$

$$\frac{\varphi_l^+(z)\,|_{\underline{\widetilde{d}}+\delta_{l-1,p}}}{\varphi_l^+(z)\,|_{\underline{\widetilde{d}}}} = v\frac{1 - z^{-1}v^{l-2}t_p^2 v^{-2d_{l-1,p}}}{1 - z^{-1}v^l t_p^2 v^{-2d_{l-1,p}}}, \tag{25}$$

$$\frac{\varphi_l^+(z)\,|_{\underline{\widetilde{d}}+\delta_{l,p}}}{\varphi_l^+(z)\,|_{\underline{\widetilde{d}}}} = v^{-2}\frac{1 - z^{-1}v^{l+2}t_p^2 v^{-2d_{l,p}}}{1 - z^{-1}v^{l-2}t_p^2 v^{-2d_{l,p}}}. \tag{26}$$

Let us compute $\varphi_i^+(z)\,|_{\underline{\widetilde{d_0}}}$, where $\underline{\widetilde{d_0}} = (d_{i,j} = 0|\forall i,\,j)$. In this case, the expression for $\varphi_i^+(z)$ reads as:

$$\begin{aligned}
\varphi_i^+(z)\,|_{\underline{\widetilde{d_0}}} &= t_{i+1}^{-1}t_i v^{-1} - t_{i+1}^{-1}t_i^{-1}v^{-1}(v^2-1)^{-1}t_i^2 \\
&\quad \times \sum_{a\geq 1}\prod_{k\leq i-1}(1 - t_i^2 t_k^{-2})^{-1}\prod_{k\leq i-1}(1 - v^2 t_k^2 t_i^{-2})^{-1} \\
&\quad \times \prod_{k\leq i-1}(1 - t_i^2 t_k^{-2})\prod_{k\leq i+1}(1 - v^2 t_k^2 t_i^{-2})(t_i^2 v^i z^{-1})^a \\
&= t_{i+1}^{-1}t_i v^{-1} - t_{i+1}^{-1}t_i v^{-1}(v^2-1)^{-1}(1-v^2)(1 - t_{i+1}^2 t_i^{-2}v^2)\frac{t_i^2 v^i z^{-1}}{1 - t_i^2 v^i z^{-1}} \\
&= t_{i+1}^{-1}t_i v^{-1}(1 - t_{i+1}^2 v^{i+2}z^{-1})(1 - t_i^2 v^i z^{-1})^{-1}.
\end{aligned}$$

So,

$$\varphi_i^+(z)\,|_{\underline{\widetilde{d_0}}} = t_{i+1}^{-1}t_i v^{-1}(1 - t_{i+1}^2 v^{i+2}z^{-1})(1 - t_i^2 v^i z^{-1})^{-1}. \tag{27}$$

From (24–27) we get a simple formula for $\varphi_i^+(z)$, explicitly:

$$\varphi_i^+(z) = t_{i+1}^{-1}t_i v^{d_{i+1}-2d_i+d_{i-1}-1}\left(a_{i+1}(zv^{-i-2})a_{i-1}(zv^{-i})a_i(zv^{-i-2})^{-1}a_i(zv^{-i})^{-1}\right)^+, \tag{28}$$

$$a_j(z) := \prod_{p\leq j}(1 - z^{-1}t_p^2 v^{-2d_{j,p}}). \tag{29}$$

Now from (28–29) and Proposition 2.15b we get $\varphi_i^+(z) = \psi_i^+(z)$. In the same way, one gets $\varphi_i^-(z) = \psi_i^-(z)$.

Theorem 2.12 is proved.

4 Parabolic sheaves and quantum toroidal algebra

In this section, we generalize the previous results to the affine setting.

4.1 Parabolic sheaves

We recall the setup of Sect. 3 of [2]. Let $\mathbf{X}$ be another smooth projective curve of genus zero. We fix a coordinate y on $\mathbf{X}$, and consider the action of $\mathbb{C}^*$ on $\mathbf{X}$ such that $c(x) = c^{-2}x$. We have $\mathbf{X}^{\mathbb{C}^*} = \{0_{\mathbf{X}}, \infty_{\mathbf{X}}\}$. Let $\mathbf{S}$ denote the product surface $\mathbf{C} \times \mathbf{X}$. Let $\mathbf{D}_\infty$ denote the divisor $\mathbf{C} \times \infty_{\mathbf{X}} \cup \infty_{\mathbf{C}} \times \mathbf{X}$. Let $\mathbf{D}_0$ denote the divisor $\mathbf{C} \times 0_{\mathbf{X}}$.

Given an n-tuple of nonnegative integers $\underline{d} = (d_0, \ldots, d_{n-1})$, we say that a *parabolic sheaf* $\mathcal{F}_\bullet$ of degree $\underline{d}$ is an infinite flag of torsion free coherent sheaves of rank n on $\mathbf{S}$: $\ldots \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots$ such that:

(a) $\mathcal{F}_{k+n} = \mathcal{F}_k(\mathbf{D}_0)$ for any k;
(b) $ch_1(\mathcal{F}_k) = k[\mathbf{D}_0]$ for any k: the first Chern classes are proportional to the fundamental class of $\mathbf{D}_0$;
(c) $ch_2(\mathcal{F}_k) = d_i$ for $i \equiv k \pmod n$;
(d) $\mathcal{F}_0$ is locally free at $\mathbf{D}_\infty$ and trivialized at $\mathbf{D}_\infty$: $\mathcal{F}_0|_{\mathbf{D}_\infty} = W \otimes \mathcal{O}_{\mathbf{D}_\infty}$;
(e) For $-n \leq k \leq 0$ the sheaf $\mathcal{F}_k$ is locally free at $\mathbf{D}_\infty$, and the quotient sheaves $\mathcal{F}_k/\mathcal{F}_{-n}$, $\mathcal{F}_0/\mathcal{F}_k$ (both supported at $\mathbf{D}_0 = \mathbf{C} \times 0_{\mathbf{X}} \subset \mathbf{S}$) are both locally free at the point $\infty_{\mathbf{C}} \times 0_{\mathbf{X}}$; moreover, the local sections of $\mathcal{F}_k|_{\infty_{\mathbf{C}} \times \mathbf{X}}$ are those sections of $\mathcal{F}_0|_{\infty_{\mathbf{C}} \times \mathbf{X}} = W \otimes \mathcal{O}_{\mathbf{X}}$ which take value in $\langle w_1, \ldots, w_{n-k} \rangle \subset W$ at $0_{\mathbf{X}} \in \mathbf{X}$.

The fine moduli space $\mathcal{P}_{\underline{d}}$ of degree $\underline{d}$ parabolic sheaves exists and is a smooth connected quasiprojective variety of dimension $2d_0 + \cdots + 2d_{n-1}$.

4.2 Fixed points

The group $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$ acts naturally on $\mathcal{P}_{\underline{d}}$, and its fixed point set is finite. In order to describe it, we recall the well known description of the fixed point set of $\mathbb{C}^* \times \mathbb{C}^*$ on the Hilbert scheme of $(\mathbf{C} - \infty_{\mathbf{C}}) \times (\mathbf{X} - \infty_{\mathbf{X}})$. Namely, the fixed points are parametrized by the Young diagrams, and for a diagram $\lambda = (\lambda_0 \geq \lambda_1 \geq \cdots)$ (where $\lambda_N = 0$ for $N \gg 0$) the corresponding fixed point is the ideal $J_\lambda = \mathbb{C}y^0 z^{\lambda_0} \oplus \mathbb{C}y^1 z^{\lambda_1} \oplus \cdots$. We will view J_λ as an ideal in $\mathcal{O}_{\mathbf{C} \times \mathbf{X}}$ coinciding with $\mathcal{O}_{\mathbf{C} \times \mathbf{X}}$ in a neighborhood of infinity.

We say $\lambda \supset \mu$ if $\lambda_i \geq \mu_i$ for any $i \geq 0$. We say $\lambda \widetilde{\supset} \mu$ if $\lambda_i \geq \mu_{i+1}$ for any $i \geq 0$.

We consider a collection $\boldsymbol{\lambda} = (\lambda^{kl})_{1 \leq k,l \leq n}$ of Young diagrams satisfying the following inequalities:

$$\lambda^{11} \subset \lambda^{21} \subset \cdots \subset \lambda^{n1} \widetilde{\subset} \lambda^{11}; \ \lambda^{22} \subset \lambda^{32} \subset \cdots \subset \lambda^{12} \widetilde{\subset} \lambda^{22}; \ \ldots; \ \lambda^{nn}$$
$$\subset \lambda^{1n} \subset \cdots \subset \lambda^{n-1,n} \widetilde{\subset} \lambda^{nn} \tag{30}$$

We set $d_k(\lambda) = \sum_{l=1}^{n} |\lambda^{kl}|$, and $\underline{d}(\lambda) = (d_0(\lambda) := d_n(\lambda), \ldots, d_{n-1}(\lambda))$.

Given such a collection λ we define a parabolic sheaf $\mathcal{F}_\bullet = \mathcal{F}_\bullet(\lambda)$, or just λ by an abuse of notation, as follows: for $1 \leq k \leq n$ we set

$$\mathcal{F}_{k-n} = \bigoplus_{1 \leq l \leq k} J_{\lambda^{kl}} w_l \oplus \bigoplus_{k < l \leq n} J_{\lambda^{kl}}(-\mathbf{D}_0) w_l \tag{31}$$

Lemma 4.3 *The correspondence $\lambda \mapsto \mathcal{F}_\bullet(\lambda)$ is a bijection between the set of collections λ satisfying (30) such that $\underline{d}(\lambda) = \underline{d}$, and the set of $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$-fixed points in $\mathcal{P}_{\underline{d}}$.*

Proof Evident.   □

4.4 Another realization of parabolic sheaves

We will now introduce a different realization of parabolic sheaves, and another parametrization of the fixed point set which is very closely related to this new realization. We first learned of this construction from A. Okounkov, though it is already present in the work of Biswas [1]. Let $\sigma : \mathbf{C} \times \mathbf{X} \to \mathbf{C} \times \mathbf{X}$ denote the map $\sigma(z, y) = (z, y^n)$, and let $G = \mathbb{Z}/n\mathbb{Z}$. Then G acts on $\mathbf{C} \times \mathbf{X}$ by multiplying the coordinate on $\mathbf{X}$ with the nth roots of unity.

A parabolic sheaf $\mathcal{F}_\bullet$ is completely determined by the flag of sheaves

$$\mathcal{F}_0(-\mathbf{D}_0) \subset \mathcal{F}_{-n+1} \subset \cdots \subset \mathcal{F}_0,$$

satisfying conditions (a)–(e) above. To $\mathcal{F}_\bullet$ we can associate a single, G invariant sheaf $\tilde{\mathcal{F}}$ on $\mathbf{C} \times \mathbf{X}$:

$$\tilde{\mathcal{F}} = \sigma^* \mathcal{F}_{-n+1} + \sigma^* \mathcal{F}_{-n+2}(-\mathbf{D}_0) + \cdots + \sigma^* \mathcal{F}_0(-(n-1)\mathbf{D}_0).$$

This sheaf will have to satisfy certain numeric and framing conditions that mimic conditions 4.1(b)–(e). Conversely, any G invariant sheaf $\tilde{\mathcal{F}}$ that satisfies those numeric and framing conditions will determine a unique parabolic sheaf.

If $\mathcal{F}_\bullet$ is a $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$ fixed parabolic sheaf corresponding to a collection λ as in the previous section, then we have

$$\tilde{\mathcal{F}} = \bigoplus_{l=1}^{n} J_{\lambda^l}(-(l-1)\mathbf{D}_0) w_l, \tag{32}$$

where $(\lambda^1, \ldots, \lambda^n)$ is a collection of partitions, given by

$$\lambda^l_{ni - n\lfloor \frac{k-l}{n} \rfloor + k - l} = \lambda^{kl}_i. \tag{33}$$

Here $\lfloor \frac{k-l}{n} \rfloor$ stands for the maximal integer smaller than or equal to $\frac{k-l}{n}$.

For $j \in \mathbb{Z}$, let $(j \bmod n)$ denote that element of $\{1, \ldots, n\}$ which is congruent to j modulo n. For $i \geq j \in \mathbb{Z}$, if we denote

$$d_{ij} = \lambda_{i-j}^{j \bmod n} \tag{34}$$

we obtain a collection $(d_{ij}) = \underline{\widetilde{d}} = \underline{\widetilde{d}}(\lambda)$ of non-negative integers with the properties that

$$d_{kj} \geq d_{ij} \ \forall i \geq k \geq j; \ d_{i+n,j+n} = d_{ij} \ \forall i \geq j; \ d_{ij} = 0 \ \text{for} \ i - j \gg 0. \tag{35}$$

For $1 \leq k \leq n$, let us write

$$d_k(\underline{\widetilde{d}}) = \sum_{j \leq k} d_{kj} = \sum_{l=1}^{n} \sum_{i \leq \lfloor \frac{k-l}{n} \rfloor} d_{k(l+ni)} = \sum_{l=1}^{n} \sum_{i \geq 0} \lambda_{ni-n\lfloor \frac{k-l}{n} \rfloor + k - l}^{l}$$

$$= \sum_{l=1}^{n} \sum_{i \geq 0} \lambda_i^{kl} = d_k(\lambda).$$

Summarizing the above discussion, we have:

Lemma 4.5 *The correspondence* $\lambda \mapsto \underline{\widetilde{d}}(\lambda)$ *is a bijection between the set of collections* λ *satisfying* (30)*, and the set* D *of collections* $\underline{\widetilde{d}}$ *satisfying* (35)*. We have* $\underline{d}(\lambda) = \underline{d}(\underline{\widetilde{d}}(\lambda))$*.*

By virtue of Lemmas 4.3 and 4.5, we will parametrize and sometimes denote the $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$-fixed points in $\mathcal{P}_{\underline{d}}$ by collections $\underline{\widetilde{d}}$ such that $\underline{d} = \underline{d}(\underline{\widetilde{d}})$.

4.6 Correspondences

If the collections $\underline{d}$ and $\underline{d}'$ differ at the only place $i \in I := \mathbb{Z}/n\mathbb{Z}$, and $d_i' = d_i + 1$, then we consider the correspondence $\mathsf{E}_{\underline{d},i} \subset \mathcal{P}_{\underline{d}} \times \mathcal{P}_{\underline{d}'}$ formed by the pairs $(\mathcal{F}_\bullet, \mathcal{F}_\bullet')$ such that for $j \not\equiv i \pmod{n}$ we have $\mathcal{F}_j = \mathcal{F}_j'$, and for $j \equiv i \pmod{n}$ we have $\mathcal{F}_j' \subset \mathcal{F}_j$.

It is a smooth quasiprojective algebraic variety of dimension $2 \sum_{i \in I} d_i + 1$.

We denote by **p** (respectively, **q**) the natural projection $\mathsf{E}_{\underline{d},i} \to \mathcal{P}_{\underline{d}}$ (resp. $\mathsf{E}_{\underline{d},i} \to \mathcal{P}_{\underline{d}'}$). For $j \equiv i \pmod{n}$ the correspondence $\mathsf{E}_{\underline{d},i}$ is equipped with a natural line bundle L_j whose fiber at $(\mathcal{F}_\bullet, \mathcal{F}_\bullet')$ equals $\Gamma(\mathbf{C}, \mathcal{F}_j / \mathcal{F}_j')$. Finally, we have a transposed correspondence $^{\mathsf{T}}\mathsf{E}_{\underline{d},i} \subset \mathcal{P}_{\underline{d}'} \times \mathcal{P}_{\underline{d}}$.

4.7 Direct sum of equivariant *K*-groups

We denote by $'V$ the direct sum of equivariant (complexified) *K*-groups: $'V = \oplus_{\underline{d}} K^{\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*}(\mathcal{P}_{\underline{d}})$. It is a module over $K^{\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*}(pt) = \mathbb{C}[\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*] =$

$\mathbb{C}[x_1, \ldots, x_n, v, u]$. Here u corresponds to a character $(x_1, \ldots, x_n, v, u) \longmapsto u$. We define $V = {}'V \otimes_{K^{\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*}(pt)} \mathrm{Frac}(K^{\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*}(pt))$.

We have an evident grading $V = \oplus_{\underline{d}} V_{\underline{d}}$, $V_{\underline{d}} = K^{\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*}(\mathcal{P}_{\underline{d}}) \otimes_{K^{\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*}(pt)} \mathrm{Frac}(K^{\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*}(pt))$.

4.8 The action of quantum affine group on V

The grading and the correspondences ${}^{\mathsf{T}}\mathsf{E}_{\underline{d},i}$, $\mathsf{E}_{\underline{d},i}$ give rise to the following operators on V (note that though $\mathbf{p}$ is not proper, $\mathbf{p}_*$ is well defined on the localized equivariant K-theory due to the finiteness of the fixed point set of $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$):

$$\mathfrak{k}_i = t_{i+1}^{-1} t_i u^{-\delta_{0,i}} v^{-2d_i + d_{i-1} + d_{i+1} - 1} : V_{\underline{d}} \to V_{\underline{d}} \tag{36}$$

$$\mathfrak{e}_i = t_{i+1}^{-1} v^{d_{i+1} - d_i - i + 1} \mathbf{p}_* \mathbf{q}^* : V_{\underline{d}} \to V_{\underline{d}-i} \tag{37}$$

$$\mathfrak{f}_i = -t_i^{-1} u^{\delta_{0,i}} v^{d_i - d_{i-1} + i - 1} \mathbf{q}_*(L_i \otimes \mathbf{p}^*) : V_{\underline{d}} \to V_{\underline{d}+i} \tag{38}$$

According to the Conjecture 3.7 of [2], the following theorem holds

Theorem 4.9 *For $n > 2$, these operators $\mathfrak{k}_i, \mathfrak{e}_i, \mathfrak{f}_i (i \in \mathbb{Z}/n\mathbb{Z})$ satisfy the relations in $U_v(\widehat{\mathfrak{sl}}_n)$, i.e. they give rise to the action of quantum affine group on V.*

Since the fixed point basis of M corresponds to the Gelfand–Tsetlin basis of the universal Verma module over $U_v(\mathfrak{gl}_n)$, we propose to call the fixed point basis of V the affine Gelfand–Tsetlin basis.

4.10 Quantum toroidal algebra

Let $(a_{kl})_{1 \leq k,l \leq n} = \widehat{A}_{n-1}$ stand for the Cartan matrix of $\widehat{\mathfrak{sl}}_n$. The double affine loop algebra $U'_v(\widehat{\mathfrak{sl}}_n)$ is an associative algebra over $\mathbb{Q}(v)$ generated by $e_{k,r}, f_{k,r}, v^{\pm h_k}, h_{k,m}$ $(1 \leq k \leq n, r \in \mathbb{Z}, m \in \mathbb{Z}\backslash\{0\})$ with the relations (9–14), where k, l are understood as residues modulo n, so that for instance if $k = n$ then $k + 1 = 1$.

The quantum toroidal algebra $\ddot{U}_v(\widehat{\mathfrak{sl}}_n)$ is an associative algebra over $\mathbb{C}(u, v)$ generated by $e_{k,r}, f_{k,r}, v^{\pm h_k}, h_{k,m}$ $(1 \leq k \leq n, r \in \mathbb{Z}, m \in \mathbb{Z}\backslash\{0\})$ with the same relations as in $U'_v(\widehat{\mathfrak{sl}}_n)$ except for relations (10, 13) for the pairs $(k, l) = (1, n), (n, 1)$. These relations are modified as follows. We introduce the shifted generating series $\hat{x}_n^{\pm}(z) := x_n^{\pm}(zv^n u^2)$, $\hat{\psi}_n^{\pm}(z) = \psi_n^{\pm}(zv^n u^2)$.

Now the new relations read

$$\hat{x}_n^{\pm}(z) x_1^{\pm}(w)(z - v^{\mp 1} w) = (v^{\mp 1} z - w) x_1^{\pm}(w) \hat{x}_n^{\pm}(z), \tag{39}$$

$$\hat{\psi}_n^s(z) x_1^{\pm}(w)(z - v^{\mp 1} w) = x_1^{\pm}(w) \hat{\psi}_n^s(z)(v^{\mp 1} z - w), \tag{40}$$

$$\psi_1^s(z) \hat{x}_n^{\pm}(w)(z - v^{\mp 1} w) = \hat{x}_n^{\pm}(w) \psi_1^s(z)(v^{\mp 1} z - w). \tag{41}$$

Thus, we have $U'_v(\widehat{\mathfrak{sl}}_n) = \ddot{U}_v(\widehat{\mathfrak{sl}}_n) / (v^n u^2 = 1)$.

Note that $\ddot{U}_v(\widehat{\mathfrak{sl}}_n)$ coincides with $\ddot{U}'$–modification of $\ddot{U}$ introduced in [16], with d not specialized to a complex number and with central element $c = 1$. The isomorphism $\Psi : \ddot{U}_v(\widehat{\mathfrak{sl}}_n) \to \ddot{U}'$ takes v to v and u to $d^{\frac{n}{2}} v^{-\frac{n}{2}}$. It is defined on the generating series as

$$\Psi(x_i^+(z)) = \mathbf{e}_{i-1}^{\pm}(d^{-i}z), \quad \Psi(x_i^-(z)) = \mathbf{f}_{i-1}^{\pm}(d^{-i}z), \quad \Psi(\psi_i^{\pm}(z)) = \mathbf{k}_{i-1}^{\pm}(d^{-i}z).$$

4.11 Main theorem

For any $m \le i \in \mathbb{Z}$ we will denote by $\underline{W}_{mi}$ the quotient $\mathcal{F}_i/\mathcal{F}_m$ of the tautological vector bundles on $\mathcal{P}_{\underline{d}} \times \mathbb{C}$. Under the Künneth decomposition, for the exterior power $\Lambda^j(\underline{W}_{mi})$ we have $\Lambda^j(\underline{W}_{mi}) =: \Lambda^j_{(j)}(\underline{W}_{mi}) \otimes 1 + \Lambda^j_{(j-1)}(\underline{W}_{mi}) \otimes \tau$ where $\Lambda^j_{(j)}(\underline{W}_{mi}), \Lambda^j_{(j-1)}(\underline{W}_{mi}) \in K^{\widetilde{T} \times \mathbb{C}^*}(\mathcal{P}_{\underline{d}})$. We introduce the generating series $\mathbf{b}_{mi}(z) := 1 + \sum_{j=1}^{\infty} \left(\Lambda^j_{(j)}(\underline{W}_{mi}) - v\Lambda^j_{(j-1)}(\underline{W}_{mi}) \right) (-z)^{-j}$.

Corollary 4.12 *The expression* $\mathbf{b}_{mi}(zv^{-i-2})^{-1}\mathbf{b}_{mi}(zv^{-i})^{-1}\mathbf{b}_{m,i-1}(zv^{-i})\mathbf{b}_{m,i+1}(zv^{-i-2})$ *is independent of* $m < i$.

The proof is similar to the proof of Corollary 2.16.
We will denote by $\psi_i^{\pm}(z)$ the common value of the expressions

$$t_{i+1}^{-1} t_i v^{d_{i+1}-2d_i+d_{i-1}-1} \left(\mathbf{b}_{mi}(zv^{-i-2})^{-1}\mathbf{b}_{mi}(zv^{-i})^{-1}\mathbf{b}_{m,i-1}(zv^{-i})\mathbf{b}_{m,i+1}(zv^{-i-2}) \right)^{\pm}. \tag{42}$$

Recall that v stands for the character of $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^* : (\underline{t}, v, u) \mapsto v$. We define the line bundle $\mathsf{L}'_k := v^k \mathsf{L}_k$ on the correspondence $\mathsf{E}_{\underline{d},k}$, that is L'_k and L_k are isomorphic as line bundles but the equivariant structure of L'_k is obtained from the equivariant structure of L_k by the twist by the character v^k.

For $1 \le k \le n$ we consider the following generating series of operators on V:

$$\psi_k^{\pm}(z) = \sum_{r=0}^{\pm\infty} \psi_{k,r}^{\pm} z^{\mp r} : V_{\underline{d}} \to V_{\underline{d}}[[z^{\mp 1}]], \tag{43}$$

$$x_k^+(z) = \sum_{r=-\infty}^{\infty} e_{k,r} z^{-r} : V_{\underline{d}} \to V_{\underline{d}-k}[[z, z^{-1}]], \tag{44}$$

$$x_k^-(z) = \sum_{r=-\infty}^{\infty} f_{k,r} z^{-r} : V_{\underline{d}} \to V_{\underline{d}+k}[[z, z^{-1}]], \tag{45}$$

$$e_{k,r} := t_{k+1}^{-1} v^{d_{k+1}-d_k+1-k} \mathbf{p}_*((\mathsf{L}'_k)^{\otimes r} \otimes \mathbf{q}^*) : V_{\underline{d}} \to V_{\underline{d}-k} \tag{46}$$

$$f_{k,r} := -t_k^{-1} v^{d_k-d_{k-1}-1+k} \mathbf{q}_*(\mathsf{L}_k \otimes (\mathsf{L}'_k)^{\otimes r} \otimes \mathbf{p}^*) : V_{\underline{d}} \to V_{\underline{d}+k} \tag{47}$$

Theorem 4.13 *These generating series of operators $\psi_k^\pm(z), x_k^\pm(z)$ on V defined in (42–47) satisfy the relations in $\ddot{U}_v(\widehat{\mathfrak{sl}}_n)$, i.e. they give rise to the action of $\ddot{U}_v(\widehat{\mathfrak{sl}}_n)$ on V.*

Now, we compute the matrix coefficients of operators $e_{i,r}$, $f_{i,r}$ and the eigenvalues of $\psi_i^\pm(z)$. For accomplishing this goal, we need to know the torus character in the tangent space to $\mathsf{E}_{d,i}$ (and $\mathcal{P}_{\underline{d}}$) at the torus fixed point given by indices $\underline{\widetilde{d}}, \underline{\widetilde{d}'}$ (and $\underline{\widetilde{d}}$ correspondingly). These characters are computed in [4] (see Propositions 4.15, 4.21 and Remark 4.17 of *loc. cit.*):

Proposition 4.14 (a) *The torus character in the tangent space to $\mathsf{E}_{d,i}$ at the torus fixed point given by indices $\underline{\widetilde{d}}, \underline{\widetilde{d}'}$ equals*

$$\sum_{k=1}^n \sum_{l \leq k}^{l' \leq k-1} \frac{t_l^2}{t_{l'}^2} \cdot v^2 \frac{(v^{2d_{(k-1)l'}} - 1)(v^{-2d_{kl}} - 1)}{v^2 - 1} \cdot u^{2\lfloor \frac{-l'}{n} \rfloor - 2\lfloor \frac{-l}{n} \rfloor}$$

$$+ \sum_{k=1}^n \sum_{l' \leq k-1} \frac{t_k^2}{t_{l'}^2} \cdot v^2 \frac{v^{2d_{(k-1)l'}} - 1}{v^2 - 1} \cdot u^{2\lfloor \frac{-l'}{n} \rfloor - 2\lfloor \frac{-k}{n} \rfloor}$$

$$- \sum_{k=1}^n \sum_{l \leq k}^{l' \leq k} \frac{t_l^2}{t_{l'}^2} \cdot v^2 \frac{(v^{2d_{kl'}} - 1)(2^{-2d_{kl}} - 1)}{v^2 - 1} \cdot u^{2\lfloor \frac{-l'}{n} \rfloor - 2\lfloor \frac{-l}{n} \rfloor}$$

$$- \sum_{k=1}^n \sum_{l \leq k} \frac{t_l^2}{t_k^2} \cdot v^2 \frac{v^{-2d_{kl}} - 1}{v^2 - 1} \cdot u^{2\lfloor \frac{-k}{n} \rfloor - 2\lfloor \frac{-l}{n} \rfloor}$$

$$+ v^2 - v^{-2d_{ij} + 2d_{(i-1)j}} + \frac{t_j^2}{t_i^2} \cdot v^{-2d_{ij} + 2d_{ii}} \cdot u^{2\lfloor \frac{-i}{n} \rfloor - 2\lfloor \frac{-j}{n} \rfloor}$$

$$+ \sum_{j \neq k \leq i-1} \frac{t_j^2}{t_k^2} \cdot v^{-2d_{ij}} \cdot (v^{2d_{ik}} - v^{2d_{(i-1)k}}) \cdot u^{2\lfloor \frac{-k}{n} \rfloor - 2\lfloor \frac{-j}{n} \rfloor}$$

    if $d'_{ij} = d_{ij} + 1$ for certain $j \leq i$.

(b) *The torus character in the tangent space to $\mathcal{P}_{\underline{d}}$ at the torus fixed point $\underline{\widetilde{d}}$ equals*

$$\sum_{k=1}^n \sum_{l \leq k}^{l' \leq k-1} \frac{t_l^2}{t_{l'}^2} \cdot v^2 \frac{(v^{2d_{(k-1)l'}} - 1)(v^{-2d_{kl}} - 1)}{v^2 - 1} \cdot u^{2\lfloor \frac{-l'}{n} \rfloor - 2\lfloor \frac{-l}{n} \rfloor}$$

$$+ \sum_{k=1}^n \sum_{l' \leq k-1} \frac{t_k^2}{t_{l'}^2} \cdot v^2 \frac{v^{2d_{(k-1)l'}} - 1}{v^2 - 1} \cdot u^{2\lfloor \frac{-l'}{n} \rfloor - 2\lfloor \frac{-k}{n} \rfloor}$$

$$- \sum_{k=1}^n \sum_{l \leq k}^{l' \leq k} \frac{t_l^2}{t_{l'}^2} \cdot v^2 \frac{(v^{2d_{kl'}} - 1)(v^{-2d_{kl}} - 1)}{v^2 - 1} \cdot u^{2\lfloor \frac{-l'}{n} \rfloor - 2\lfloor \frac{-l}{n} \rfloor}$$

$$- \sum_{k=1}^n \sum_{l \leq k} \frac{t_l^2}{t_k^2} \cdot v^2 \frac{v^{-2d_{kl}} - 1}{v^2 - 1} \cdot u^{2\lfloor \frac{-k}{n} \rfloor - 2\lfloor \frac{-l}{n} \rfloor}$$

So analogously to Theorem 3.17 ([4]) we get the following proposition

Proposition 4.15 *Define* $p_{i,j} := t_{j \ (\text{mod } n)}^2 v^{-2d_{ij}} u^{-2\lfloor \frac{-j}{n} \rfloor} = t_{j \ (\text{mod } n)}^2 v^{-2d_{ij}} u^{2\lceil \frac{j}{n} \rceil}.$

(a) *The matrix coefficients of the operators* $f_{i,r}$, $e_{i,r}$ *in the fixed points basis* $\{[\widetilde{\underline{d}}]\}$ *of* V *are as follows*:

$$f_{i,r[\widetilde{\underline{d}}, \widetilde{\underline{d}}']} = - t_i^{-1} v^{d_i - d_{i-1} - 1 + i} \, p_{i,j} (p_{i,j} v^i)^r (1 - v^2)^{-1}$$

$$\times \prod_{j \neq k \leq i} (1 - p_{i,j} p_{i,k}^{-1})^{-1} \prod_{k \leq i-1} (1 - p_{i,j} p_{i-1,k}^{-1})$$

if $d'_{i,j} = d_{i,j} + 1$ *for certain* $j \leq i$;

$$e_{i,r[\widetilde{\underline{d}}, \widetilde{\underline{d}}']} = t_{i+1}^{-1} v^{d_{i+1} - d_i + 1 - i} (p_{i,j} v^{i+2})^r (1 - v^2)^{-1}$$

$$\times \prod_{j \neq k \leq i} (1 - p_{i,k} p_{i,j}^{-1})^{-1} \prod_{k \leq i+1} (1 - p_{i+1,k} p_{i,j}^{-1})$$

if $d'_{i,j} = d_{i,j} - 1$ *for certain* $j \leq i$;
All the other matrix coefficients of $e_{i,r}$, $f_{i,r}$ *vanish.*

(b) *The eigenvalue of* $\psi_i^{\pm}(z)$ *on* $\{[\widetilde{\underline{d}}]\}$ *equals*

$$t_{i+1}^{-1} t_i v^{d_{i+1} - 2d_i + d_{i-1} - 1} \prod_{j \leq i} (1 - z^{-1} v^{i+2} p_{i,j})^{-1} (1 - z^{-1} v^i p_{i,j})^{-1}$$

$$\times \prod_{j \leq i+1} (1 - z^{-1} v^{i+2} p_{i+1,j}) \prod_{j \leq i-1} (1 - z^{-1} v^i p_{i-1,j}).$$

Remark 4.16 These formulas are the same as in Proposition 2.15 with the change $s_{i,j} \to p_{i,j}$.

Proof For arbitrary $k \in \mathbb{Z}$, we define $x_k^{\pm}(z)$, $\psi_k^{\pm}(z)$ by the same formulas (42–47).

The formula for the eigenvalue of $\psi_i^{\pm}(z)$ on $\{[\widetilde{\underline{d}}]\}$ from Proposition 4.15b shows that $\psi_0^{\pm}(z) = \psi_n^{\pm}(zv^n u^2) = \hat{\psi}_n^{\pm}(z)$.

From the formulas of Proposition 4.15a we get $x_{k-n}^{\pm}(z) = x_k^{\pm}(zv^n u^2)$. In particular, $x_0^{\pm}(z) = x_n^{\pm}(zv^n u^2) = \hat{x}_n^{\pm}(z)$.

Now the relations (39–41) follow again from Theorem 2.12 and the remark above.
□

4.17 Specialization of Gelfand–Tsetlin basis

We fix a positive integer K (a level). We consider an n-tuple $\mu = (\mu_{1-n}, \ldots, \mu_0) \in \mathbb{Z}^n$ such that $\mu_0 + K \geqq \mu_{1-n} \geq \mu_{2-n} \geq \cdots \geq \mu_{-1} \geq \mu_0$. We view μ as a dominant (integrable) weight of $\widehat{\mathfrak{gl}}_n$ of level K. We extend μ to a nonincreasing sequence $\widetilde{\mu} = (\widetilde{\mu}_i)_{i \in \mathbb{Z}}$ setting $\widetilde{\mu}_i := \mu_{i \ (\text{mod } n)} + \lfloor \frac{-i}{n} \rfloor K$.

We define a subset $D(\mu)$ (*affine Gelfand–Tsetlin patterns*) of the set D of all collections $\underline{\widetilde{d}}$ satisfying the conditions (35) as follows:

$$\underline{\widetilde{d}} \in D(\mu) \text{ iff } d_{ij} - \widetilde{\mu}_j \le d_{i+l,j+l} - \widetilde{\mu}_{j+l} \ \forall \ j \le i, \ l \ge 0. \tag{48}$$

We specialize the values of $t_1, \ldots, t_n, v, u$ so that

$$u = v^{-K-n}, \ t_j = v^{\widetilde{\mu}_j - j + 1}. \tag{49}$$

We define the renormalized vectors

$$\langle \underline{\widetilde{d}} \rangle := C_{\underline{\widetilde{d}}}^{-1} [\underline{\widetilde{d}}] \tag{50}$$

where $C_{\underline{\widetilde{d}}}$ is the product $\prod_{w \in T_{\underline{\widetilde{d}}} \mathcal{P}_d} w$ of the weights of $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$ in the tangent space to $\mathcal{P}_d$ at the point $\underline{\widetilde{d}}$. The explicit formula for the multiset $\{w\}$ is given in Proposition 4.14b).

Proposition 4.18 *The only nonzero matrix coefficients of the operators $f_{i,r}, e_{i,r}$ in the renormalized fixed points basis $\{\langle \underline{\widetilde{d}} \rangle\}$ of V are as follows*:

$$e_{i,r\langle \underline{\widetilde{d}}', \underline{\widetilde{d}} \rangle} = t_{i+1}^{-1} v^{d_{i+1} - d_i - i} (p_{i,j} v^i)^r (1 - v^2)^{-1}$$
$$\times \prod_{j \neq k \le i} (1 - p_{i,j} p_{i,k}^{-1})^{-1} \prod_{k \le i-1} (1 - p_{i,j} p_{i-1,k}^{-1})$$

if $d'_{i,j} = d_{i,j} + 1$ for certain $j \le i$;

$$f_{i,r\langle \underline{\widetilde{d}}', \underline{\widetilde{d}} \rangle} = -t_i^{-1} v^{d_i - d_{i-1} - 2 + i} p_{i,j} v^2 (p_{i,j} v^{i+2})^r (1 - v^2)^{-1}$$
$$\times \prod_{j \neq k \le i} (1 - p_{i,k} p_{i,j}^{-1})^{-1} \prod_{k \le i+1} (1 - p_{i+1,k} p_{i,j}^{-1})$$

if $d'_{i,j} = d_{i,j} - 1$ for certain $j \le i$;

Proof According to Proposition 4.15 the matrix elements $e_{i,r[\underline{\widetilde{d}}', \underline{\widetilde{d}}]}$ $(f_{i,r[\underline{\widetilde{d}}', \underline{\widetilde{d}}]})$ are nonzero only if $\underline{\widetilde{d}}' = \underline{\widetilde{d}} + \delta_{i,j}$ $(\underline{\widetilde{d}}' = \underline{\widetilde{d}} - \delta_{i,j})$ for some $j \le i$. According to the Bott–Lefschetz fixed point formula:

$$e_{i,r[\underline{\widetilde{d}}', \underline{\widetilde{d}}]} = t_{i+1}^{-1} v^{d_{i+1} - d_i - i} (t_j^2 v^{-2d_{ij}} u^{2\lceil \frac{i}{n} \rceil} v^i)^r \frac{\prod_{w \in T_{\underline{\widetilde{d}}'} \mathcal{P}'_d} w}{\prod_{w \in T_{(\underline{\widetilde{d}}, \underline{\widetilde{d}}')} \mathsf{E}_{\underline{d},i}} w};$$

$$f_{i,r[\underline{\widetilde{d}}', \underline{\widetilde{d}}]} = -t_i^{-1} v^{d_i - d_{i-1} - 2 + i} (t_j^2 v^{-2d_{ij} + 2} u^{2\lceil \frac{i}{n} \rceil}) (t_j^2 v^{-2d_{ij} + 2} u^{2\lceil \frac{i}{n} \rceil} v^i)^r \frac{\prod_{w \in T_{\underline{\widetilde{d}}'} \mathcal{P}'_d} w}{\prod_{w \in T_{(\underline{\widetilde{d}}, \underline{\widetilde{d}}')} \mathsf{E}_{\underline{d},i}} w}.$$

So after renormalizing vectors according to (50) we have:

$$e_{i,r\langle\widetilde{\underline{d}}',\widetilde{\underline{d}}\rangle} = -f_{i,r[\widetilde{\underline{d}},\widetilde{\underline{d}}']}t_i t_{i+1}^{-1} v^{d_{i+1}-2d_i+d_{i-1}+2-2i},$$
$$f_{i,r\langle\widetilde{\underline{d}},\widetilde{\underline{d}}'\rangle} = -f_{i,r[\widetilde{\underline{d}}',\widetilde{\underline{d}}]}t_i^{-1} t_{i+1} v^{-d_{i+1}+2d_i-d_{i-1}+2i}.$$

Now the proposition follows from Proposition 4.15. □

We define $V(\mu)$ as the $\mathbb{C}(v)$–linear span of the vectors $\langle\widetilde{\underline{d}}\rangle$ for $\widetilde{\underline{d}} \in D(\mu)$.

Theorem 4.19 *The formulas of Theorem 4.13 give rise to the action of* $\ddot{U}_v(\widehat{\mathfrak{sl}}_n)/$ $(u - v^{-K-n})$ *in* $V(\mu)$.

Proof The proof is exactly the same as the proof of Theorem 3.20 of [4].
We have to check two things:

(a) for $\widetilde{\underline{d}} \in D(\mu)$ the denominators of the matrix coefficients $e_{i,r\langle\widetilde{\underline{d}},\widetilde{\underline{d}}'\rangle}$, $f_{i,r\langle\widetilde{\underline{d}},\widetilde{\underline{d}}'\rangle}$ do not vanish;
(b) for $\widetilde{\underline{d}} \in D(\mu)$, $\widetilde{\underline{d}}' \notin D(\mu)$ the numerators of the matrix coefficients $e_{i,r\langle\widetilde{\underline{d}},\widetilde{\underline{d}}'\rangle}$, $f_{i,r\langle\widetilde{\underline{d}},\widetilde{\underline{d}}'\rangle}$ do vanish.

Both are straightforward. □

Restricting $V(\mu)$ to the subalgebra of $\ddot{U}_v(\widehat{\mathfrak{sl}}_n)$, generated by $\{e_{i,0}, f_{i,0}, v^{\pm h_i}\}_{1\le i\le n}$ which is isomorphic to $U_v(\widehat{\mathfrak{sl}}_n)$ (called *horizontal* in [16]) we obtain the same named $U_v(\widehat{\mathfrak{sl}}_n)$-module with the Gelfand–Tsetlin basis parameterized by $D(\mu)$. Recall that in the proof of Theorem 3.22 [4] there was constructed a bijection between $D(\mu)$ and Tingley's crystal $\mathfrak{B}_\mu$ of cylindric plane partitions model of section 4 [14]. This answers Tingley's *Question 1* ([14], p. 38).
Finally we formulate a conjecture:

Conjecture 4.20 $\ddot{U}_v(\widehat{\mathfrak{sl}}_n)/(u-v^{-K-n})$–*module* $V(\mu)$ *is isomorphic to Uglov-Takemura module, constructed in* [15].

It seems likely that these $\ddot{U}_v(\widehat{\mathfrak{sl}}_n)$–modules are obtained by the application of the *Schur* functor ([7]) to the irreducible $\mathfrak{X}$-semisimple modules over the double affine Cherednik algebra $\ddot{H}_n(v)$ of type A_{n-1}, see [13].

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