Functional Analysis (325b/525b) Practice Final exam questions

Michael Magee, michael.magee@yale.edu

Question 1.

a. Prove a normed space is complete if and only if every absolutely convergent series is convergent.
b. Prove ℓ^p is complete for $1 \leq p < \infty$. (You can assume it is a well-defined normed vector space.)

Question 2.

a. State and prove the Hahn-Banach Theorem. (You can and should use Zorn's lemma.)
b. Show that if X and Y are normed vector spaces and $T \in B(X, Y)$ then $\|T\| = \|T^*\|$.

Question 3.

Show that a normed vector space is finite dimensional if and only if its closed unit ball is compact.

Question 4.

a. State and prove the Baire Category Theorem in some form.
b. Let $C^{\mathbf{R}}([0,1])$ denote the real valued continuous functions on $[0,1]$. By considering for each $n \geq 1$ the set

$$F_n = \{ f \in C([0,1]) \; : \; \frac{|f(x) - f(y)|}{|x-y|} \leq n \, \forall \, 0 \leq x, y \leq 1, x \neq y \},$$

deduce that the continuous nowhere-differentiable functions are dense in $C^{\mathbf{R}}([0,1])$.

Question 5.

a. State and prove the open mapping Theorem, and deduce the closed graph Theorem. (You may assume any version of the Baire Category Theorem provided it is stated clearly.)

b. Let H be a Hilbert space and $T \in B(H)$. Show if for all $y \in H$ there is some $\eta = \eta(y)$ such that

$$\langle Tx, y\rangle = \langle x, \eta\rangle$$

for all $x \in H$, then T is bounded.

Question 6.

a. State and prove the Arzelà-Ascoli Theorem.

b. Sketch a proof of Montel's Theorem: a uniformly bounded set of holomorphic functions on an open set $\Omega \subset \mathbf{C}$ contains a subsequence that is *uniformly convergent on all compact subset of* Ω.

Question 7.

a. State and prove the Alexander sub-base Theorem.

b. Using the sub-base Theorem, show that $[0,1]$ is compact. (Do **not** use Heine-Borel or Bolzano-Weierstrass).

Question 8.

a. State the 'Geometric' Hahn Banach Theorem. Prove that a convex subset of a normed space is closed if and only if it is weak closed.

b. Prove that the closure of the convex hull of a non-empty subset S of a normed vector space is the intersection of all the closed half spaces containing S. That is, prove for non-empty S

$$\overline{\text{co}}S = \{x \in X \;:\; f(x) \le \sup_{s\in S} f(s) \text{ for all } f \in X^*\}.$$

Question 9.

a. Show that if F is a complete subspace of a Euclidean space E that $E = F \oplus F^{\perp}$.

b. Give (with proof) an example of a Euclidean space E and a closed subset $A \subset E$ for which the distance to A is not always attained: there is some $b \in E$ such that for all $a \in A$

$$d(b, a) > d(b, A) = \inf_{a' \in A} d(b, a').$$

Question 10.

a. Show if X is a Banach space and Y a normed vector space, then $T \in B(X, Y)$ is invertible if and only if T and T^* are bounded below.

b. Show that if X and Y are Banach spaces then $\mathrm{Im}T$ is closed if and only if $\mathrm{Im}T^*$ is closed.