1. (a) Evaluate the integral \( \int \frac{x + 1}{x^3 - 3x^2 + 2x} \, dx \).

First, observe

\[ x^3 - 3x^2 + 2x = x(x^2 - 3x + 2) = x(x - 1)(x - 2) \]

and so

\[ \frac{x + 1}{x^3 - 3x^2 + 2x} = A \frac{x}{x - 1} + B \frac{1}{x - 1} + C \frac{1}{x - 2} \]

This gives

\[ x + 1 = A(x - 1)(x - 2) + Bx(x - 2) + Cx(x - 1) \]

Then

\[
x = 1 \quad \text{gives } B = -2 \\
x = 2 \quad \text{gives } C = 3/2 \\
x = 0 \quad \text{gives } A = 1/2
\]

Then

\[
\int \frac{x + 1}{x^3 - 3x^2 + 2x} \, dx = \frac{1}{2} \int \frac{1}{x} \, dx - 2 \int \frac{1}{x - 1} \, dx + \frac{3}{2} \int \frac{1}{x - 2} \, dx \\
= \frac{1}{2} \ln |x| - 2 \ln |x - 1| + \frac{3}{2} \ln |x - 2| + C
\]

(b) \( \int x^2 \cos(x) \, dx \)

Integrate by parts twice

\[
\int x^2 \cos(x) \, dx = x^2 \sin(x) - 2 \int x \sin(x) \, dx \\
= x^2 \sin(x) - 2 \left( -x \cos(x) - \int -\cos(x) \, dx \right) \\
= x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + C
\]

For the first integration by parts, take \( u = x^2 \) and \( dv = \cos(x) \, dx \).

For the second integration by parts, take \( u = x \) and \( dv = \sin(x) \, dx \).

(c) \( \int \frac{\cos(x)}{\sin(x) \sqrt{1 + \sin^2(x)}} \, dx \)

Substituting \( u = \sin(x) \) so \( du = \cos(x) \, dx \), we find
\[
\int \frac{\cos(x)}{\sin(x)\sqrt{1 + \sin^2(x)}} \, dx = \int \frac{du}{u\sqrt{1 + u^2}}
\]

\[
= -\ln\left|\frac{\sqrt{1 + u^2} + 1}{u}\right| + C \quad \text{by 22, integral table}
\]

\[
= -\ln\left|\frac{\sqrt{1 + \sin^2(x)} + 1}{\sin(x)}\right| + C
\]

2. Say \( E_1 \) represents all people under age 60 with colorectal cancer, \( E_2 \) those under 60 without colorectal cancer, and \( A \) those with an APC mutation. The data provided are

\[ P(A|E_1) = 0.95 \quad P(A|E_2) = 0.5 \quad P(E_1) = 0.1 \quad \text{and so } P(E_2) = 0.9. \]

We are asked to find \( P(E_1|A) \). Apply Bayes’ theorem,

\[
P(E_1|A) = \frac{P(A|E_1) \cdot P(E_1)}{P(A)}
\]

To find \( P(A) \), use the law of conditioned probabilities

\[
P(A) = P(A|E_1) \cdot P(E_1) + P(A|E_2) \cdot P(E_2)
\]

Substituting in the values gives

\[
P(A) = 0.95 \cdot 0.1 + 0.5 \cdot 0.9 = 0.545
\]

Then Bayes’ theorem gives

\[
P(E_1|A) = \frac{0.95 \cdot 0.1}{0.545} = 0.174
\]

3. Apply the ratio test with \( a_n = \frac{2^n}{n^2 + 4}(x - 2)^n \). Then

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{2^{n+1}}{2^n} \frac{n^2 + 4}{(n + 1)^2 + 4} |x - 2| = 2|x - 2|
\]

Then convergence is given by \( \lim_{n \to \infty} |a_{n+1}/a_n| < 1 \), that is, \( 2|x - 2| < 1 \), and \( |x - 2| < 1/2 \), so the radius of convergence is \( R = 1/2 \). To find the interval of convergence, first find the endpoints, then test each separately.

\[
|x - 2| < 1/2, \quad \text{so } -1/2 < x - 2 < 1/2, \quad \text{so } 3/2 < x < 5/2
\]

Test the endpoints to determine the interval of convergence.
Substituting $x = 3/2$ in the series becomes $\sum (-1)^n/(n^2 + 4)$, which converges by the alternating series test.

Substituting $x = 5/2$ in the series becomes $\sum 1/(n^2 + 4)$, which converges by comparison with $\sum 1/n^2$.

Then the interval of convergence of $\sum_{n=1}^{\infty} \frac{2^n}{n^2 + 4} (x - 2)^n$ is $[3/2, 5/2]$.

4. For the system

$$\begin{align*}
\frac{dx}{dt} &= y - x^3 \\
\frac{dy}{dt} &= x - y^2
\end{align*}$$

(a) the $x$-nullcline is the curve $y - x^3 = 0$, that is, $y = x^3$. The $y$-nullcline is the curve $x - y^2 = 0$, that is, $x = y^2$.

(b) The fixed points are circled in the diagram. To find their coordinates, solve $x = y^2 = (x^3)^2 = x^6$. That is, $0 = x^6 - x = x(x^5 - 1)$. The solutions are $x = 0$ and $x = 1$. (The equation $x^5 - 1 = 0$ has 5 solutions, but only 1 is real.) The fixed points are $(0,0)$ and $(1,1)$.

(c) To test the stability of these fixed points, first compute the derivative matrix

$$DF(x,y) = \begin{bmatrix} -3x^2 & 1 \\ 1 & -2y \end{bmatrix}$$

Then we see

$$DF(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad DF(1,1) = \begin{bmatrix} -3 & 1 \\ 1 & -2 \end{bmatrix}$$

The eigenvalues of $DF(0,0)$ are $\pm 1$, so the origin is unstable because (at least) one eigenvalue is positive.
The eigenvalues of $D\vec{F}(1, 1)$ are $(-5 \pm \sqrt{5})/2$, both negative, so the fixed point $(1, 1)$ is asymptotically stable.

5. First draw the transition graph of the Markov process with this transition matrix

$$
\begin{bmatrix}
0.5 & 0.2 & 0.8 & 0 \\
0.5 & 0.8 & 0.1 & 0.9 \\
0 & 0 & 0.1 & 0 \\
0 & 0 & 0 & 0.1
\end{bmatrix}
$$

From the graph we see that states 3 and 4 eventually are emptied, leaving the population to shift between states 1 and 2, governed by the matrix

$$
\begin{bmatrix}
0.5 & 0.2 \\
0.5 & 0.8
\end{bmatrix}
$$

This is a stochastic matrix with all entries positive, so the larger eigenvalue is $\lambda = 1$. An eigenvector of $\lambda = 1$ is a solution of

$$
\begin{bmatrix}
0.5 & 0.2 \\
0.5 & 0.8
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix} = 1 \cdot 
\begin{bmatrix}
u \\
v
\end{bmatrix}
$$

That is, $u = (2/5)v$. The values of the distribution must sum to 1, so $1 = u + v = (2/5)v + v$. This gives

- fraction in state 1 = $u = 2/7$
- fraction in state 2 = $v = 5/7$

6. Recall $rr' = xx' + yy'$ and substitute in the expressions for $x'$ and $y'$ from the system

$$
x' = 2x - 2y - x(x^2 + y^2)
y' = x + 2y - y(x^2 + y^2)
$$

we obtain

$$
rr' = x(2x - 2y - x(x^2 + y^2)) + y(x + 2y - y(x^2 + y^2))
\quad = 2(x^2 + y^2) - xy - (x^2 + y^2)^2
\quad = 2r^2 - r^2 \cos(\theta) \sin(\theta) - r^4
$$
where the last equality was obtained by the polar coordinate substitution $x = r \cos(\theta)$, $y = r \sin(\theta)$. This gives

$$r' = 2r - r \cos(\theta) \sin(\theta) - r^3$$

Now certainly $-1 \leq \cos(\theta) \sin(\theta) \leq 1$. In fact, recalling $\cos(\theta) \sin(\theta) = \sin(2\theta)/2$, we get the stronger bounds $-1/2 \leq \cos(\theta) \sin(\theta) \leq 1/2$. In fact, the weaker bounds suffice.

From $-1 \leq \cos(\theta) \sin(\theta) \leq 1$ we obtain $-r \leq r \cos(\theta) \sin(\theta) \leq r$. Adding $2r - r^3$ across the inequality,

$$r - r^3 \leq r' \leq 3r - r^3$$

At $r = 1/2$, the lower bound gives $3/8 \leq r'$; at $r = 2$ the upper bound gives $r' < -2$. Then the annulus $1/2 \leq r \leq 2$ is a trapping region. Because the origin is the only fixed point, it follows from the Poincaré-Bendixson theorem that this annulus contains a limit cycle.

7. Assuming $x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots$, the condition $x(0) = 1$ gives $a_0 = 1$. To solve the equation

$$x'(t) = tx(t) + t, \quad x(0) = 1$$

we need series expressions for $x'(t)$ and for $tx(t) + t$. Differentiating term-by-term we find

$$x'(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + \cdots$$

and

$$tx(t) = a_0 t + a_1 t^2 + a_2 t^3 + a_3 t^4 + a_4 t^5 + \cdots$$

Then the series for $tx(t) + t$ is obtained from the series for $tx(t)$ by adding 1 to the coefficient of the $t$ term in that series:

$$tx(t) + t = (a_0 + 1)t + a_1 t^2 + a_2 t^3 + a_3 t^4 + a_4 t^5 + \cdots$$

Next, equate the coefficients of like powers of $t$ in the series for $x'(t)$ and the series for $tx(t) + t$.

\[
\begin{array}{c|c|c}
\hline
n & x^n & tx + t \\
\hline
0 & a_1 & a_1 = 0 \\
1 & 2a_2 & 2a_2 = 1 + 1 \text{ so } a_2 = 1 \\
2 & 3a_3 & 3a_3 = a_1 \text{ so } a_3 = 0 \\
3 & 4a_4 & 4a_4 = a_2 \text{ so } a_4 = 1/4 \\
4 & 5a_5 & 5a_5 = a_3 \text{ so } a_5 = 0 \\
5 & 6a_6 & 6a_6 = a_4 \text{ so } a_6 = 1/(6 \cdot 4) \\
\vdots & \vdots & \vdots \\
\end{array}
\]
We see all the odd subscript coefficients are 0, that is, \( a_{2k+1} = 0 \) for \( k = 0, 1, 2, \ldots \).

The even coefficients are a bit more complicated:

\[
a_2 = 1, \ a_4 = 1/4, \ a_6 = 1/(6 \cdot 4), \ a_8 = 1/(8 \cdot 6 \cdot 4), \ldots
\]

The denominators are the products of even numbers starting with 4, so factoring a 2 from each factor in the denominator, we have

\[
a_4 = a_{2 \cdot 2} = \frac{1}{2 \cdot 2} = \frac{1}{2!} \\
a_6 = a_{2 \cdot 3} = \frac{1}{2 \cdot 3 \cdot 2} = \frac{1}{2 \cdot 3!} \\
a_8 = a_{2 \cdot 4} = \frac{1}{2 \cdot 4 \cdot 3 \cdot 2} = \frac{1}{2 \cdot 4 \cdot 3!}
\]

\[
\ldots
\]

In fact, \( a_2 \) fits this pattern: \( a_2 = a_{2 \cdot 1} = 1/(2^0 \cdot 1!) \). That is, all the even subscript coefficients have the form

\[
a_{2k} = \frac{1}{2k-1 \cdot k!} = \frac{2}{2^k \cdot k!}
\]

for \( k \geq 1 \). Except we have seen that \( a_0 = 1 \), so to fit the pattern that includes the 2 in the numerator of \( a_{2k} \), we must write \( a_0 = 2 - 1 \). Then

\[
x(t) = a_0 + a_2 t^2 + a_4 t^4 + a_6 t^6 + a_8 t^8 + \ldots
\]

\[
= (2 - 1) + 2 \frac{t^2}{2} + 2 \frac{t^4}{2 \cdot 2} + 2 \frac{t^6}{3 \cdot 2} + 2 \frac{t^8}{4 \cdot 3 \cdot 2} + \ldots
\]

\[
= 2 \left( 1 + \left( \frac{t^2}{2} \right) + \frac{t^4}{2 \cdot 2} + \frac{t^6}{3 \cdot 2} + \frac{t^8}{4 \cdot 3 \cdot 2} + \ldots \right) - 1
\]

\[
= 2e^{t^2/2} - 1
\]

To check this is correct, first observe

\[
x(0) = 2e^0 - 1 = 1
\]

Next, by the chain rule

\[
x' = 2e^{t^2/2} \cdot (t^2/2)' = 2e^{t^2/2} \cdot t
\]

and

\[
x(t) = t \left( 2e^{t^2/2} - 1 \right) + t = 2e^{t^2/2} \cdot t - t + t = 2e^{t^2/2} \cdot t
\]

8. For the population equation

\[
P_{n+1} = rP_n(1 - P_n)
\]
with $r = 6$, the left side of the figure illustrates by graphical iteration that every $P_0 < a$ iterates to 0. These populations become extinct.

(b) The right side of the figure illustrates that $P_0$ near 1 iterates to a point less than $a$, thence to 0.