Lyapunov exponents, Schrödinger cocycles, and Avila's global theory.

W. Schlag (Yale University)

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Spectrum of periodic Schrödinger operators

Periodic self-adjoint operator

$$(H\psi)_n = \psi_{n+1} + \psi_{n-1} + v_n\psi_n, \quad n \in \mathbb{Z}$$

with v_n a real-valued periodic potential, e.g. $v_n = f(x + np/q)$ and $f : \mathbb{T} \to \mathbb{R}$ measurable, x fixed.

Gaps and bands: Floquet operator

$$S_q(E) = \prod_{j=q}^{1} \begin{bmatrix} v_j - E & -1 \\ 1 & 0 \end{bmatrix} \in SL(2, \mathbb{R}), \quad \Delta_q(E) := \mathrm{trace} S_q(E)$$

Spectrum equals all E for which $S_q(E)$ is elliptic, i.e., $|\Delta_q(E)| \leq 2$. These are the bands, separated by gaps (can collapse).

Bloch-Floquet waves $\psi(n) = a(n, E)e^{ik(E)n}$ solve $H\psi = E\psi$, where $e^{\pm ik(E)}$ eigenvalues of $S_q(E)$, amplitude $a(n + q, E) = a(n, E)$.

Spectrum of ergodic Schrödinger operators

Consider self-adjoint operators

$$(H_x\psi)_n = \psi_{n+1} + \psi_{n-1} + v_n(x)\psi_n, \quad n \in \mathbb{Z}$$

with $v_n(x)$ an "ergodic potential", i.e., $v_n(x) = V(T^n x)$ and $T : X \to X$ ergodic transformation on a probability space X, $V : X \to \mathbb{R}$ measurable. There exists *fixed* compact set $K \subset \mathbb{R}$ with $\mathrm{spec}(H_x) = K$ for a.e. $x \in X$: ergodic theorem and property of the spectral resolution E_x of H_x

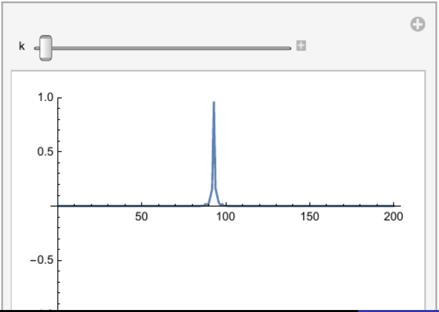
$$E_x = S^{-1} \circ E_{Tx} \circ S, \quad S = \text{ right shift}$$

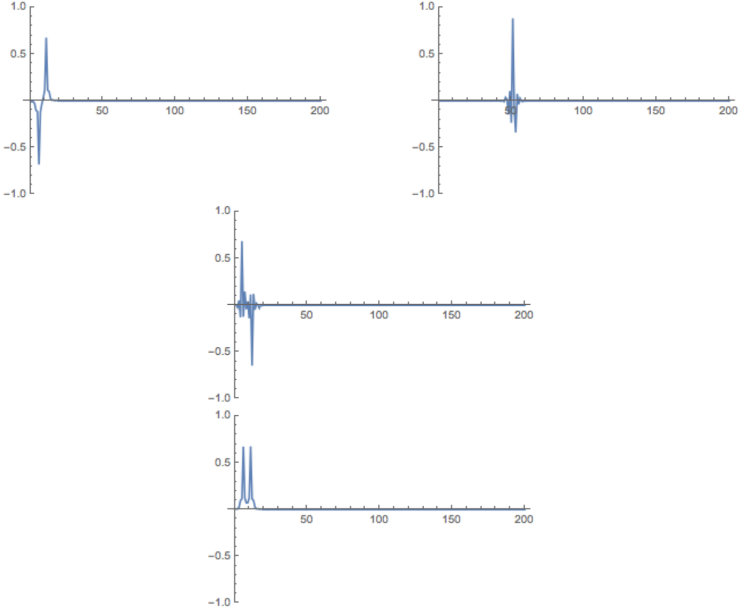
Moreover, $\mathrm{spec}_{ac}(H_x), \mathrm{spec}_{pp}(H_x), \mathrm{spec}_{sc}(H_x)$ deterministic. Eigenvalues are **NOT** deterministic, but their **closure** is. Anderson localization means that $\mathrm{spec}_{pp}(H_x) = \mathrm{spec}(H_x)$, eigenfunctions decay exponentially. *Open problem:* Anderson conjecture on extended states in 3-dim. random model.

Numerically computed eigenfunctions

```
In[404]:= n = 200; epsilon = 0.3; omega = Sqrt[2]; theta = -17 * omega / 2;

In[405]:= Laplace = epsilon * Total[
            {DiagonalMatrix[Array[-1 &, n - 1], 1], DiagonalMatrix[Array[-1 &, n - 1], -1]}];

In[406]:= f = Cos[2 * Pi * (theta + # * omega)] &;

In[407]:= V = DiagonalMatrix[Array[f, n]];

In[408]:= H = Laplace + V;

In[409]:= With[{eigs = Eigenvectors[H]},
           Manipulate[ListLinePlot[eigs[[k]], ImageSize → 4 * 72, PlotRange → {-1, 1}],
            {{k, 1}, 1, Length[eigs], 1}]]
```

Out[409]=




1.0
0.5
−0.5
−1.0
50
100
150
200

Let V be analytic, real-valued on $\mathbb{T}^d$, and $Tx := x + \omega$ ergodic shift. Consider the Schrödinger equation on $\mathbb{Z}$

$$(H_x\, \psi)(n) = -\psi(n+1) - \psi(n-1) + V(T^n x)\psi(n) = E\psi(n) \quad (1)$$

Rewrite as a system (linear cocycle):

$$\begin{bmatrix} \psi(n+1) \\ \psi(n) \end{bmatrix} = A(T^n x, E) \begin{bmatrix} \psi(n) \\ \psi(n-1) \end{bmatrix},$$

$$A(x,E) = \begin{bmatrix} V(x) - E & -1 \\ 1 & 0 \end{bmatrix}.$$

Propagator $M_n(x,E) := A(T^n x, E) \ldots A(Tx, E)$. Lyapunov exp.:

$$L_n(E) := \frac{1}{n} \int_{\mathbb{T}^d} \log \|M_n(x,E)\| \, dx$$

Subadditivity: $L_n(E) \to L(E)$ exists. Since $\det A = 1$, one has $L(E) \geq 0$. Pointwise (Fürstenberg-Kesten): for a.e. x

$$L(E) = \lim_{n\to\infty} n^{-1} \log \|M_n(x,E)\|$$

Localization via Oseledec

How to establish AL? Assume positive Lyapunov exponent $\inf_E L(E) > 0$, ω fixed irrational.
By *multiplicative ergodic theorem (Oseledec theorem)*, for every energy and almost every $x \in \mathbb{T}$ there exist directions $\mathbf{v}_x^{\pm}(E)$ which are **contracting** as $n \to \pm\infty$, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \log \|M_n(x,E)\mathbf{v}_x^{+}(E)\| = -L(E) \tag{2}$$

and same for $n \to -\infty$. If these directions coincide we obtain a globally exponentially decaying solution. If these directions do not coincide, then on one side the solution will grow exponentially, **and thus the energy will not belong to the spectrum.**
Conclusion: The spectrum consists purely of eigenvalues with exponentially decaying eigenfunctions. So why haven't we proved AL?
Fallacy: We need to remove the zero measure sets in x for all energies. This is not allowed.

Oseledec theorem

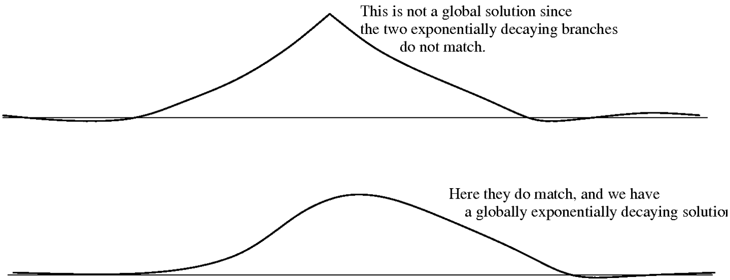


Figure: The solutions of $H_x\psi = E\psi$

Localization theorem by Bourgain-Goldstein

"Localization is a game of resonances" (Phil Anderson). What this means is that we need to make sure that we cannot have infinite tunneling, as this would lead to extended states.

Theorem (Bourgain-Goldstein, Annals, 2000)

Let V be a trigonometric polynomial. Assume that $L(E, \omega) > 0$ for all (ω, E). Then for almost every $\omega \in \mathbb{T}$ the operator H_0 exhibits AL.

Assumption holds for $V(x) = \lambda f(x)$, trigonometric polynomial, λ large by Herman's '83 subharmonic argument. Random i.i.d. case: was known long before, Fürstenberg's product of random matrices theorem '60s, Goldsheid, Molchanov, Pastur '77, Froehlich, Spencer '83 AL for large random potentials in any dimension by multi-scale method, Aizenman, Molchanov '90 fractional moments

remarks on the localization theorem

- Froehlich, Spencer, Wittwer '90: perturbative KAM-type proof of AL for even cosine-like potentials for large disorder, first and second order perturbation theory for eigenvalues, eigenfunctions. Location of resonances known exactly: $\theta = -k\omega/2 \mod 1$
- Forman, VandenBoom '21: removed even assumption.
- Preceded by seminal key result by S. Jitomirskaya for Harper operator, $V(x) = \lambda \cos(2\pi x)$, $|\lambda| > 1$
- By Fubini, we also have AL for almost every (ω, x) for H_x.
- In the argument, eliminate zero measure set from Diophantine class precisely to prevent tunneling (eliminate double resonances).
- Extends to more than one frequency, but hard problem to extend to other dynamics such as the skew shift (only know for large disorder).

A quantitative version of subadditive ergodic theorem under a Diophantine condition: $\|n\omega\| > n^{-1}(\log n)^{-2}$ for all $n \geq n_0(\omega)$. A.e. $\omega \in \mathbb{T}$ satisfies such a condition.

Theorem

Let $\omega \in \mathbb{T}$ satisfy a Diophantine condition. Then there exists $\sigma > 0$ such that

$$\left|\left\{x \in \mathbb{T} : |n^{-1} \log \|M_n(x,E)\| - L_n(E)| > n^{-\frac{1}{4}}\right\}\right| < e^{-n^{\sigma}} \quad \text{(LDT)}$$

for all $n \geq n_0(E, V, \omega)$.

On strip around $\mathbb{R}$ $u(z) := n^{-1} \log \|M_n(z,E)\|$ subharmonic, ≥ 0, and 1-periodic, and of size $\lesssim 1$. *Riesz representation theorem*

$$u(x) = \int \log |e(x) - \zeta| \, \mu(d\zeta) + h(x) \quad \forall\, x \in \mathbb{R} \qquad (3)$$

h harmonic. Combine with almost invariance $\|u - u(\cdot + \omega)\|_\infty \leq \frac{C}{n}$.

Plot of log of norm for almost Mathieu

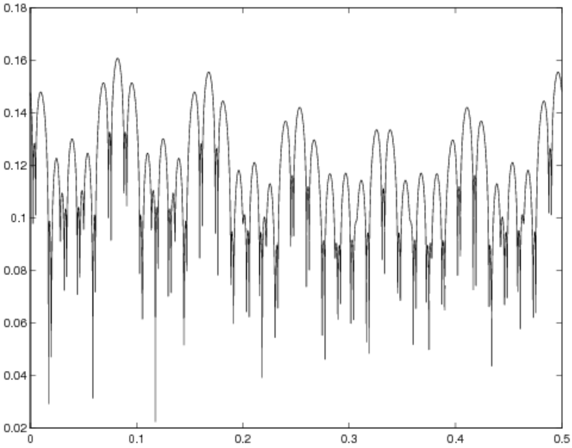


Figure: The graph of $\frac{1}{100} \log \|M_{100}(x)\|$, $\omega = \sqrt{2}$, $\lambda = 2.2$

Good (nonresonant), bad (resonant) Green functions

Definition

We say that for any interval $\Lambda \subset \mathbb{Z}$ the Green function $G_\Lambda(E,\omega) = (H_\Lambda(0,\omega) - E)^{-1}$ is **non-resonant** iff

1. $\|G_\Lambda(E,\omega)\| < e^{|\Lambda|^{b_1}}$
2. $|G_\Lambda(E,\omega)(n,m)| < \exp(-L(E)|n-m| + |\Lambda|^{b_2}) \quad \forall\, n,m \in \Lambda$

where $0 < b_1, b_2 < 1$. Otherwise, Green function is **resonant**.

For $\Lambda = [n, n+N]$,

$$H_\Lambda(0,\omega) = H_{[0,N]}(n\omega,\omega)$$

If LDT holds for $(\theta = n\omega, \omega)$, then $G_{[0,N]}(E,\theta,\omega)$ is good. Λ is resonant at energy E with probability at most e^{-N^σ}. Depends on Diophantine properties of ω.

Good and bad Green functions

The property of good Green functions is intrinsic to Anderson Localization: Indeed, suppose H_Λ has eigenbasis $\{\psi_j\}_{j\in\mathbb{Z}}$ of exponentially decaying eigenfunctions with eigenvalues E_j on some finite volume Λ. Assume

$$\mathrm{dist}(E, \mathrm{spec}(H_\Lambda)) > \exp(-|\Lambda|^b)$$

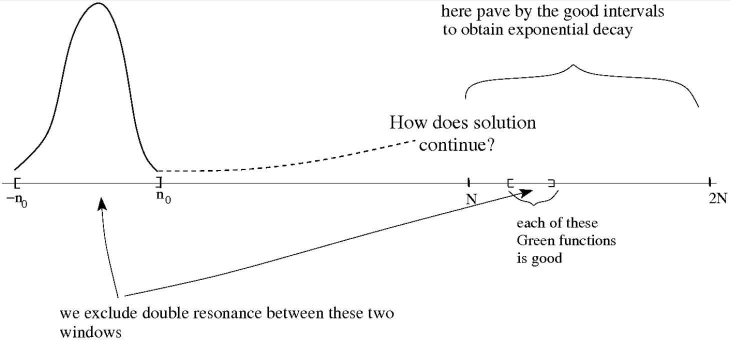
Then

$$(H_\Lambda - E)^{-1}(n, m) = \sum_j \frac{\psi_j(n)\psi_j(m)}{E_j - E}$$

satisfies

$$|(H_\Lambda - E)^{-1}(n, m)| \lesssim \exp(-\gamma|n - m| + |\Lambda|^b)$$

The AL strategy



- polynomially growing solution of $H_0\psi = E\psi$, resonance window about the origin
- exclusion of double resonances: LDT and semi-algebraic set techniques (Seidenberg-Tarski, Milnor-Thom bound on number of connected components)
- paving and resolvent identity

Removing double resonances

We need to remove the set

$$\{\omega \in \mathbb{T} : (\omega, \ell\omega) \in S_n \mod \mathbb{Z}^2 \text{ for some } N \leq \ell \leq 2N\}$$

ℓ gives position of small interval to the right of $[-n_0, n_0]$.

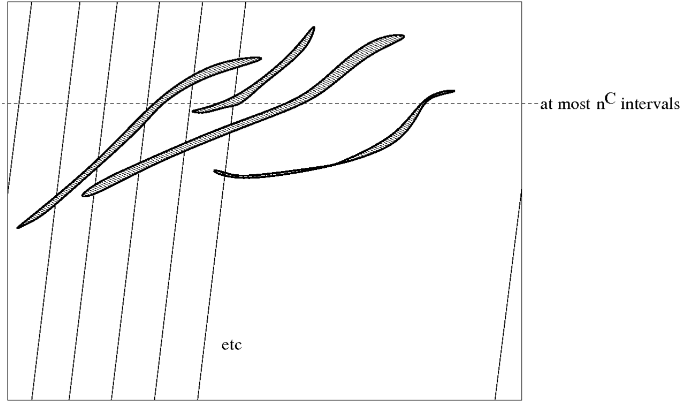


Figure: Eliminating "bad" ω

The lemma on steep lines

Lemma (Bourgain-Goldstein 2000)

Let $S \subset \mathbb{T}^2$ be a measurable set with the following properties:

- *For each $\theta \in \mathbb{T}$ the horizontal section S_θ is covered by at most M intervals.*
- $|S| < N^{-3}$, *where* $N > M$

Then

$$|\{\omega \in \mathbb{T} : (\omega, \ell\omega) \in S \mod \mathbb{Z}^2 \text{ for some } N \leq \ell \leq 2N\}|$$
$$\lesssim N^{\frac{3}{2}} |S|^{\frac{1}{2}} + MN^{-1}$$

We apply this to S_n with $M = n^B$, $N = n^{2B}$, $|S| < \exp(-n^\varepsilon)$. So we eliminate a set $\mathcal{B}_n$ of bad ω of measure $|\mathcal{B}_n| \lesssim n^{-2}$, say. This is summable, and we can apply Borel-Cantelli to conclude that we just need to remove a measure zero set.

Absolutely continuous spectrum

Theorem (Bourgain-Jitomirskaya 2002)

For $|\lambda| < \lambda_0(v)$ any Diophantine ω, a.e. x the quasi-periodic operator has absolutely continuous spectrum.

Proof based on duality (Aubry) and an analysis of the determinant in the denominator of the Green function (random walk expansion). The strongest results in this setting were obtained by **Avila, Jitomirskaya**, "*Almost localization and almost reducibility*", Journal EMS 12 (2010), 93–131. Their approach goes via conjugation of the transfer-matrix cocycle with fixed E to a constant co-cycle (this is so called reducibility). As a striking application they establish the optimal $1/2$- Hölder continuity of the Lyapunov exponent $L(E)$.

Avalanche Principle

Proposition (Goldstein-S. '99)

Let $A_1, \ldots, A_n \in SL(2, \mathbb{R})$ *so that*

$$\min_{1 \le j \le n} \|A_j\| \ge \mu > n$$

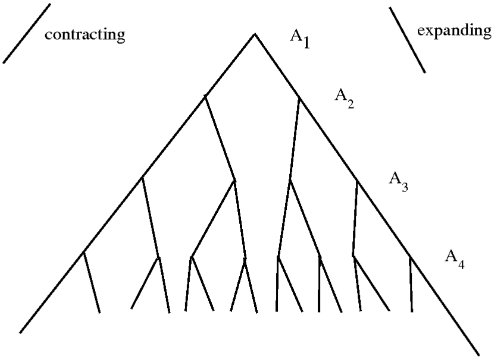
$$\max_{1 \le j < n} [\log \|A_{j+1}\| + \log \|A_j\| - \log \|A_{j+1} A_j\|] < \frac{1}{2} \log \mu$$

Then

$$\left| \log \|A_n \cdot \ldots \cdot A_1\| + \sum_{j=2}^{n-1} \log \|A_j\| - \sum_{j=1}^{n-1} \log \|A_{j+1} A_j\| \right| < C \frac{n}{\mu}$$

Obstruction $AA^{-1}AA^{-1} \ldots$ excluded. Singular value decomposition, expanding/contracting directions.
Duarte, Klein '16: more general and powerful formulation, any size matrices, largest singular value is simple (gap)

Binary structure in the proof of the Avalanche Principle



- Expanding, contracting structure in the original proof
- Duarte, Klein have a similar point of view but do not require large norm of each factor in terms of length of chain.
- reformulation in terms of quasi-geodescis in the hyperpolic plane: Oregon-Reyes '19, Sampaio '22

Rates of convergence

Averaging AP using LDT gives with $k \simeq (\log N)^C$

$$|L_{2N}(E) + L_k(E) - 2L_{2k}(E)| < \frac{k}{N}$$

same with N in place of $2N$. Subtracting we obtain

$$0 \leq L_N(E) - L_{2N}(E) < \frac{(\log N)^C}{N}$$

whence $L_N(E) - L(E) \lesssim (\log N)^C N^{-1}$ (works for any Diophantine condition). By a more careful rendition of the same argument we see that in fact

$$L_N(E) - L(E) \lesssim N^{-1} \qquad (*)$$

Moreover, convergence is uniform on any compact interval on which $L > 0$. Gives also Hölder continuity of IDS (general regularity theory by Duarte, Klein). (*) also holds when $L(E) = 0$.

Cartan estimate

How do we control the large negative values of a sub-harmonic function? Cartan's estimate reduces it to $\|\mu\| \log |z|$.

Theorem

Fix $0 < \varepsilon \leq 1$. *Let*

$$u(z) = \int_{\mathbb{C}} \log |z - \zeta| \, d\mu(\zeta) \tag{4}$$

for some positive finite measure μ. *For any* $0 < H < 1$ *there* $\exists \{D(z_j, r_j)\}_{j=1}^{\infty}$ *disjoint so that*

$$\sum_j r_j^{\varepsilon} \leq H^{\varepsilon} \tag{5}$$

$$u(z) > -\|\mu\| \left[\varepsilon^{-1} + \log \frac{1}{H} \right] \quad \forall \, z \in \mathbb{C} \setminus \bigcup_{j=1}^{\infty} D(z_j, 5r_j). \tag{6}$$

Cartan theorem

- For $P(z) = \prod_{j=1}^{n}(z - z_j)$ one has $|P(z)| \geq (H/e)^n$ outside disks D_j with $\sum_j r_j \leq 5H$. Due to maximum principle can assume that each disk contains a zero.
- Typically can set $\varepsilon = 1$. However, sending $\varepsilon \to 0$ we get that $\dim\{u = -\infty\} = 0$, where dim refers to Hausdorff dimension.
- Example 1: $\mu = n\delta_0$. Then

$$u(z) = n \log |z - 1|$$

$$|\{x \in \mathbb{T} : \frac{1}{n} u(e(x)) < -\lambda\}| \leq \exp(-\lambda)$$

- Example 2: $\mu = \sum_{j=1}^{n} \delta_{\zeta_j}$ where ζ_j are n^{th} roots of unity. Then

$$u(z) = \log |z^n - 1|$$

$$|\{x \in \mathbb{T} : \frac{1}{n} u(e(x)) < -\lambda\}| \leq \exp(-n\lambda)$$

Avila's global theory of $SL(2,\mathbb{C})$ cocycles

$A : \{1-\rho < |z| < 1+\rho\} \to SL(2,\mathbb{C})$ analytic, cocycle $\Phi : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{T} \times \mathbb{R}^2, (x,v) \mapsto (x+\omega, A(x))$. Define $A_\epsilon(x) = A(x+i\epsilon)$, Lyapunov exponent $L(A_\epsilon, \omega)$ convex in ϵ, even function in Schrödinger case $\epsilon \mapsto L(\varepsilon; E)$.

Acceleration:

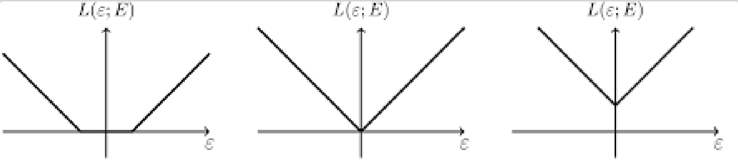
$$\kappa(A_\epsilon) := \lim_{\delta \to 0+} \delta^{-1}(L(A_{\epsilon+\delta}, \omega) - L(A_\epsilon, \omega))$$

upper-semicontinuous in ϵ.

Quantization, Avila '15: $\kappa(A_\epsilon) \in \mathbb{Z}$, uniform hyperbolicity of Schrödinger cocycle equivalent to $\kappa(\varepsilon, E) = 0$ near $\varepsilon = 0$, equivalent (Johnson's theorem '80s) to E not in the spectrum.

In the UHYP case, quantization due to interpretation of $\kappa(\varepsilon, E)$ as a winding number.

The three cases for $L(\varepsilon; E)$ in Avila's theory



(A) subcritical behavior (B) critical behavior (C) supercritical behavior

- *Subcritical* means $\|M_n(x + i\epsilon, E)\|$ grows sub-exponentially in n uniformly on $\mathbb{T} \times [-\epsilon_0, \epsilon_0]$. Almost reducibility conjecture (Avila, Jitomirskaya '11): $\|B(x + \omega)^{-1}A(x)B(x) - A_*\| < \epsilon$, almost conjugation to a constant cocycle, analytic A, B
- *Critical*: physically relevant (Harper model) but non-generic (Avila '15)
- *Supercritical:* $L(E) > 0$ and E in the spectrum
- *Phase transition, mobility edge*: increase factor in front of potential function from small to large to move from (A) to (C). Can depend on energy.

Quantization in the UHYP case

Analytic stable/unstable splitting $\mathbb{C}^2 = \ell_u(z) \oplus \ell_s(z)$, with $A(z)\ell_u(z) = \lambda(z)\ell_u(z+\omega)$, $\lambda \neq 0$. Ergodic theorem:

$$L(\varepsilon; E) = \int_{\mathbb{T}} \log |\lambda(x + i\varepsilon)| \, dx$$

Cauchy-Riemann equations relate acceleration to the winding number of λ around origin:

$$\begin{aligned} \kappa(\varepsilon; E) &= \frac{1}{2\pi} \mathrm{Re} \int_0^1 \partial_\varepsilon \log \lambda(x + i\varepsilon) \, dx \\ &= \frac{1}{2\pi} \mathrm{Im} \int_0^1 \partial_x \log \lambda(x + i\varepsilon) \, dx \\ &= \frac{1}{2\pi} \mathrm{Im} \int_{|z| = \exp(\varepsilon)} \frac{\lambda'(z)}{\lambda(z)} \, dz = N(\lambda; 0) \end{aligned}$$

In Schrödinger case, $\kappa = 0$.

Regularity and domination

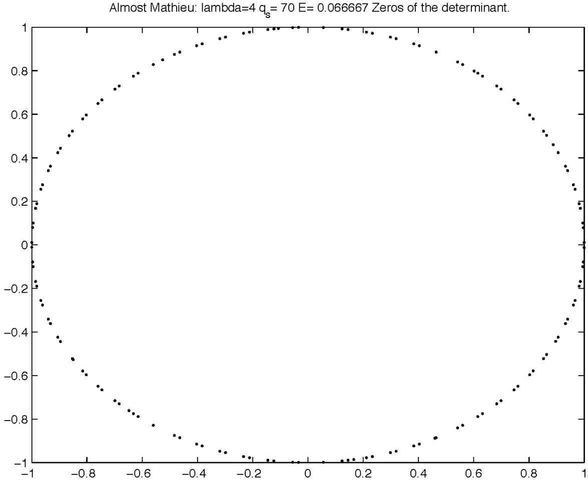
Avila, Jitomirskaya, Sadel JEMS '17

$A : \mathbb{T} \to SL(d, \mathbb{C})$, $d \geq 2$ analytic, Lyapunov exponents $L_k(\omega, A(\cdot + it))$, $\omega \in \mathbb{R} \setminus \mathbb{Q}$

- $L_k(\omega, A) > L_{k+1}(\omega, A)$ then (ω, A) is k-regular iff this cocycle is k-dominated (admits a dominated splitting)
- If $L_k(\omega, A) > L_{k+1}(\omega, A)$, then for a.e. small t one has k-domination of $(\omega, A(\cdot + it))$
- there exists $1 \leq \ell \leq d - 1$ so that $\ell\omega_k$ are integers for all $1 \leq k \leq d$
- $L_k(\omega, A)$ jointly continuous

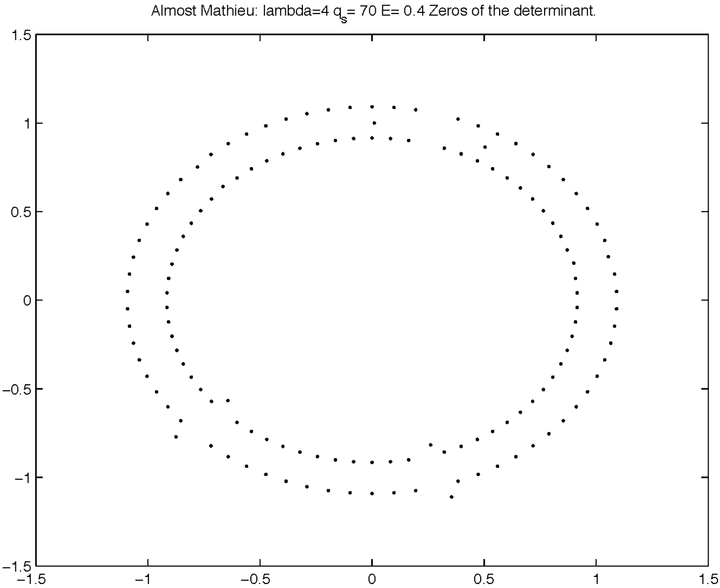
The final property allows for periodic approximations. Earlier proof of joint continuity by Bourgain, Jitomirskaya using LDT, AP technology (see book by Duarte, Klein '16).

Zeros of determinants in finite volume: E in the spectrum

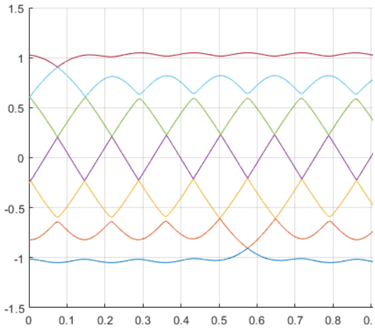


Consider zeroes of $f_N(z, E) = \det(H_{[1,N]}(z) - E)$ in the complex phase z near $|z| = 1$.

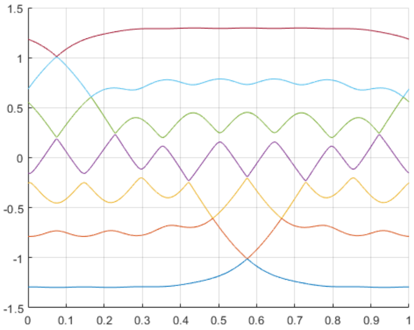
Zeros of determinants: E outside the spectrum



Zeros, Rellich graphs, spectral gaps



(a) $\varepsilon = 0.1$



(b) $\varepsilon = 0.3$

- Eigenvalue parameterizations for the cosine potential
- Apparent crossings are not crossings: simplicity of eigenvlaues
- gap formation, Cantor structure of the spectrum

Multi frequency vs. one frequency potentials

Consider potential as in Bourgain-Goldstein localization theorem, but on $\mathbb{T}^d$ with $d \geq 2$:

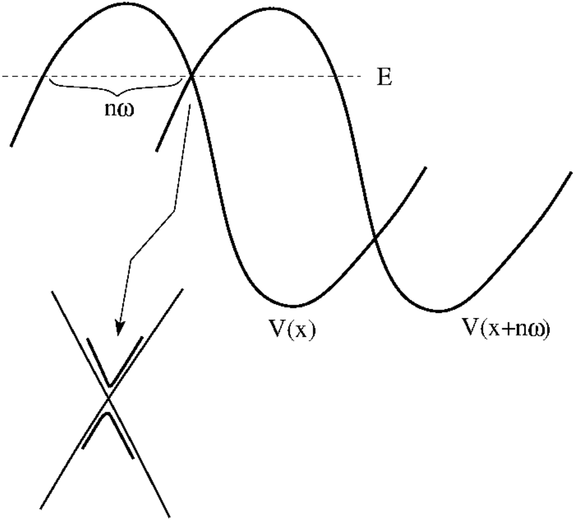
$$V_{x_1,x_2}(n) = v(x_1 + n\omega_1, x_2 + n\omega_2)$$

v analytic on $\mathbb{T}^2$. Chulaevsky-Sinai conjectured around 1988 that for typical V the spectrum is an interval, because forbidden zones cannot form.
For one frequency, these arise as follows (perturbatively):

$$\begin{aligned} &\det \begin{pmatrix} \lambda_1(x) - E & \varepsilon \\ \varepsilon & \lambda_2(x) - E \end{pmatrix} = 0, \quad \lambda_1(x_0) = \lambda_2(x_0) = E_0 \\ &E_{\pm}(x) = \frac{1}{2}(\lambda_1(x) + \lambda_2(x)) \pm \sqrt{(\lambda_1(x) - \lambda_2(x))^2 + 4\varepsilon^2} \end{aligned} \tag{7}$$

This is a reflection of the fact that for the Dirichlet problem eigenvalues are simple.

Forbidden zones



Crossing of graphs of eigenvalues creates a gap in dimension=1

Cantor spectrum, one frequency

Analytic potentials $V(x + n\omega)$ with $L(E, \omega) > 0$ a.e. ω
Goldstein-S. '08: spectrum is a Cantor set, induction on scales, constructive, uses crossings of graphs of eigenvalues. Tools: AP, sharp LDT, elimination of resonant ω via resultants, Cartan estimates, phase-energy duality, Weierstrass preparation theorem, quantitative separation of eigenvalues.

Puig 2002 for Diophantine ω, and Avila-Jitomirskaya with ω any irrational around '06 showed this for $V(x) = \lambda \cos(2\pi x)$, $\lambda \neq 0$. Solution of ten-Martini problem of Marc Kac.

Two frequencies: forbidden zones should not form (for generic potentials) because graph of $V(x)$ intersects that of $V(x + n\omega)$ in a non-horizontal curve, so all energies are still covered.
Goldstein-S-Voda '19: Spectrum is an interval for large disorder and "generic" analytic potential function, spectrum contains an interval if $L > 0$

Zeros: from regularity of the IDS to localization

Integrated Density of States (IDS): limiting distribution of the eigenvalues

$$\mathcal{N}(\lambda) = \lim_{N\to\infty} N^{-1} \int_{\mathbb{T}} \#\{1 \le j \le N \ : \ E_{j,N}(x,\omega) \le \lambda\}\, dx$$

Cantor staircase function. Hilbert transform of Lyapunov exponent, Thouless formula

$$L(E,\omega) = \int \log|E - E'|\,\mathcal{N}(dE')$$

Hölder regularity for one frequency known, see general regularity theory by Duarte, Klein via AP, LDT. Multi frequency case poorly understood, AP, LDT get worse, Hölder unknown. Natural to conjecture (almost) Lipschitz property of IDS (by a similar mechanism that leads to *spectrum=interval*, i.e., no forbidden zones)

Zeros of determinants, acceleration, and Hölder exponent of the IDS

Goldstein-S. GAFA '08: At energy E the IDS is Hölder C^α with $\alpha < \frac{1}{k_0}$ where $k_0 \geq 1$ is the largest integer with

$$k_0 \leq \lim_{\epsilon \to 0+} \limsup_{N \to \infty} N^{-1} \#\{z \in \mathcal{A}_\epsilon \,:\, \det(H_{1,N]}(z,\omega) - E) = 0\}$$

On the spectrum, this is well-defined. Off the spectrum, the IDS is constant. If potential function $V(x) = \sum_{k=-d}^{d} c_k e(kx)$, $c_{-k} = \bar{c}_k$, then $k_0 \leq 2d$.

Theorem (Rui Han-S '22): $k_0 = 2\kappa(E,0)$, Avila's acceleration.

The Hölder regularity can only be sharp on a zero Hausdorff dimensional set, representing the edges of the spectral gaps ([GS '08, '11]). Requires elimination of double resonances, and one has that the IDS is close to Lipschitz off a sparse set of energies.