YALE MATH DIRECTED READING PROGRAM FALL 2018
PROJECT DESCRIPTIONS

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ABSTRACT. The Directed Reading Program (DRP) at Yale Department of Mathematics pairs undergraduate students with graduate student mentors to read and work through a mathematics text over the course of one semester. Each group or pair meets at least once each week for at least one hour, with the undergraduates expected to do about 4 hours of independent reading per week. Over the semester, the DRP hosts tea events and symposiums that would facilitate communication among different reading groups. Undergraduates are expected to have a high level of mathematical maturity and eagerness to learn the topic. Note that for graduate mentors with more than one proposed projects, not all projects will be offered.

CONTENTS

1. ALGORITHMIC ASPECTS OF COMBINATORIAL DISCREPANCY

1.1. **Mentor.** Christopher Harshaw (`crharshaw@gmail.com`)

1.2. **Primary Text(s).**

- *Geometric Discrepancy: An Illustrated Guide* by Jiri Matousek
- *The Discrepancy Method: Randomness and Complexity* by Bernard Chazelle
- *A Panorama of Discrepancy Theory* by many authors

1.3. **Description.** Discrepancy theory is also called the theory of irregularities of distribution. Here are some typical questions: What is the "most uniform" way of distributing n points in the unit square? How big is the "irregularity" necessarily present in any such distribution? The field of combinatorial discrepancy is interested in these questions when the setting is more discrete. Combinatorial discrepancy studies vertex colorings of hypergraphs and asks similar questions regarding their irregularity. There has been a rich body of mathematics to show the existence of low discrepancy colorings and also to design efficient algorithms for producing such low discrepancy colorings. Combinatorial discrepancy theory has connections to other branches of mathematics as well as practical applications in computer science.

1.4. **Syllabus.** In the first thrust, we will cover the basic results in combinatorial discrepancy such as discrepancy of random colorings, Spencer's theorem and the entropy method, set systems with bounded degree, and eigenvalue lower bounds. In the second thrust, there are several possible next steps and they depend on the student's interest. These possibilities include algorithms and complexity of producing low discrepancy colorings, discrepancy of set systems with low VC dimension, hereditary and linear discrepancy, connections to geometric discrepancy, or the application of discrepancy theory to other problems in computer science such as numerical integration. Running experiments / collecting evidence of various results through computer simulation will be encouraged, but not required.

1.5. **Primary Goal(s).** The main goal will be to understand the proof techniques for demonstrating tight upper and lower bounds on set discrepancy. An "icing on the cake" will be to understand algorithmic aspects for producing such low discrepancy colorings or applications to other areas of mathematics.

1.6. **Expected Output.** The expected output depends on the student's creative inclinations. An end of semester report including an exposition on discrepancy bounds and one algorithm, along with an implementation of the algorithm and experimental results would be fantastic. Students may also decide to present a discrepancy tutorial to a group or make a clever animated video describing discrepancy theory.

1.7. **Prerequisites.** One semester courses in both discrete mathematics and linear algebra are required. Some familiarity with theoretical analysis of algorithms would be helpful, but not required. We'll pick it up as needed. Experience with computer programming is completely optional.

2. A TOUR OF GEOMETRIC GROUP THEORY

2.1. **Mentor.** Aaron Calderon (`aaron.calderon@yale.edu`).

2.2. **Primary Texts.**

- *Office Hours with a Geometric Group Theorist* (`https://press.princeton.edu/titles/11042.html`); other titles will be provided as the student's interests develop.

2.3. **Description.** The amount of research currently being done in geometric group theory (GGT) belies its relative infancy as a field. While algebraic geometry and topology employ the machinery of algebra to investigate non-algebraic objects when intuition breaks down, GGT uses geometric and combinatorial intuitions to investigate algebraic objects.

While the topic can trace its history back to Dehn in the 1910s, its modern incarnation has only been around for around 35 years. While GGT is interesting in its own right, many of the groups that people are interested in for other reasons also admit geometric investigation (e.g. reflection groups, hyperbolic groups, fundamental groups of 3-manifolds, etc.)!

2.4. **Syllabus.** We will cover the basics of geometric group theory, with a focus on examples and general concepts. We will spend a fair amount on the free group, but students will have the opportunity to explore many other directions thanks to the book's wealth of (mostly independent!) examples (Part 4). Further reading will mostly depend on the interests of the student(s), but one possibility is the large-scale geometry of groups, including ends of a group, asymptotic dimension, and quasi-isometric rigidity of surface groups (via Kapovich's notes)

2.5. **Primary Goal(s).** The goal of this project is to get a feel for the tools used in GGT, and to apply these to a chosen example. In particular, by the end of the project students will have collected a zoo of examples and understand how the basic methods of GGT apply to these examples.

2.6. **Expected Output.** The students will each choose one example (i.e. one class of geometrically interesting groups) and write up a summary of its salient points. Ideally, these projects will also result in some useful visualization of these groups as geometric objects, either by making helpful pictures or building physical models of (pieces of) the relevant spaces!

2.7. **Prerequisites.** One semester of abstract algebra (group theory). Some experience with topology or geometry would be helpful, but certainly not required.

3. Riemann surfaces and dessins d'enfants

3.1. **Mentor.** Aaron Calderon (`aaron.calderon@yale.edu`).

3.2. **Primary Text(s).**

- Introduction to Compact Riemann Surfaces and Dessins d'Enfants (`http://www.cambridge.org/us/academic/subjects/mathematics/geometry-and-topology/introduction-compact-riemann-surfaces-and-dessins-denfants?format=PB&isbn=9780521740227`)

3.3. **Description.** Riemann surfaces (which equivalently masquerade as algebraic curves, 1-D complex manifolds, hyperbolic surfaces, or Fuchsian groups of finite type) are one of the most fundamental objects of all of mathematics. Sitting at the intersection of algebraic geometry, complex geometry, geometric topology, number theory, and Galois theory, they provide a glimpse into many different fields, and their relatively low complexity allows for fascinating connections between all of these fields.

This course will study the development of the theory of Riemann surfaces up through Grothendieck's *dessins d'enfants*, or "childs' drawings." These combinatorial objects allow one to encode the algebraic structure of a Riemann surface defined over a number field in a very small amount of data, and by a theorem of Belyi, *every* such structure arises from this sort of data. Along the way to understanding this theorem, we will trace through the development of the theory of Riemann surfaces, with a particular emphasis on understanding their connections to other areas of mathematics.

3.4. **Syllabus.** We will begin by understand the equivalences between Riemann surfaces, (complex) algebraic curves, and hyperbolic surfaces, with particular focus on the examples in the book. We will also spend some time investigating the Galois action on a (branched) cover of a Riemann surface. From there we will cover Belyi's theorem, and if time permits, begin to understand the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of dessins.

3.5. **Primary Goals.** The goal of the project is to understand exactly what a dessin d'enfant *is*. The main mathematical content of the project is to prove Belyi's theorem relating dessins d'enfants and branched covers of the Riemann sphere.

3.6. **Expected Outputs.** By the end of the project, I hope that the students will have produced an expository article written at the advanced undergraduate level (think Math Magazine).

3.7. **Prerequisites.** Complex analysis, abstract algebra (groups, fields and field extensions). Ideally the student(s) will also have some knowledge of topology (fundamental groups, covering spaces) and Galois theory. Some basic notions from hyperbolic and/or algebraic geometry would come in handy, but neither are required.

4. Lattices in Lie groups

4.1. **Mentor.** Alex Rasmussen (`alexander.rasmussen@yale.edu`)

4.2. **Primary Text(s).**

- *Introduction to Arithmetic Groups* by Dave Morris.

4.3. **Description.** Lattices are discrete subgroups of Lie groups that in some sense capture the structure of the Lie groups that they sit inside of. They are fundamental objects of study in mathematics. Being fundamental objects, lattices have been studied for well over 100 years. However, some of the most striking properties of lattices were discovered only more recently by researchers including (Yale's own) Grigory Margulis. They proved numerous important theorems about the algebraic, dynamical, and geometric properties of lattices in higher rank semisimple Lie groups. Among these are the normal subgroup theorem (a strong restriction on the kinds of normal subgroups a lattice can have), Margulis superrigidity (which says roughly that finite dimensional representations of a lattice extend to the entire ambient Lie group), and Property (T) (a restriction on the types of probability measure-preserving actions a lattice can have).

In this project I would like to discuss the basic properties of lattices with the goal of eventually understanding some of the above theorems. Morris's book is an excellent place to learn this material. It covers a large amount of material while avoiding the many technicalities present in other books. It is very readable and contains numerous exercises to guide the reader's understanding. If the students are interested, we could also discuss mapping class groups, another geometrically interesting class of groups that in some senses resemble lattices in higher rank Lie groups, and in other senses diverge strongly from them.

4.4. **Syllabus.** The schedule will be flexible. In the first two months I would like to get through most of Part II of Morris's book, which discusses basic properties and constructions of lattices. In the last month or so, I would like to discuss some of the tools from Part III of Morris's book, and one or two of the major results from Part IV.

4.5. **Primary Goals.** The main goal is to understand at least one of the striking theorems on lattices in higher rank semisimple Lie groups. Which theorem(s) we try to tackle will depend on the tastes and background of the readers. Ideally, I would also like discuss mapping class groups, and how some of the main results apply or do not apply to them.

4.6. **Expected Outputs.** It will depend on the tastes of the students, but it would be good write a short expository paper, maybe discussing a bit the analogy between lattices and mapping class groups.

4.7. **Prerequisites.** The student should have some knowledge of the basics of Lie groups. This includes the definition and basic examples of Lie groups, Lie algebras, the exponential map, and the adjoint representation.

5. $CAT(0)$ CUBE COMPLEXES, COXETER GROUPS, AND THE DAVIS COMPLEX

5.1. **Mentor.** Alex Rasmussen (`alexander.rasmussen@yale.edu`)

5.2. **Primary Text(s).**

- *The Geometry and Topology of Coxeter Groups* by Michael Davis.

5.3. **Description.** Cube complexes are metric spaces built out of Euclidean cubes glued along their faces. They are thus in a sense simple combinatorial objects. On the other hand, they can have complicated geometric structures. They have become ubiquitous examples in geometry and topology in the last twenty years and have been used to prove spectacular results about groups, metric spaces, and manifolds. As an example, the virtual Haken conjecture says that most compact 3-manifolds have finite covers that contain nicely embedded surfaces. This statement has nothing to do with cube complexes, and yet Ian Agol proved it in 2012 using the technology of cube complexes – a result that won him the 2016 Breakthrough Prize in Mathematics.

The key ingredient that allowed Agol to prove his result is a construction that associates to any group acting on a space that is cut up into half spaces in many different ways, an action on a $CAT(0)$ cube complex. For the first part of this project I would like us to understand several classes of groups via their actions on $CAT(0)$ cube complexes. These classes include right-angled Artin groups and right-angled Coxeter groups.

For the latter part of the project I would like to look at Coxeter groups more generally. Coxeter groups are a classically studied class of groups which generalize groups generated by reflections through hyperplanes in Euclidean space. Such groups do not act on $CAT(0)$ cube complexes in general. However, they act on related $CAT(0)$ spaces. I would like for us to understand and discuss the construction of these spaces.

All of these topics are prominent in current research in geometry and topology. If the students are interested, we can also discuss some of the results that have appeared, as well as some open questions.

5.4. **Syllabus.** In the first month or so, I would like to discuss some basics of geometric group theory and to read through part of Jason Manning's notes for background on $CAT(0)$ cube complexes. After that we will read through part of Davis's book to understand Coxeter groups and their Davis complexes.

5.5. **Primary Goals.** The main goal is to understand various groups via their actions on $CAT(0)$ complexes. Ideally we will understand these concepts well enough to discuss some current results in geometric group theory and topology.

5.6. **Expected Outputs.** It will depend on the tastes of the students, but it would be good write a short expository paper, illustrating some examples of right-angled Artin groups and their Salvetti complexes, and Coxeter groups and their Davis complexes.

5.7. **Prerequisites.** It would be helpful for the students to have some knowledge of the fundamental group of a topological space. It would also be helpful if the students have seen the hyperbolic plane before. We can review these concepts as necessary.

6. Basics of structures and representations of Lie algebras

6.1. **Mentor.** Yaochen Wu (`yaochen.wu@yale.edu`)

6.2. **Primary Text(s).**

- GTM 9 Introduction to Lie Algebras and Representation Theory;
- GTM 129 Representation Theory: A First Course

6.3. **Description.** The theory of Lie algebras is a classical topic, yet it is fundamental to many branches of current researches in mathematics and physics. The aim of this reading project is to provide the student(s) basic knowledge about the structure of semisimple Lie algebras over an algebraically closed field of characteristic 0 (e.g. $\mathbb{C}$), and essential results of their representations, e.g. the highest weight theorem. Important constructions, including Serre's relations, universal enveloping algebra and PBW basis will be developed too. These are preliminaries for many advanced and modern topics, such as Kac-Moody algebra and quantum groups.

6.4. **Syllabus.** (Depending on the students' progress) First term: Basic definitions of Lie algebra; Engel's and Lie's theorem; the Killing form; criterion for semisimplicity; Weyl's theorem; representations of type A_1; root space decomposition. Second/third term: Abstract root spaces; Cartan subalgebras and conjugacy theorems; PBW theorem; Serres' theorem; multiplicity formula and character formula. Classical Lie algebras will be main examples.

6.5. **Primary Goals.** This project is kind of elementary: to gain some familiarity with Lie algebras and their representations, and see how various concepts are realized in classical Lie algebras. It would be ideal if the mentee(s) try to understand some of the technical proofs.

6.6. **Expected Outputs.** I want my mentee(s) to work through some examples and computations of representations of classical Lie algebras, as well as their structures. They will illustrate many concepts (e.g. the Killing form, root systems) and formulas.

6.7. **Prerequisites.** Linear algebra (abstract vector spaces, Jordan canonical form, tensor products etc); some knowledge about groups, rings and modules, and fields. Knowledge about representations of finite groups or Lie groups is not really necessary.

7. High Dimensional Probability: Theory and Applications

7.1. **Mentor.** George Linderman (`george.linderman@yale.edu`)

7.2. **Primary Text(s).**

- High Dimensional Probability by Roman Vershynin

7.3. **Description.** A large number of statistical and computational techniques in "data science," particularly those applied to high dimensional problems, have rich mathematical theory behind them. The goal of this reading course will be to explore the theory behind some of these methods. More details can be found in Vershynin's book, which we will follow closely. Student(s) will be encouraged to perform numerical experiments for certain theorems, which can give crucial insight into how they work.

7.4. **Syllabus.** Progress depends on the students. I would expect to go through the first 4-6 chapters of the book.

7.5. **Primary Goals.** Short-term goal (one semester): To cover the first 4-6 chapters, including concentration inequalities, random vectors in high dimensions, random matrices, and concentration without independence. Progress would depend significantly on the level and interest of the student(s).

Long-term goal (two semesters): Second semester would be catered to the interest of the student(s). Two options would be to either finish Vershynin's book or switch to An Introduction to Matrix Concentration Inequalities by Joel Tropp.

7.6. **Expected Outputs.** For example ,students could apply techniques and approaches that we cover to real-life datasets.

7.7. **Prerequisites.** Strong background in probability theory and linear algebra. Some rudimentary functional analysis (e.g. some familiarity with Hilbert spaces).

8. Introduction to Tropical Geometry

8.1. **Mentor.** Shiyue Li (`shiyue.li@yale.edu`)

8.2. **Primary Text(s).**

- *Introduction to Tropical Geometry* by Diane Maclagan and Bernd Sturmfels.

8.3. **Description.** Tropical Geometry is an emerging subfield of algebraic geometry. Loosely speaking, it provides a technique to degenerate algebraic varieties with combinatorial objects. Classical algebraic geometers study the interplay between polynomials and their zeros. But zeros could be hard to study. In 1990s, people discovered that transforming all our polynomials into tropical semiring, which is $\mathbb{R} \cup \{\infty\}$ with the usual addition and multiplication replaced with taking the min/max and addition, will turn polynomials into piecewise linear functions and their zeros into polyhedra. Now we can do combinatorics to tackle these algebraic geometry problems.

8.4. **Syllabus.** We will be mainly following Chapter 1 and 2 of the main text. If time permits, we will read selected sections in Chapter 3 and 4 on Grassmannians, matroid, and hyperplane arrangements.

8.5. **Primary Goal(s).** The primary goal is to understand basic arithmetic in tropical semiring, tropical plane curves, Bezout's Theorem for tropical plane curves, tropical varieties, connections to Grassmannians, matroids and hyperplane arrangements.

8.6. **Expected Output.** We will work out selected exercises and examples of the text over the term. Students can choose from writing expository paper or present on a result that they like, within or outside the text, is related to the reading and reflects their understanding of the text.

8.7. **Prerequisites.** Ring and field theory, commutative algebra, projective geometry, some basic knowledge about algebraic geometry will be useful but we can talk about what we need as we go.

9. Arithmetic of Elliptic Curves

9.1. **Mentor.** Shiyue Li (`shiyue.li@yale.edu`)

9.2. **Primary Text(s).** *The Arithmetic of Elliptic Curves* by Joseph Silverman.

9.3. **Project Description.** The study of Diophantine equations, or finding the integral or rational solutions of polynomial equations has been fascinating to mathematicians since ancient Greece. On the geometry side, families of polynomial equations describe algebraic varieties which are geometric objects whose properties can be studied using geometric tools. Understanding arithmetic properties of algebraic curves of genus one or elliptic curves has been very central and fruitful in solving Diophantine equations. In this course, we will be starting off with basic theory of algebraic curves in the language of algebraic geometry such as maps between algebraic curves, divisors, differentials and so on. We will then dive into the geometry of elliptic curves and touch the surface of Mordell-Weil Theorem that tells us about structure of rational points on these elliptic curves if time permits.

9.4. **Syllabus.** We will roughly follow Silverman's book and skip sections as we see necessary.

9.5. **Primary Goals.** The main goal is to have some foundational knowledge about algebraic curves of genus 1 in the language of algebraic geometry, and to understand divisors, differentials, Riemann-Roch Theorem, Frobenius maps, and group law of elliptic curves. The students will be equipped with the language to read further on the related subjects.

9.6. **Expected Output.** We will work out selected exercises and examples of the text over the term. Students can choose from writing expository paper or present on a result that they like, within or outside the text, is related to the reading and reflects their understanding of the text.

9.7. **Prerequisites.** Ring and field theory, a course in commutative algebra or the equivalents, projective geometry, some basic knowledge about algebraic geometry, and algebraic number theory will be useful but we can talk about what we need as we go.

10. Basic Theory of Affine Group Schemes

10.1. **Mentor.** Elad Zelingher (`elad.zelingher@yale.edu`)

10.2. **Primary Text(s).**

- Basic Theory of Affine Group Schemes - James Milne, (`https://www.jmilne.org/math/CourseNotes/AGS.pdf`

10.3. **Description.** (Edited from Milne's site) The notion of an algebraic group is fundamental for many topics in math. The goal of this project to learn the essential theory of algebraic group schemes (especially reductive groups) with the minimum of prerequisites and the minimum of effort. In particular, it should not be necessary to learn the subject twice, once using 1950s style algebraic geometry, and then again using modern algebraic geometry. Nor should it be necessary to read EGA first.

10.4. **Syllabus.** We follow Milne's text, concentrate on examples and exercises.

10.5. **Primary Goals.** The primary goal is to understand the fundamental notions in algebraic group theory, via affine group schemes. These include affine schemes, affine group schemes, subgroups, normal subgroups, center. The associated Lie algebra. Eventually, our goal is to understand what is a reductive group. This should open the door for further studies (for instance the correspondence root datums and reductive groups).

10.6. **Expected Output.** I expect the students to be able to study more advanced topics in algebraic group theory after this course. Another output is solutions for some of Milne's exercises.

10.7. **Prerequisites.** A course on commutative algebra (for instance Math 380).

11. Differential Geometry

11.1. **Mentor.** Fernando Al Assal.

11.2. **Primary Text(s).**

- **Track 1:** *Differential Geometry of Curves and Surfaces* by Do Carmo.
- **Track 2:** *Riemannian Geometry*

11.3. **Description.** Differential geometry is an old and important area of math, which studies shapes (manifolds) in which you can measure angles and distances. A potential track of study is to focus on 2-dimensional manifolds (surfaces) and concretely understand their geometry, with plenty of computations. More advanced students can jump in the study of general Riemannian manifolds.

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Track 2: More advanced students can jump in the study of general Riemannian manifolds

11.4. **Primary Goals.** Short-term goals: **Track 1:** Curves, surfaces, notions of curvature and the Theorema Egregium of Gauss. **Track 2:** Understanding Riemmanian metrics, Riemannian connections, geodesics and curvature, with plenty of examples.

Long-term goals: **Track 1:** Gauss-Bonnet theorem, complete surfaces and the theorem of Hopf-Rinow, variations of energy, Jacobi Fields etc. **Track 2:** Similar as above in the Riemannian context, also Bochner formulae, the sphere theorem and topics chosen by the student.

11.5. **Prerequisites.** **Track 1:** Multivariable calculus at a rigorous level and linear algebra. **Track 2:** Notions of differentiable manifolds.

12. Stochastic Processes

12.1. **Mentor.** Fernando Al Assal.

12.2. **Primary Text(s).**

- *Introduction to Stochastic Processes* by Lawler.
- *An Introduction to Stochastic Differential Equations* by Evans.
- Online lecture notes by Lalley.

12.3. **Description.** In this project we study random processes evolving with time! These objects are important in pure as well as applied math. On the one hand, we can study processes that evolve with a tick of a discrete clock, such as Markov chains — random processes whose evolution only depends on the previous instant. Examples include random walks and some ecological models. On the other hand, with more analytical machinery (or faith in such machinery) we can study processes that evolve continuously, such as Brownian motion, which models things such as the price of a stock and can also be used to prove theorems in complex analysis (!).

12.4. **Primary Goals. Track 1:** Markov chains, discrete-time martingales, optimal stopping problems and a satisfying glimpse of Brownian motion and stochastic differential equations. **Track 2:** Discrete and continuous-time martingales, construction of Brownian motion, its relation to harmonic functions and complex analysis, stochastic differential equations and Itô's formula. Long-term goals could include the Girsanov theorem, Lyons' theory of rough paths etc.

12.5. **Prerequisites. Track 1:** Multivariable calculus and linear algebra. **Track 2:** Real and functional analysis; probability theory helpful but strictly optional.