Counting circles and Ergodic theory of Kleinian groups

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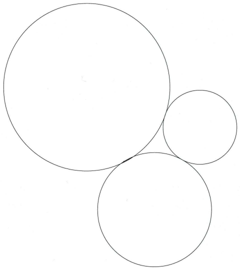
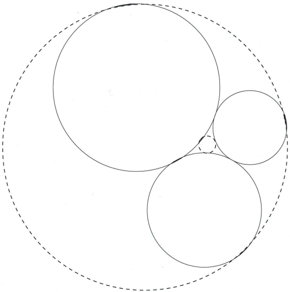
Dec 2009

Apollonius' theorem

Theorem (Apollonius 262-190 BC)

Given 3 mutually tangent circles, there are exactly two circles tangent to all three.

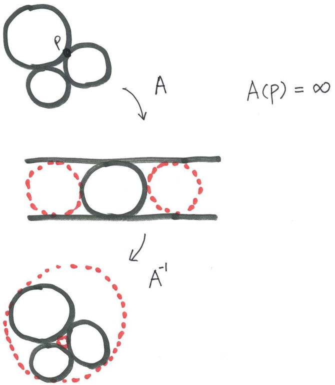




We give a proof using *Möbius transformations*: For $a, b, c, d \in \mathbb{C}$, $ad - bc = 1$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}, \quad z \in \mathbb{C} \cup \{\infty\}.$$

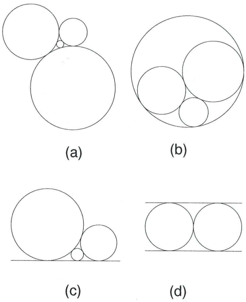
A *Möbius* transformation maps circles (including lines) to circles, preserving angles between them.



Apollonian circle packing

Start with 4 mutually tangent circles.

Four possible configurations



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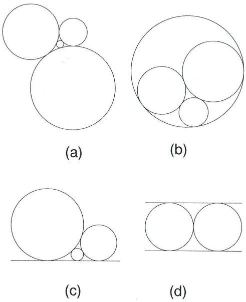

Add (four) new circles tangent to three of the initial four circles. Continuing to add newer circles tangent to three of the previous circles, we arrive at an infinite circle packing called an

Apollonian circle packing .

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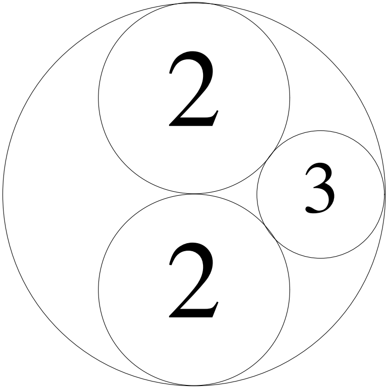
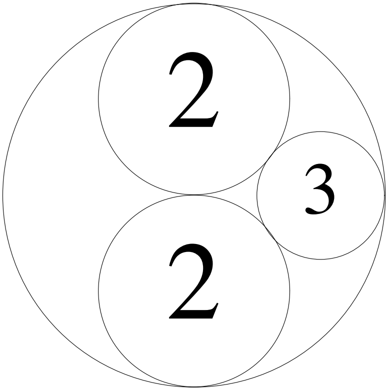
Initial stage

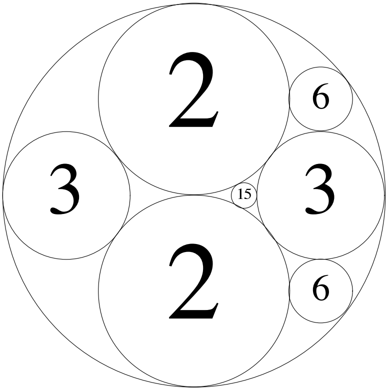


Each circle is labeled with the reciprocal of its radius (=the curvature), and the greatest circle has radius one and hence curvature -1 (oriented to have disjoint interiors).

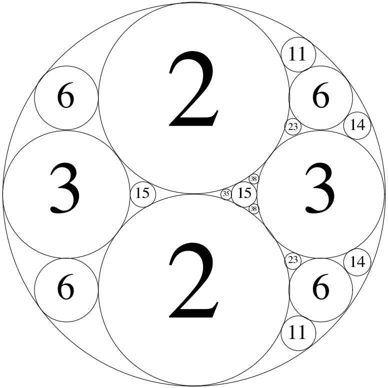
First generation



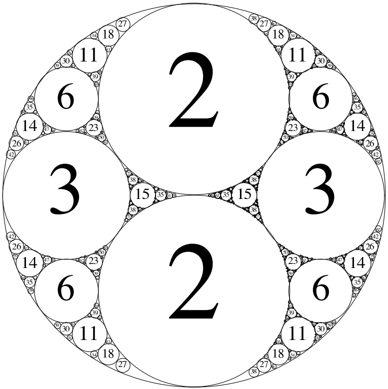
Second generation



Third generation



Example of bounded Apollonian circle packing



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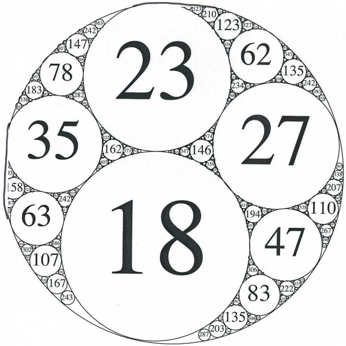

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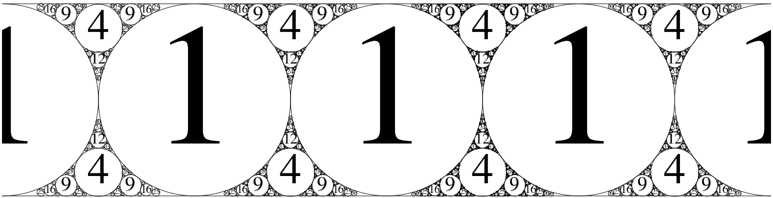


There are also other **unbounded** Apollonian packings containing either only one line or no line at all.

Counting question

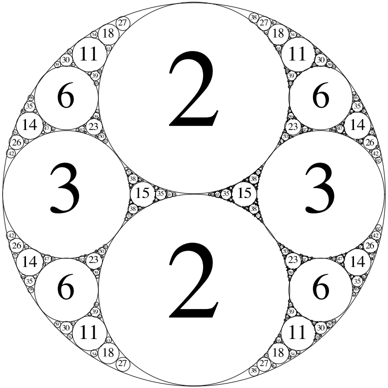
For a bounded Apollonian packing $\mathcal{P}$, there are only finitely many circles of radius bigger than a given number.
Set

$$N_{\mathcal{P}}(T) := \#\{C \in \mathcal{P} : \mathrm{curv}(C) < T\} < \infty.$$

Question

- Is there an asymptotic of $N_{\mathcal{P}}(T)$ as $T \to \infty$?
- If so, can we compute?

Apollonian circle packing

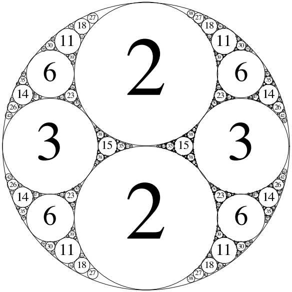


Residual set and its fractal dimension

Definition (Residual set of $\mathcal{P}$)

$$\text{Residual}(\mathcal{P}) := \overline{\cup_{C \in \mathcal{P}} C}.$$

In other words, the residual set of $\mathcal{P}$ is what is left in the plane after removing all the open disks enclosed by circles in $\mathcal{P}$.



Residual dimension

The Hausdorff dimension (or the fractal dimension) of the residual set of $\mathcal{P}$ is called the Residual dimension of $\mathcal{P}$, which we denote by δ ($1 \leq \delta \leq 2$).

We observe

1. δ is independent of $\mathcal{P}$: any two Apollonian packings are equivalent to each other by a *Möbius* transformation.
2. $\delta = 1.30568(8)$ (McMullen 1998)

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Using elementary methods, Boyd showed:

Theorem (Boyd 1982)

$$\lim_{T\to\infty} \frac{\log N_{\mathcal{P}}(T)}{\log T} = \delta.$$

Boyd also made many numerical experiments, which led him to wonder that perhaps

$$N_{\mathcal{P}}(T) \sim c \cdot T^{\delta}(\log T)^{\beta}$$

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Theorem (Kontorovich-O. 08)

For a **bounded** *Apollonian packing* $\mathcal{P}$*, there exists a constant* $c_{\mathcal{P}} > 0$ *such that*

$$N_{\mathcal{P}}(T) \sim c_{\mathcal{P}} \cdot T^{\delta}$$

where $\delta = 1.30568(8)$ *is the residual dimension of* $\mathcal{P}$*.*

Theorem (Descartes 1643)

A quadruple (a, b, c, d) *is the curvatures of four mutually tangent circles if and only if it satisfies the quadratic equation:*

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2.$$

E.g: $2((\text{-}1)^2 + 2^2 + 2^2 + 3^2) = 36 = (-1 + 2 + 2 + 3)^2$

Given three mutually tangent circles of curvatures a, b, c, the curvatures, say, d and d', of the two circles tangent to all three must satisfy

$$d + d' = 2(a + b + c).$$

So, if a, b, c, d are integers, so is d'.

Integral Apollonian packings

Corollary (Soddy 1937)

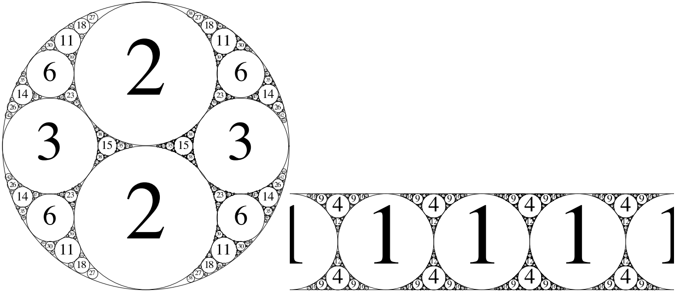
If the initial 4 circles in an Apollonian packing $\mathcal{P}$ have integral curvatures, then every circle in $\mathcal{P}$ has an integral curvature as well.

Such a packing is called integral .

By Descartes' theorem, for any integral solution of $2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2$, there exists an integral Apollonian packing!

Integral Apollonian circle packings

Any integral Apollonian packing is either bounded or lies between two parallel lines:



Primes and Twin primes

It is natural to inquire about the Diophantine properties of an integral Apollonian packing $\mathcal{P}$ such as

Question

How many circles in $\mathcal{P}$ have prime curvatures?

Assume that $\mathcal{P}$ is primitive, i.e.,$\gcd_{C \in \mathcal{P}}(\mathrm{curv}(C)) = 1$.

Definition

1. A circle is *prime* if its curvature is a prime number.
2. A pair of tangent prime circles is *twin primes*.

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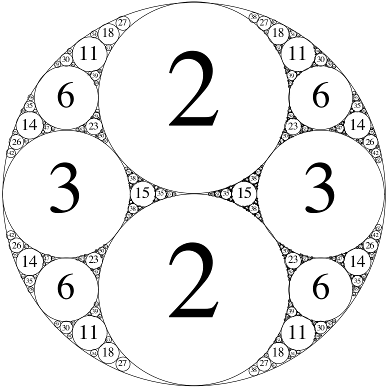
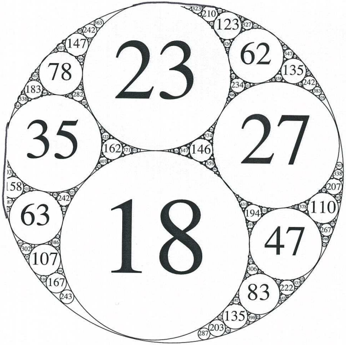


Twin primes: (2,3), (2,11), (3, 23), ...



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135

Theorem (Sarnak 07)

There are infinitely many twin primes in any primitive integral $\mathcal{P}$.

Using the recent work of Bourgain, Gamburd, Sarnak on the uniform spectral gap property together with Selberg's upper bound sieve, we prove:

Theorem (Kontorovich-O. 08)

1. $\#\{\textit{prime } C \in \mathcal{P} : \mathrm{curv}(C) < T\} \ll \frac{T^\delta}{\log T}$
2. $\#\{\textit{twin primes } C_1, C_2 \in \mathcal{P} : \mathrm{curv}(C_i) < T\} \ll \frac{T^\delta}{(\log T)^2}$

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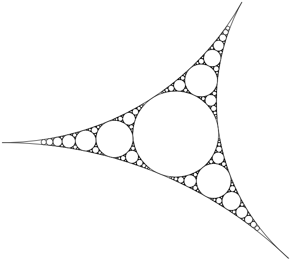
More general counting function

For unbounded Apollonian packing $\mathcal{P}$, $N_{\mathcal{P}}(T) = \infty$ in general.

Consider a curvilinear triangular region $\mathcal{R}$ in **any** Apollonian packing (either bounded or unbounded):



Set

$$N_{\mathcal{R}}(T) := \#\{C \in \mathcal{R} : \mathrm{curv}(C) < T\} < \infty.$$

Counting inside Triangle

Theorem (O.-Shah 09)

For a curvilinear triangle $\mathcal{R}$ of any Apollonian packing $\mathcal{P}$,

$$N_{\mathcal{R}}(T) \sim c_{\mathcal{R}} \cdot T^{\delta}.$$

Hidden symmetries

Question

How are we able to count circles in an Apollonian packing?

We exploit the fact that

an Apollonian packing has lots of hidden symmetries.

Explaining these hidden symmetries will lead us to explain the relevance of the packing with **"the limit set of a Kleinian group"**.

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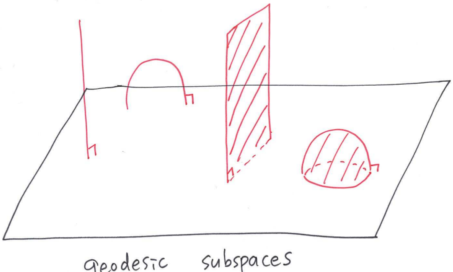
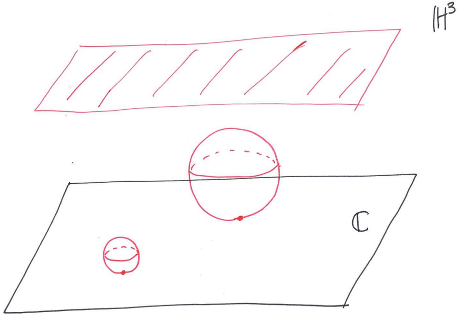
Basic notions in 3 dimensional hyperbolic geometry

Consider the upper-half space model for the hyperbolic 3 space $\mathbb{H}^3$:

$$\mathbb{H}^3 = \{(x_1, x_2, y) : y > 0\} \text{ with } \quad ds = \frac{\sqrt{dx_1^2 + dx_2^2 + dy^2}}{y}$$

and $\partial_\infty(\mathbb{H}^3) = \mathbb{C} \cup \{\infty\} = \hat{\mathbb{C}}$.

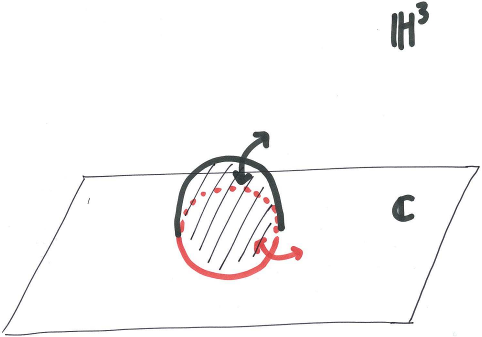




Horospheres

$$\mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}_+(\mathbb{H}^3) \quad \text{via the Poincare extension}$$

Here $\mathrm{PSL}_2(\mathbb{C})$ acts on $\hat{\mathbb{C}}$ as *Möbius* transformations and an inversion w.r.t a circle C in $\hat{\mathbb{C}}$ corresponds to the inversion w.r.t the vertical hemisphere in $\mathbb{H}^3$ bounded by C.



Kleinian group

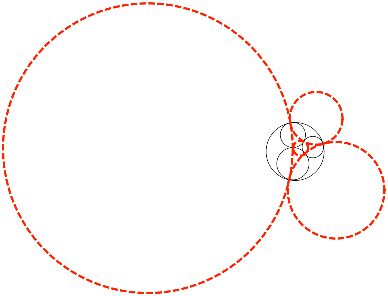
Definition

- A Kleinian group Γ is a torsion-free discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$.
- The limit set $\Lambda(\Gamma) \subset \hat{\mathbb{C}}$ is the set of all accumulation points of an orbit $\Gamma(z)$, $z \in \hat{\mathbb{C}}$.
- A Kleinian group Γ is geometrically finite if Γ has a finite sided fundamental domain in $\mathbb{H}^3$.

In fact, the residual set of an Apollonian packing is the limit set of certain geometrically finite Kleinian group, called the Apollonian group.

Apollonian group

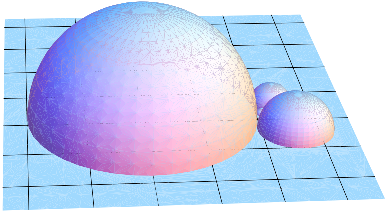
The Apollonian group $\mathcal{A}$ is gen. by inversions w.r.t the dual circles of fixed 4 mutually tangent circles in $\mathcal{P}$:



- Inverting the initial four (black) circles in $\mathcal{P}$ w.r.t the (red) dual circles gen. the whole packing $\mathcal{P}$;
- the orbit of a tangent pt under $\mathcal{A}$ gives rise to all tangent pts in $\mathcal{P}$;

$$\text{Residual}(\mathcal{P}) = \Lambda(\mathcal{A}).$$

The quotient mfd $\mathcal{A}\backslash\mathbb{H}^3$ (in fact, an orbifold) has a fund. domain given by the exterior of the four corresp. hemispheres (the domes over the 4 dual circles):



In particular, $\mathcal{A}\backslash\mathbb{H}^3$ has infinite volume.

Counting circles in bdd Apol. packing

- $$Q = 2(a^2+b^2+c^2+d^2)-(a+b+c+d)^2 : \text{ Descartes quad.form.}$$

  Has signature $(3,1)$, and hence $\mathrm{SO}(Q) = \mathrm{Isom}_+(\mathbb{H}^3)$.
- Set
  $$V = \{Q = 0\};$$
  known to be the space of horospheres in $\mathbb{H}^3$.

Let $\mathcal{P}$ be a **bounded** Apollonian packing.

1. { quad. of 4 mut. tang. circles in $\mathcal{P}$} $= \mathcal{A}(X_0)$;
2. By Descartes' thm, $\mathcal{A}(X_0) \subset V$ discrete subset
3. $N_{\mathcal{P}}(T) = \#\{X \in \mathcal{A}(X_0) : \|X\|_{\max} < T\}$.

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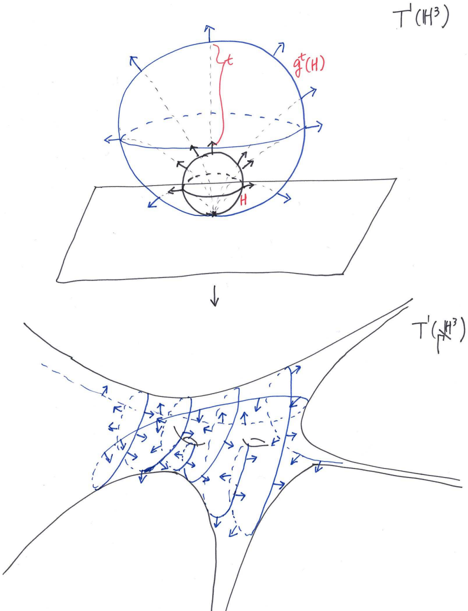
Hence the circle counting problem in $\mathcal{P}$ reduces to:

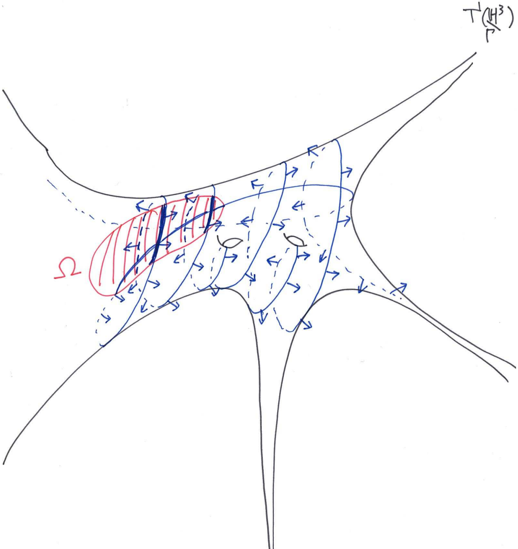
- Count a discrete orbit $\mathcal{A}(X_0)$ in the space of horospheres of $\mathbb{H}^3$; or equivalently,
- Understand the dist. of an expanding closed horosphere in the unit tangent bundle $\mathrm{T}^1(\mathcal{A}\backslash\mathbb{H}^3)$.

Expanding horosphere



Distribution of Expanding horosphere

For a cpt $\Omega \subset \mathrm{T}^1(\Gamma\backslash\mathbb{H}^3)$, how much proportion of $g^t(H)$ intersects Ω as $t \to \infty$?



Equi-distribution

If $\exists$ a Borel measure μ in $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^3)$ s.t. $\forall$ nice cpt $\Omega \subset \mathrm{T}^1(\Gamma\backslash\mathbb{H}^3)$,

$$|\Omega \cap g^t(H)| \sim \mu(\Omega)$$

we say $g^t(H)$ is **equi-distributed** w.r.t μ.

Theorem (Sarnak 81, Eskin-McMullen 93)

Let $\mathrm{Vol}(\Gamma\backslash\mathbb{H}^3) < \infty$ *and* $H \subset \Gamma\backslash\mathbb{H}^3$ *be closed horosphere. Then the expanding horospheres* $g^t(H)$ *become equi-distri. w.r.t the Liouville measure of* $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^3)$ *(=locally Riem. volume form* $\times$ *angular measure).*

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Weighted equi-distribution

Let Γ: geom. finite Kleinian gp, non-elementary (no abelian subgp of fin. index), $\delta > 0 :=$ the H. dim of $\Lambda(\Gamma)$.

Patterson-Sullivan showed that $\delta = 2$ iff $\mathrm{Vol}(\Gamma\backslash\mathbb{H}^3) < \infty$.

Theorem

For any nice cpt $\Omega \subset \mathrm{T}^1(\Gamma\backslash\mathbb{H}^3)$,

- *for $\delta < 2$, $|\Omega \cap g^t(H)| \to 0$;*
-
$$e^{(2-\delta)t} \cdot |\Omega \cap g^t(H)| \sim \mu_{BR}(\Omega).$$

Here μ_{BR} is the Burger-Roblin measure of $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^3)$: an inf. Radon meas. (unless $\delta = 2$) supp. on the union of horospheres based on the limit set $\Lambda(\Gamma)$.

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Proved by

- Kontorovich-O. for $\delta > 1$: based on the spectral theory of Laplacian (Patterson-Sullivan, Lax-Phillips)
- O.-Shah for $\delta > 0$: based on the mixing of the geodesic flow w.r.t Bowen-Margulis-Sullivan measure (Rudolph).

Going back to counting circle theorems, [Kontorovich-O.] approach relied on the Descartes theorem in translating the circle counting problem into a statement about horospheres in a hyperbolic 3 manifold.

For an unbounded packing $\mathcal{P}$, this translation is not possible, since the Descartes quadruples from $\mathcal{P}$ form a **non-discrete** subset.

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In [O.-Shah], we translate the circle counting problem into a weighted equdistribution statement about distribution of the orthogonal translates of a totally geodesic surface in a hyperbolic 3 manifold, NOT using the Descartes theorem.

This was an important point for generalizations beyond Apollonian circle packings, as there is **no analogue of Descartes theorem in general**.

Circles in the limit set of a Kleinian group

We are now led to ask even more general counting question:

Let $\mathcal{P}$ be a circle packing in the plane. That is, a union of circles with disjoint interiors.
Suppose that the residual set of $\mathcal{P}$ is the limit set of some finitely generated Kleinian group Γ.

Question

Can one count circles in $\mathcal{P}$ of curvature at most T?

Before stating a theorem, we present some pictures of the limit sets of Kleinian groups:

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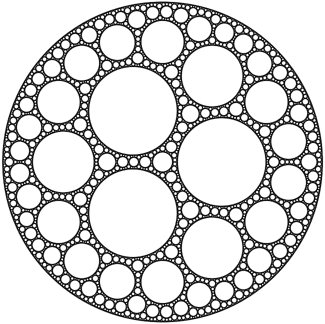
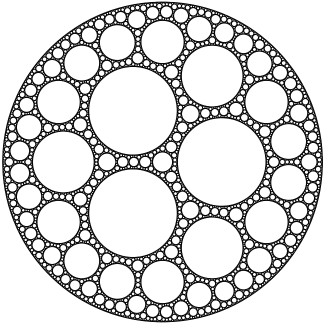
Ex. of Sierpinski curve (McMullen)



Here $\Gamma = \pi_1$(cpt. hyp. 3-mfd with non-empty tot. geod. bdry);
The limit set of an embedding of Γ into $\mathrm{PSL}_2(\mathbb{C})$ is a Sierpinski curve.

The next pictures are copied from the book "Indra's pearls" by Mumford, Series and Wright, illustrating the limits set of Schottky groups.

Ex. of Sierpinski curve (McMullen)



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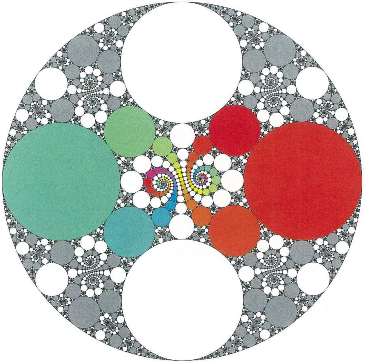
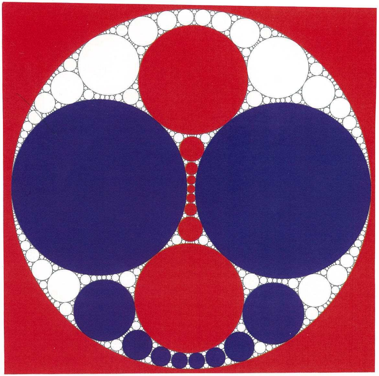
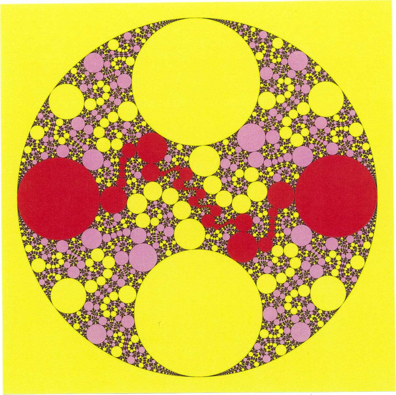
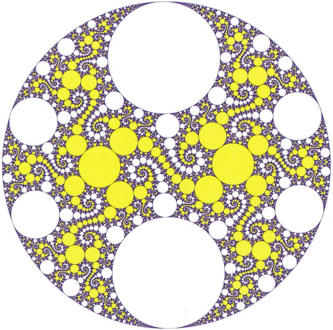
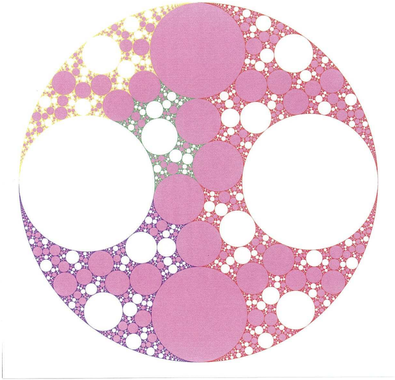
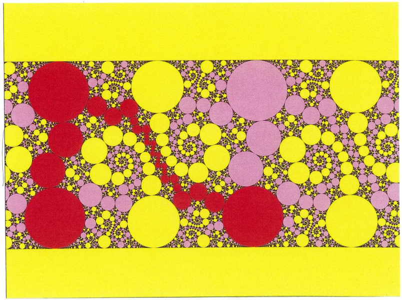






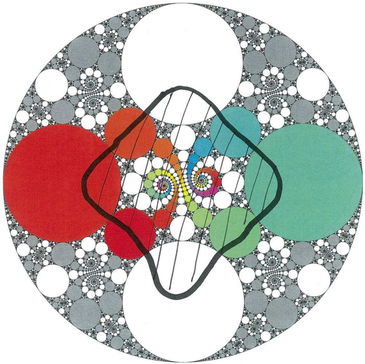


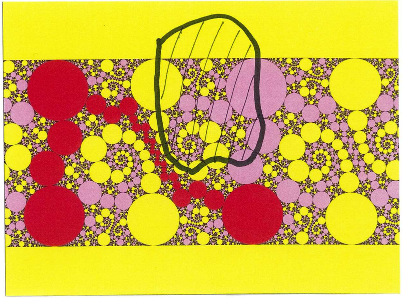




Let $\mathcal{P}$ be a circle packing whose residual set is the limit set of a finitely generated Kleinian group Γ. Let $\mathcal{R} = \mathcal{P}$ if $\mathcal{P}$ is bounded, or more generally, $\mathcal{R}$ can be any bounded domain such that

$$\#\partial(\mathcal{R}) \cap \text{Residual}(\mathcal{P}) < \infty, \;\; \mathcal{R}^{int} \cap \text{Residual}(\mathcal{P}) \neq \emptyset.$$





Theorem (O.-Shah 09)

Then

$$N_{\mathcal{R}}(T) := \#\{C \in \mathcal{R} : \mathrm{curv}(C) < T\} \sim c_{\mathcal{R}}\, T^{\delta_{\mathcal{P}}}$$

where $\delta_{\mathcal{P}}$ is the residual dimension of $\mathcal{P}$.

Circle packings on the sphere

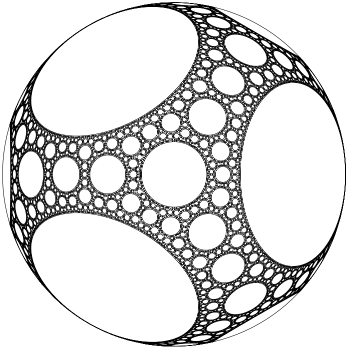
Consider the unit ball model of the hyperbolic 3 space:

$$\mathbb{B}^3 := \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 < 1\}$$

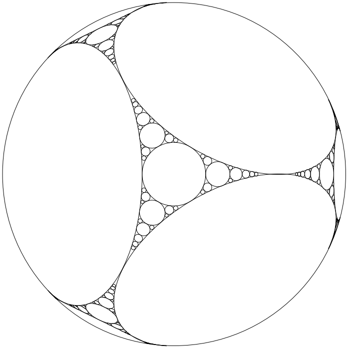
with the metric $ds = \frac{2\sqrt{dx_1^2 + dx_2^2 + dx_3^2}}{1 - (x_1^2 + x_2^2 + x_3^2)}$.

As the unit sphere S^2 is the natural boundary of $\mathbb{B}^3$, we obtain many circle packings on the sphere S^2 whose residual sets are the limit sets of Kleinian groups.

Sierpinski curve on the sphere (McMullen)



Apollonian packing on the sphere (McMullen)



Circle packings of the sphere

Given a circle packing $\mathcal{P}$ on S^2, we may ask

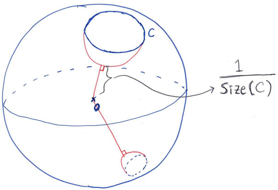
Question

How many circles can you see standing at the origin?

Here

$$\text{Visual size}(C) = d(o, \hat{C})^{-1}$$

where $\hat{C}$ is the orthogonal hemisphere bounded by C.



Theorem (O.-Shah 09)

Let $\mathcal{P}$ be a circle packing on the sphere whose residual set is the limit set of a finitely generated Kleinian group Γ.
Then

$$\#\{C \in \mathcal{P} : \mathrm{size}(C) > T^{-1}\} \sim c_{\mathcal{P}}\, e^{\delta_{\mathcal{P}} T}$$

where $\delta_{\mathcal{P}}$ is the residual dimension of $\mathcal{P}$.

Sierpinski curve on the sphere (McMullen)