

SCHAUDER AND L^p ESTIMATES FOR PARABOLIC SYSTEMS VIA CAMPANATO SPACES

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1 Introduction

The following note deals with classical Schauder and L^p estimates in the setting of parabolic systems. For the heat equation these estimates are usually obtained via potential theoretic methods, i.e. by studying the fundamental solution (see e.g. [3], [8], and, for the elliptic case, [7]). For systems, however, it has become customary to base both Schauder and L^p theory on Campanato's technique. For the elliptic case, this is explained in [4] and [5] (see also the references therein to Campanato's original papers), and for the stationary Stokes system, in [6]. The purpose of this note is to show how this technique can be applied to parabolic systems. In many ways it is very similar to Giaquinta's exposition—in spirit as well as in detail. However, the parabolic case does offer certain peculiarities, most notably the global Schauder estimates, where we have to deal with a compatibility condition on the data (c.f. (22) on p. 9). Due to the importance of the estimates involved and the elegance of Campanato's technique as compared to the potential theoretic approach, one would naturally expect that all of this should appear in the classical literature. And indeed, in [1], Campanato develops the theory of the function spaces $L^{p,\lambda}$ and $\mathcal{L}^{p,\lambda}$, introduced by him previously to study elliptic regularity, in the context of the heat equation. To the best of the author's knowledge, however, the proof of the global estimates in [1], which is based on reflection across flat boundary pieces, does not directly apply to systems. Our main goal is therefore to show how to make appropriate modifications of Campanato's technique in the case of systems, and, in particular, how to deal with the aforementioned compatibility condition. For a Campanato type approach to nonlinear systems see Struwe's survey paper [12].

This note is organized as follows. In section 2 we reproduce the well known characterization of Hölder continuity via mean square oscillations (see e.g. [1]). Section 3 is devoted to establishing some basic and well known estimates for solutions u of homogeneous, constant coefficient parabolic systems. The decay estimates for the mean square oscillations of Du , which are the basis for Campanato's method, are contained in Proposition 1. The

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formulation of Proposition 1 shows clearly that the form of the estimates depends on the size and position of the parabolic cylinders relative to the boundary of the domain. Typically, the average of a function is not included in the mean square integral if that function vanishes on a piece of the boundary (compare (13) to (14)). Section 4 contains the essence of Campanato's technique, i.e. perturbing off the homogeneous, constant coefficient case. The main estimates are contained in Proposition 2. The most interesting case in the proof is probably the boundary portion $\{x_n = 0\} \times (0, T)$, where the aforementioned compatibility condition enters. When dealing with estimates for elliptic systems $A_{ij}^{\alpha\beta} D_{\alpha\beta} u^j = f^i$ on the boundary portion $\{x_n = 0\}$, all second order derivatives, with the exception of $D_{nn}u$, are typically estimated as in the interior case, and $D_{nn}u$ can be controlled directly from the equation. In the parabolic case, however, we are faced with an additional u_t term (in the scalar case this difficulty can be circumvented by a reflection across the boundary, as in [1], p. 75). Fortunately, this term can again be estimated using Campanato's method, and it is here that the compatibility condition becomes important. In Proposition 2, all parabolic cylinders are assumed to have special sizes and positions. This restriction is removed by a simple geometric argument, which finally leads to global Schauder estimates in Theorem 2. In section 5 we combine Theorem 2 with standard interpolation arguments to establish Schauder estimates for general, symmetric, parabolic systems. Symmetric here means that $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$ (see (43)). Systems of this type arise naturally when one considers gradient flows associated with variational problems.

The remaining sections are devoted to establishing L^p estimates by interpolating between L^2 and BMO , as in [1]. Unfortunately, we do not know how to deal with the case $1 < p < 2$ (in [1] the same restriction applies). The BMO estimates (in the constant coefficient case) are derived in section 6. The proof of Proposition 4 is only sketched, since it is very similar (and easier than) the one of Proposition 2. Lemma 4 deals with the technical issue of breaking up a domain $\Omega_T = \Omega \times [0, T]$ into small pieces in a controlled way. A decomposition of this type is needed to exploit the continuity of the leading order coefficients in the proof of theorem 4. For the sake of completeness, we rederive Stampacchia's interpolation theorem in the context of parabolic scaling in the last section. The argument given there is a straightforward adaptation of one from Stein [11].

2 An integral characterization of Hölder continuity

The following definition introduces the notation which we shall use.

Definition 1 *Let $r > 0$, $x_0 \in R^n$, and $T > 0$. Then*

$$B_r(x_0) \equiv \{x = (x^1, \dots, x^n) \in R^n \mid \max |x^i - x_0^i| < r\}$$

$$B_r^+ \equiv \{x = (x^1, \dots, x^n) \in R^n \mid \max |x^i| < r, x^n \geq 0\}$$

$$Q_1 \equiv B_1^+ \times [0, T], Q_2 \equiv B_2^+ \times [0, T].$$

We shall call

$$\Gamma \equiv \overline{Q_2} \cap (\{t = 0\} \cup \{x^n = 0\})$$

the boundary of Q_2 . It consists of $\Gamma_1 \equiv \Gamma \cap \{t > 0\}$ (the wall), $\Gamma_2 \equiv \Gamma \cap \{x^n > 0\}$ (the lower boundary), and $\Gamma_3 \equiv \Gamma \cap \{t = 0, x^n = 0\}$ (the corner).

The set

$$P_r(x_0, t_0) \equiv \{(x, t) \in Q_2 \mid \max |x^i - x_0^i| \leq r, |t - t_0| \leq r^2\}$$

will be referred to as the parabolic cylinder with center at $(x_0, t_0) \in Q_1$. We shall frequently write P_r if it is clear from the context what the center is. The parabolic boundary of P_r is defined to be

$$\partial P_r(x_0, t_0) \equiv P_r \cap (\{t = t_0 - r^2\} \cup \{\max |x^i - x_0^i| = r\}).$$

For $f \in L^1(Q_2)$ we let

$$(f)_r \equiv \oint_{P_r} f \equiv \frac{1}{|P_r|} \int_{P_r} f,$$

where integration is always understood to be over x and t . Let $K \subseteq Q_2$. Then

$$|f|_{0,K} \equiv \sup_{(x,t) \in K} |f(x, t)|$$

$$[f]_{\mu,K} \equiv \sup \left\{ \frac{|f(x, t) - f(y, s)|}{d((x, t), (y, s))^\mu} \mid (x, t), (y, s) \in K, d((x, t), (y, s)) \leq 1 \right\},$$

where $d((x, t), (y, s)) \equiv \max(|x - y|, |t - s|^{1/2})$ is the parabolic distance, and $\mu \in (0, 1)$ (we restrict the distances to be less than 1 in order to keep the constants in the following theorem independent of T). The space corresponding to $[\cdot]_{\mu,K}$ will be denoted by $C^{\mu, \mu/2}(K)$. ■

Parabolic Hölder spaces can be characterized by the following version of Campanato's theorem.

Theorem 1 *Let $f \in L^2(Q_2)$ and assume that*

$$\oint_{P_r(x_0, t_0)} |f - (f)_r|^2 \leq A^2 r^{2\mu} \tag{1}$$

for all $(x_0, t_0) \in Q_1$ and all $0 < r < 1$. Then $f \in C^{\mu, \mu/2}(Q)$ and

$$[f]_{\mu, Q_1} \leq CA,$$

with $C = C(n)$.

Proof: Fix $(x_0, t_0) \in Q_1$ and $0 < r < 1$. It is easy to see that (1) implies

$$|(f)_{2^{-j}r} - (f)_{2^{-j-1}r}| \leq 2^{n+2} A 2^{-j\mu} r^\mu \quad (2)$$

for all $j = 0, 1, 2, \dots$. Hence $\lim_{j \rightarrow \infty} (f)_{2^{-j}r}$ exists and, by Lebesgue's differentiation theorem in the context of nonisotropic scaling (see [11], p. 10 and p. 13), it equals f a.e. in Q_1 . Moreover, we conclude from (1), (2), and the triangle inequality that

$$\int_{P_r} |f - f(x_0, t_0)|^2 \leq C A^2 r^{2\mu}.$$

Thus we have for any two points $(x_0, t_0), (x_1, t_0) \in Q_1$, with $r \equiv |x_0 - x_1| \leq 1$,

$$\begin{aligned} |f(x_0, t_0) - f(x_1, t_0)|^2 &\leq 2 \int_{P_r(x_0, t_0) \cap P_r(x_1, t_0)} |f - f(x_0, t_0)|^2 + \\ &\quad + 2 \int_{P_r(x_0, t_0) \cap P_r(x_1, t_0)} |f - f(x_1, t_0)|^2 \\ &\leq C \int_{P_r(x_0, t_0)} |f - f(x_0, t_0)|^2 + \\ &\quad + C \int_{P_r(x_1, t_0)} |f - f(x_1, t_0)|^2 \\ &\leq C A^2 r^{2\mu} = C A^2 d((x_0, t_0), (x_1, t_0))^2. \end{aligned} \quad (3)$$

Now let $0 < t_1 < T$ and suppose that $r^2 \equiv |t_0 - t_1| \leq 1$. A calculation very similar to the one above then shows that

$$|f(x_0, t_0) - f(x_0, t_1)|^2 \leq C A^2 r^{2\mu} = C A^2 |t_0 - t_1|^\mu = C A^2 d((x_0, t_0), (x_0, t_1))^2. \quad (4)$$

The theorem now follows from (3) and (4). ■

3 The homogeneous, constant coefficient case

In this section we shall study the parabolic system

$$\begin{cases} (L(u))^\alpha \equiv u_t^\alpha - A_{ij}^{\alpha\beta} D_{ij} u^\beta = 0 & \text{in } Q_2 \\ u = 0 & \text{on } \Gamma \end{cases} \quad (5)$$

where $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha} = \text{const}$ and $\lambda|\xi|^2 \leq A_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j \leq \Lambda|\xi|^2$.

The following lemma contains some well known estimates for solutions of equation (5). Unless otherwise indicated, all constants in this section will depend only on n, λ , and Λ .

Lemma 1 *Let $(x_0, t_0) \in Q_1$ and $0 < r \leq 1$. Then any smooth solution u of (5) satisfies, with $P_r = P_r(x_0, t_0)$,*

i.

$$\int_{P_{r/2}} |u_t|^2 \leq Cr^{-2} \int_{P_r} |Du|^2 \quad (6)$$

ii. for $k = 1, 2, 3, \dots$

$$\int_{P_{r/2}} |D^k u|^2 \leq C_k r^{-2k} \int_{P_r} |u|^2 \quad (7)$$

iii. for any $0 < \rho < r \leq 1$

$$\oint_{P_\rho} |u|^2 \leq C \oint_{P_r} |u|^2; \quad (8)$$

if $(x_0, t_0) \in \Gamma_1$, then

$$\oint_{P_\rho} |u|^2 \leq C \left(\frac{\rho}{r} \right)^2 \oint_{P_r} |u|^2. \quad (9)$$

Proof: We shall assume throughout that $r = 1$. The general case then follows by rescaling.

First we introduce some smooth cutoff functions. Fix $0 < \rho < 1$ and let $\eta : R^n \rightarrow R$, $\text{supp}(\eta) \subseteq B_1(x_0)$ and $\eta = 1$ on $B_\rho(x_0)$; $\tau : R \rightarrow R$, $\text{supp}(\tau) \subseteq (t_0 - 1, t_0 + 1)$ and $\tau = 1$ on $[t_0 - \rho^2, t_0 + \rho^2]$. Finally, set $\zeta(x, t) = \eta(x)\tau(t)$.

To prove (6) we compute

$$\begin{aligned} 0 &= \int_{P_1} (u_t^\alpha - A_{ij}^{\alpha\beta} D_{ij} u^\beta) u_t^\alpha \zeta^2 = \\ &= \int_{P_1} |u_t|^2 \zeta^2 + (A_{ij}^{\alpha\beta} D_j u^\beta D_i u^\alpha \zeta^2 / 2)_t + 2A_{ij}^{\alpha\beta} D_j u^\beta u_t^\alpha \zeta D_i \zeta - A_{ij}^{\alpha\beta} D_j u^\beta D_i u^\alpha \zeta \zeta_t \\ &\geq \frac{1}{2} \int_{P_1} |u_t|^2 \zeta^2 - C \int_{P_1} |Du|^2 (\zeta |\zeta_t| + |D\zeta|^2). \end{aligned}$$

Hence

$$\int_{P_\rho} |u_t|^2 \leq C(n, \lambda, \Lambda, \rho) \int_{P_1} |Du|^2.$$

Next we turn to the case $k = 1$ of inequality (7).

$$\begin{aligned} 0 &= \int_{P_1} (u_t^\alpha - A_{ij}^{\alpha\beta} D_{ij} u^\beta) u^\alpha \zeta^2 \\ &= \int_{P_1} (|u|^2 \zeta^2 / 2)_t - |u|^2 \zeta \zeta_t + A_{ij}^{\alpha\beta} D_j u^\beta D_i u^\alpha \zeta^2 + 2A_{ij}^{\alpha\beta} D_j u^\beta u^\alpha \zeta D_i \zeta \\ &\geq \frac{\lambda}{2} \int_{P_1} |Du|^2 \zeta^2 - C \int_{P_1} |u|^2 (\zeta |\zeta_t| + |D\zeta|^2). \end{aligned}$$

Thus

$$\int_{P_\rho} |Du|^2 \leq C(n, \lambda, \Lambda, \rho) \int_{P_1} |u|^2.$$

Induction now shows that, for $k = 1, 2, 3, \dots$,

$$\int_{P_\rho} |D^k u|^2 \leq C(n, \lambda, \Lambda, \rho, k) \int_{P_1} |u|^2. \quad (10)$$

As induction hypothesis, we assume that (10) holds up to some $k \geq 1$. Since $D_i u$ also solves (5) for $1 \leq i \leq n-1$, we obtain, by applying a rescaled version of (10) to $D_i u$,

$$\sum_{i=1}^{n-1} \int_{P_\rho} |D^k D_i u|^2 \leq C \sum_{i=1}^{n-1} \int_{P_{\rho_1}} |D_i u|^2 \leq C \int_{P_1} |u|^2, \quad (11)$$

where $\rho_1 \equiv (1 + \rho)/2$. Note that the left hand side does not contain $D_n^{k+1} u$. Since u_t solves equation (5), we can use inequalities (10), and (6), in conjunction with the equation, to estimate this missing term as follows.

$$\begin{aligned} \int_{P_\rho} |D_n^{k+1} u|^2 &\leq C \int_{P_\rho} |D_n^{k-1} u_t|^2 + C \sum_{i+j < 2n} \int_{P_\rho} |D^{k-1} D_{ij} u|^2 \\ &\leq C \int_{P_{\rho_1}} |u_t|^2 + C \sum_{i=1}^{n-1} \int_{P_{\rho_1}} |D^k D_i u|^2 \\ &\leq C \int_{P_{\rho_2}} |Du|^2 + C \int_{P_1} |u|^2 \\ &\leq C \int_{P_1} |u|^2, \end{aligned}$$

with $\rho_2 \equiv (1 + \rho_1)/2$.

The monotonicity formula (8) is a consequence of (7) and Sobolev's imbedding theorem. Namely, let $0 < \rho < \frac{1}{2}$ and compute

$$\begin{aligned} \int_{P_\rho} |u|^2 &\leq \sup_{P_{\frac{1}{2}}} |u|^2 \leq C \left\{ \sum_{k+j \leq n+1} \int_{P_{\frac{1}{2}}} |D^k \partial_t^j u|^2 + \int_{P_{\frac{1}{2}}} |u|^2 \right\} \\ &\leq C \left\{ \int_{P_{\frac{1}{2}}} |D^{2(n+1)} u|^2 + \int_{P_1} |u|^2 \right\} \\ &\leq C \int_{P_1} |u|^2, \end{aligned}$$

where we have used the equation to remove the time-derivative as well as a standard interpolation result in Sobolev spaces. To prove (9), we note that $u(x_1, x_2, \dots, x_{n-1}, 0) = 0$ implies that

$$\sup_{0 < x_n < \rho} |u|^2 \leq \rho \int_0^\rho |D_n u(x_1, \dots, x_{n-1}, t)|^2 dt.$$

Now we obtain as above

$$\begin{aligned}
\int_{P_\rho} |u|^2 &\leq \rho^2 \int_{P_\rho} |D_n u|^2 \leq \rho^2 \sup_{P_{\frac{1}{2}}} |D_n u|^2 \leq \\
&\leq C \rho^2 \left\{ \sum_{k+j \leq n+1} \int_{P_{\frac{1}{2}}} |D^k \partial_t^j D_n u|^2 + \int_{P_{\frac{1}{2}}} |D_n u|^2 \right\} \\
&\leq C \rho^2 \left\{ \int_{P_{\frac{1}{2}}} |D^{2n+3} u|^2 + \int_{P_1} |u|^2 \right\} \\
&\leq C \rho^2 \int_{P_1} |u|^2. \blacksquare
\end{aligned}$$

Remark: The above calculations require the vanishing of u on a boundary portion only if P_r intersects that portion. We shall make frequent use of this observation in what follows.

The following lemma will play a crucial role in the derivation of Campanato-type decay estimates for the inhomogeneous equation.

Proposition 1 *Let u be a smooth solution of (5). Then*

i. $(x_0, t_0) \in Q_1 \setminus \Gamma$ implies

$$\int_{P_\rho} |Du - (Du)_\rho|^2 \leq C \left(\frac{\rho}{r} \right)^2 \int_{P_r} |Du - (Du)_r|^2, \quad (12)$$

provided $0 < \rho \leq r \leq 1$ and $P_r \subseteq Q_2 \setminus \Gamma$.

ii. $(x_0, t_0) \in \Gamma_1$ implies

$$\int_{P_\rho} |D_i u|^2 \leq C \left(\frac{\rho}{r} \right)^2 \int_{P_r} |D_i u|^2, \quad \text{for } 1 \leq i \leq n-1, \quad (13)$$

and

$$\int_{P_\rho} |D_n u - (D_n u)_\rho|^2 \leq C \left(\frac{\rho}{r} \right)^2 \int_{P_r} |D_n u - (D_n u)_r|^2, \quad (14)$$

provided $0 < \rho \leq r \leq 1$ and $P_r \subseteq Q_2 \setminus \Gamma_2$.

iii. $(x_0, 0) \in \Gamma_2 \cup \Gamma_3$ implies

$$\int_{P_\rho} |Du|^2 \leq C \left(\frac{\rho}{r} \right)^2 \int_{P_r} |Du|^2, \quad (15)$$

provided $0 < \rho \leq r \leq 1$.

Proof: As in the previous proof, we shall assume throughout that $r = 1$. Moreover, since the above estimates are obvious in case $\frac{1}{4} < \rho < 1$, we may assume w.l.o.g. that $\rho < \frac{1}{4}$. To prove the first part of the lemma we compute, using Poincaré's inequality,

$$\begin{aligned}
\int_{P_\rho} |u - (u)_\rho|^2 &\leq C \left\{ \rho^2 \int_{P_\rho} |Du|^2 + \rho^4 \int_{P_\rho} |u_t|^2 \right\} \\
&\leq C \rho^2 \int_{P_{2\rho}} |Du|^2 \leq C \rho^2 \int_{P_{\frac{1}{2}}} |Du|^2 \\
&\leq C \rho^2 \int_{P_1} |u - (u)_1|^2.
\end{aligned} \tag{16}$$

Here we applied (6) to u , (8) to Du , and (7) to $u - (u)_1$. This is justified, since we assumed that $P_r \cap \Gamma = \emptyset$. For the same reason, we may replace u by Du in (16), which demonstrates (12).

To prove (ii), we let $v = D_i u$, where $1 \leq i \leq n-1$. Since $L(v) = 0$, $v = 0$ on Γ , (13) follows immediately from (9). To obtain (14), we let $w = u - x^n (D_n u)_1$. Note that $L(w) = 0$ and $w = 0$ on Γ_1 . By Poincaré's inequality and (7), applied to w_t (keeping in mind that $P_\rho \cap \Gamma_2 = \emptyset$),

$$\begin{aligned}
\int_{P_\rho} |D_n u - (D_n u)_\rho|^2 &= \int_{P_\rho} |D_n w - (D_n w)_\rho|^2 \\
&\leq C \left\{ \rho^2 \int_{P_\rho} |D D_n w|^2 + \rho^4 \int_{P_\rho} |D_n w_t|^2 \right\} \\
&\leq C \rho^2 \int_{P_{2\rho}} |D^2 w|^2 + C \rho^2 \int_{P_{2\rho}} |w_t|^2.
\end{aligned} \tag{17}$$

The first term on the right hand side can again be estimated by Sobolev's imbedding theorem.

$$\begin{aligned}
\int_{P_{2\rho}} |D^2 w|^2 &\leq \sup_{P_{\frac{1}{2}}} |D^2 w|^2 \leq C \left\{ \int_{P_{\frac{1}{2}}} |D^{2(n+1)} D^2 w|^2 + \int_{P_{\frac{1}{2}}} |D^2 w|^2 \right\} \\
&\leq C \int_{P_1} |w|^2 \leq C \int_{P_1} |D_n w|^2.
\end{aligned} \tag{18}$$

To estimate the second term on the right hand side of (17) we first apply (8) to w_t , then (6) and (7) to w .

$$\begin{aligned}
\int_{P_{2\rho}} |w_t|^2 &\leq C \int_{P_{\frac{1}{2}}} |w_t|^2 \leq C \int_{P_{\frac{3}{4}}} |Dw|^2 \\
&\leq C \int_{P_1} |w|^2 \leq C \int_{P_1} |D_n w|^2.
\end{aligned} \tag{19}$$

Combining (17), (18), and (19), finally yields

$$\int_{P_\rho} |D_n u - (D_n u)_\rho|^2 \leq C \rho^2 \int_{P_1} |D_n w|^2 = C \rho^2 \int_{P_1} |D_n u - (D_n u)_1|^2,$$

which finishes the proof of (14).

As for the third part of the lemma we compute, using $u = 0$ on Γ_2 , the fact that u_t solves (5), and Lemma 1:

$$\begin{aligned} \int_{P_\rho(x_0,0)} |u|^2 &\leq C \rho^4 \int_{P_\rho} |u_t|^2 \leq C \rho^4 \int_{P_{\frac{1}{2}}} |u_t|^2 \\ &\leq C \rho^4 \int_{P_{\frac{3}{4}}} |Du|^2 \leq C \rho^4 \int_{P_1} |u|^2. \end{aligned} \quad (20)$$

If $P_1(x_0, 0) \cap \Gamma_1 = \emptyset$ we simply apply (20) to Du . Otherwise, the following calculation finishes the proof.

$$\begin{aligned} \int_{P_\rho} |D_n u|^2 &\leq C \rho^4 \int_{P_\rho} |D_n u_t|^2 \leq C \rho^2 \int_{P_{2\rho}} |u_t|^2 \leq C \rho^2 \int_{P_{\frac{1}{2}}} |u_t|^2 \\ &\leq C \rho^2 \int_{P_{\frac{3}{4}}} |Du|^2 \leq C \rho^2 \int_{P_1} |u|^2 \\ &\leq C \rho^2 \int_{P_1} |D_n u|^2, \end{aligned}$$

where we have used $P_1(x_0, 0) \cap \Gamma_1 \neq \emptyset$ in the last step. ■

4 The inhomogeneous case with variable coefficients of the highest order

Next we consider the inhomogeneous parabolic system

$$\begin{cases} u_t^\alpha(x, t) - A_{ij}^{\alpha\beta}(x, t) D_{ij} u^\beta(x, t) = f^\alpha(x, t) & \text{in } Q_2 \\ u = 0 & \text{on } \Gamma, \end{cases} \quad (21)$$

with $A_{ij}^{\alpha\beta} \in C^{\mu, \mu/2}(Q_2)$, $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$, and $\lambda|\xi|^2 \leq A_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j \leq \Lambda|\xi|^2$. We assume that $f \in C^{\mu, \mu/2}(Q_2)$ and that it satisfies the compatibility condition

$$f = 0 \quad \text{on } \Gamma_3. \quad (22)$$

The main result in this section will be an a priori Hölder estimate of solutions of (21). First we introduce some notation.

Definition 2 Let $(x_0, t_0) \in Q_1 \setminus \Gamma$. We then denote by $\rho_0 = \rho_0(x_0, t_0)$ the l.u.b. of all $\rho \in (0, 1)$ such that $P_\rho(x_0, t_0) \subseteq Q_2 \setminus \Gamma$. Similarly, for $(x_0, 0) \in \Gamma_2$, $\rho_1 = \rho_1(x_0, 0)$ is defined to be the l.u.b. of all $\rho \in (0, 1)$ such that $P_\rho(x_0, 0) \subseteq Q_2 \setminus \Gamma_1$, and, for $(x_0, t_0) \in \Gamma_1$, $\rho_2 = \rho_2(x_0, t_0)$ is defined to be the l.u.b. of all $\rho \in (0, 1)$ such that $P_\rho \subseteq Q_2 \setminus \Gamma_2$. Finally, we set $\rho_3 = 1$.

The following result generalizes Proposition 1 to the inhomogeneous problem. Constants will now also depend on $[A]_\mu$.

Proposition 2 Let $M^2 \equiv [f]_\mu^2 + |D^2 u|_{0, Q_2}^2$. Then for any solution $u \in C^{2,1}(Q_2)$ of (21),

i. $(x_0, t_0) \in Q_1 \setminus \Gamma$ implies

$$\int_{P_\rho(x_0, t_0)} |D^2 u - (D^2 u)_\rho|^2 \leq C \rho^{2\mu} \left(\rho_0^{-2\mu} \int_{P_{\rho_0}} |D^2 u - (D^2 u)_{\rho_0}|^2 + M^2 \right) \quad (23)$$

for all $0 < \rho \leq \rho_0(x_0, t_0)$.

ii. $(x_0, t_0) \in \Gamma_1$ implies

$$\sum_{i,j=1}^{n-1} \int_{P_\rho(x_0, t_0)} |D_{ij} u|^2 \leq C \rho^{2\mu} \left(\rho_2^{-2\mu} \sum_{i,j=1}^{n-1} \int_{P_{\rho_2}} |D_{ij} u|^2 + M^2 \right), \quad (24)$$

$$\sum_{i=1}^{n-1} \int_{P_\rho(x_0, t_0)} |D_{in} u - (D_{in} u)_\rho|^2 \leq C \rho^{2\mu} \left(\rho_2^{-2\mu} \sum_{i=1}^{n-1} \int_{P_{\rho_2}} |D_{in} u|^2 + M^2 \right), \quad (25)$$

and

$$\int_{P_\rho(x_0, t_0)} |D_{nn} u - (D_{nn} u)_\rho|^2 \leq C \rho^{2\mu} \left(\rho_2^{-2\mu} \int_{P_{\rho_2}} |D^2 u|^2 + M^2 \right), \quad (26)$$

for all $0 < \rho \leq \rho_2(x_0, t_0)$.

iii. $(x_0, 0) \in \Gamma_2 \cup \Gamma_3$ implies

$$\int_{P_\rho(x_0, t_0)} |D^2 u|^2 \leq C \rho^{2\mu} \left(\rho_i^{-2\mu} \int_{P_{\rho_i}} |D^2 u|^2 + M^2 \right) \quad (27)$$

for $0 < \rho \leq \rho_i$. Here $i = 1$ if $(x_0, 0) \in \Gamma_2$, and $i = 3$ if $(x_0, 0) \in \Gamma_3$.

Proof: Let (x_0, t_0) be any point in Q_1 . Then

$$u_t^\alpha(x, t) - A_{ij}^{\alpha\beta}(x_0, t_0)D_{ij}u^\beta(x, t) = f^\alpha(x, t) + (A_{ij}^{\alpha\beta}(x, t) - A_{ij}^{\alpha\beta}(x_0, t_0))D_{ij}u^\beta(x, t),$$

which we shall write symbolically as

$$L_0(u) \equiv u_t - A_0 D^2 u = F \equiv f + (A - A_0) D^2 u.$$

Let $(x_0, t_0) \in Q_1 \setminus \Gamma$ and $0 < \rho < r \leq \rho_0 = \rho_0(x_0, t_0)$. Fix $1 \leq k \leq n$ and let $v = D_k u$. We define w to be the weak solution of

$$\begin{cases} w_t - A_0 D^2 w &= D_k F & \text{in } P_r \\ w &= 0 & \text{on } \partial P_r, \end{cases} \quad (28)$$

and set $z = v - w$. By a weak solution we mean a function $w \in L^2(0, T; H_0^1(P_r))$ with time derivative $w_t \in L^2(0, T; H^{-1}(P_r))$ (see [9]). Note that $L_0(z) = 0$ in the weak sense, which implies that z is smooth.

First we estimate Dw .

$$\begin{aligned} \int_{P_r} w_t^\alpha w^\alpha - A_{ij}^{\alpha\beta}(x_0, t_0) D_{ij} w^\beta w^\alpha &= \int_{P_r} \frac{1}{2} |w_t|^2 + A_{ij}^{\alpha\beta}(x_0, t_0) D_j w^\beta D_i w^\alpha \\ &= \int_{P_r} D_k F^\alpha w^\alpha = - \int_{P_r} (F^\alpha - (F^\alpha)_r) D_k w^\alpha. \end{aligned}$$

Hence

$$\int_{P_r} |Dw|^2 \leq C \int_{P_r} |F - (F)_r|^2 \leq C([f]_\mu^2 + |D^2 u|_{0, Q_2}^2) r^{n+2+2\mu} = CM^2 r^{n+2+2\mu}. \quad (29)$$

In connection with (12), this yields

$$\begin{aligned} \int_{P_\rho} |Dv - (Dv)_\rho|^2 &\leq 2 \int_{P_\rho} |Dz - (Dz)_\rho|^2 + 2 \int_{P_\rho} |Dw - (Dw)_\rho|^2 \\ &\leq C \left(\frac{\rho}{r} \right)^{n+4} \int_{P_r} |Dz - (Dz)_r|^2 + 4 \int_{P_r} |Dw|^2 \\ &\leq C \left(\frac{\rho}{r} \right)^{n+4} \int_{P_r} |Dv - (Dv)_r|^2 + CM^2 r^{n+2+2\mu} \end{aligned} \quad (30)$$

for all $0 < \rho < r \leq \rho_0$. Since k was arbitrary, (23) follows from (30) via Lemma 2 (see p. 13).

If $(x_0, t_0) \in \Gamma_1$ we can no longer differentiate u with respect to x^n . However, by differentiating u with respect to x^k , where $1 \leq k \leq n-1$, and using the same kind of argument as above (with (13) and (14) replacing (12)) we easily obtain (24) and (25).

The missing term, i.e. $D_{nn}u$, can be estimated via u_t . To this end let ϕ solve

$$\begin{cases} L_0(\phi) = \phi_t - A_0 D^2 \phi &= f(x_0, t_0) & \text{in } P_{\rho_2} \\ \phi &= 0 & \text{on } \partial P_{\rho_2} \end{cases} \quad (31)$$

in the weak sense, and set $v = u - \phi$, $g = F - f(x_0, t_0)$. Note that $L_0(v) = F - f(x_0, t_0) = g$. Furthermore, we take w to be the weak solution of

$$\begin{cases} w_t - A_0 D^2 w &= g & \text{in } P_r \\ w &= 0 & \text{on } \partial P_r. \end{cases} \quad (32)$$

Since $g \in L^2(P_r)$, we have $w_t \in L^2(P_r)$ together with the estimate

$$\int_{P_r} |w_t|^2 \leq \int_{P_r} |g|^2$$

(see [9]). Recalling the definition of F , we now obtain

$$\int_{P_r} |w_t|^2 \leq \int_{P_r} |F - f(x_0, t_0)|^2 \leq C r^{n+2+2\mu} M^2,$$

which illustrates why we introduced the auxiliary function ϕ . Let $z = v - w$. Then $L_0(z) = L_0(v) - L_0(w) = g - g = 0$, and $z = 0$ on Γ_1 . In particular, z is smooth by standard regularity theory (see [9]). Hence, we can apply the monotonicity formula (9) to z_t to wit

$$\begin{aligned} \int_{P_\rho} |v_t|^2 &\leq 2 \int_{P_\rho} |z_t|^2 + 2 \int_{P_\rho} |w_t|^2 \\ &\leq C \left(\frac{\rho}{r} \right)^{n+4} \int_{P_r} |z_t|^2 + 2 \int_{P_r} |w_t|^2 \\ &\leq C \left(\frac{\rho}{r} \right)^{n+4} \int_{P_r} |v_t|^2 + C M^2 r^{n+2+2\mu} \end{aligned} \quad (33)$$

for all $0 < \rho < r \leq \rho_2(x_0, t_0)$. From Lemma 2 we conclude that

$$\int_{P_\rho} |v_t|^2 \leq C \rho^{2\mu} \left\{ \rho_2^{-2\mu} \int_{P_{\rho_2/2}} |v_t|^2 + M^2 \right\}. \quad (34)$$

In order to obtain an estimate for u_t from (34), we observe that ϕ will satisfy

$$\rho_2^{\mu+2} [\phi_t]_{\mu, P_{\rho_2/2}} \leq C \rho_2^2 |f|_{0, P_{\rho_2}}. \quad (35)$$

To establish (35) one simply rescales equation (31) to the unit cylinder P_1 , using the fact that the interior of P_{ρ_2} does not intersect Γ_2 . Since $\phi = 0$ on Γ_1 , we infer from

inequalities (34) and (35) that

$$\begin{aligned}
\int_{P_\rho(x_0, t_0)} |u_t|^2 &\leq 2 \int_{P_\rho} |v_t|^2 + 2 \int_{P_\rho} |\phi_t|^2 \\
&\leq C \rho^{2\mu} \left\{ \rho_2^{-2\mu} \int_{P_{\rho_2/2}} |v_t|^2 + M^2 \right\} + C \rho^{2\mu} \rho_2^{-2\mu} |f|_{0, P_{\rho_2}}^2 \\
&\leq C \rho^{2\mu} \left\{ \rho_2^{-2\mu} \int_{P_{\rho_2}} |u_t|^2 + M^2 + C \rho_2^{-2\mu} |f|_{0, P_{\rho_2}}^2 \right\}.
\end{aligned} \tag{36}$$

The estimate for $D_{nn}u$ can now be obtained as follows. Equation (21) implies

$$A_{nn}^{\alpha\beta} D_{nn} u^\beta = u_t^\alpha - \sum_{i+j < 2n} A_{ij}^{\alpha\beta} D_{ij} u^\beta - f^\alpha.$$

Since $\{A_{nn}^{\alpha\beta}\}_{\alpha\beta=1}^N$ is a positive matrix, we can solve for $D_{nn}u$, and it is readily seen that

$$\begin{aligned}
\int_{P_\rho} |D_{nn}u - (D_{nn}u)_\rho|^2 &\leq C \int_{P_\rho} |u_t|^2 + C \sum_{i+j < 2n} \int_{P_\rho} |D_{ij}u - (D_{ij}u)_\rho|^2 + \\
&\quad + C \int_{P_\rho} |f - (f)_\rho|^2 \\
&\quad + C \rho^{2\mu} (|D^2u|_{0, Q_2}^2 + |f|_{0, P_\rho}^2) \\
&\leq C \rho^{2\mu} \left\{ \rho_2^{-2\mu} \int_{P_{\rho_2}} (|u_t|^2 + |D^2u|^2) + M^2 + \right. \\
&\quad \left. + \rho_2^{-2\mu} |f|_{0, P_{\rho_2}}^2 \right\},
\end{aligned} \tag{37}$$

where we have applied (36), (24), and (25). Using equation (21) again to express u_t on the right hand side of (37) in terms of f and D^2u , we finally obtain

$$\int_{P_\rho} |D_{nn}u - (D_{nn}u)_\rho|^2 \leq C \rho^{2\mu} \left\{ \rho_2^{-2\mu} \int_{P_{\rho_2}} |D^2u|^2 + M^2 + \rho_2^{-2\mu} |f|_{0, P_{\rho_2}}^2 \right\}. \tag{38}$$

Since $P_{\rho_2} \cap \Gamma_3 \neq \emptyset$, we conclude from the compatibility condition (22) that

$$|f|_{0, P_{\rho_2}} \leq C \rho_2^\mu [f]_\mu.$$

Substituting this into (38) finishes the proof of (26).

The argument for a boundary point $(x_0, 0) \in \Gamma_2$ is very much like the one for an interior point—in particular, we can differentiate u with respect to x_k , $1 \leq k \leq n$. The only difference is that we have to use (15) instead of (12).

For a corner-point $(x_0, 0) \in \Gamma_3$, we are faced with the same difficulty as in the case of a wall-point—namely, estimating $D_{nn}u$ in terms of u_t . However, in this case, we can

take $\phi = 0$. In fact, by the compatibility condition (22), the only solution of (31) is 0. Otherwise, the argument is completely analogous to the one above and shall therefore be omitted. ■

The following technical lemma is standard and we refer to [4], p. 86, for a proof.

Lemma 2 *Let Φ be a nonnegative, nondecreasing function on $(0, r_0]$ such that*

$$\Phi(\rho) \leq A \left(\frac{\rho}{r} \right)^\alpha \Phi(r) + Br^\beta$$

for all $0 < \rho < r \leq r_0$, where $0 < \beta < \alpha$ are fixed constants. Then

$$\Phi(r) \leq Cr^\beta(r_0^{-\beta}\Phi(r_0) + B)$$

with a constant $C = C(A, \alpha, \beta)$.

By the proof of Theorem 1 the first estimate of Proposition 2 implies an interior a priori $C^{\mu, \mu/2}$ estimate of D^2u . In order to obtain global Hölder estimates we need an inequality of the form (23) to hold for $0 < \rho \leq 1$.

Proposition 3 *Suppose $u \in C^{2,1}(Q_2)$ solves (21). Then for any $(x_0, t_0) \in Q_1$ and $0 < \rho \leq 1$,*

$$\int_{P_\rho(x_0, t_0)} |D^2u - (D^2u)_\rho|^2 \leq C\rho^{2\mu}M^2. \quad (39)$$

Proof: The idea of the proof is to expand P_ρ until it hits the boundary, say on Γ_1 . The expanded cylinder is then replaced by one of comparable size and with its center on Γ_1 . This new cylinder is then expanded until it hits Γ_2 , upon which it is replaced by a cylinder of comparable size and with a corner point as its center. Finally, this new cylinder is expanded to radius 1. Applying Proposition 2 at each step leads to (39). Clearly, these expansions will be necessary only if the cylinders in questions do not already intersect the respective boundary portions.

Let $(x_0, t_0) \in Q_1 \setminus \Gamma$ and $\rho_0 = \rho_0(x_0, t_0)$. We shall need to consider several cases:

1) $\rho \leq \rho_0$

P_{ρ_0} intersects either Γ_1 or Γ_2 . For simplicity we shall assume that it intersects Γ_2 , the other case being similar. Obviously, $P_{\rho_0}(x_0, t_0) \subseteq P_{2\rho_0}(x_0, 0)$. By Proposition 2,

$$\begin{aligned} \int_{P_\rho(x_0, t_0)} |D^2u - (D^2u)_\rho|^2 &\leq C\rho^{2\mu} \left(\rho_0^{-2\mu} \int_{P_{\rho_0}(x_0, t_0)} |D^2u - (D^2u)_{\rho_0}|^2 + M^2 \right) \\ &\leq C\rho^{2\mu} \left(\rho_0^{-2\mu} \int_{P_{2\rho_0}(x_0, 0)} |D^2u|^2 + M^2 \right). \end{aligned} \quad (40)$$

We need to distinguish between two further cases.

(a) $2\rho_0 \leq \rho_1(x_0, 0) = \rho_1$

Let $(x_1, 0) \in \Gamma_3$ be the closest corner point to $(x_0, 0)$. Then

$$P_{\rho_1}(x_0, 0) \subseteq P_{2\rho_1}(x_1, 0),$$

and Proposition 2 yields

$$\begin{aligned} \rho_0^{-2\mu} \int_{P_{2\rho_0}(x_0, 0)} |D^2 u|^2 &\leq C \left(\rho_1^{-2\mu} \int_{P_{\rho_1}(x_0, 0)} |D^2 u|^2 + M^2 \right) \\ &\leq C \left(\rho_1^{-2\mu} \int_{P_{2\rho_1}(x_1, 0)} |D^2 u|^2 + M^2 \right) \\ &\leq C \left(\int_{P_2(x_1, 0)} |D^2 u|^2 + M^2 \right). \end{aligned} \quad (41)$$

Combining (40) and (41) establishes (39) in this case.

(b) $2\rho_0 > \rho_1(x_0, 0)$

Then $P_{2\rho_0}(x_0, 0) \cap \Gamma_1 \neq \emptyset$ and we have, with x_1 as above,

$$P_{2\rho_0}(x_0, 0) \subseteq P_{\rho_0}(x_1, 0).$$

By Proposition 2,

$$\rho_0^{-2\mu} \int_{P_{2\rho_0}(x_0, 0)} |D^2 u|^2 \leq C \rho_0^{-2\mu} \int_{P_{4\rho_0}(x_1, 0)} |D^2 u|^2 \leq C \left(\int_{P_2(x_1, 0)} |D^2 u|^2 + M^2 \right). \quad (42)$$

Substituting this into (40) leads to (39).

2) $\rho > \rho_0(x_0, t_0)$

In this case $P_\rho(x_0, t_0) \cap \Gamma \neq \emptyset$. We shall again assume for simplicity that $P_\rho \cap \Gamma_2 \neq \emptyset$.

Then

$$P_\rho(x_0, t_0) \subseteq P_{2\rho}(x_0, 0).$$

(a) $2\rho \leq \rho_1(x_0, 0)$

By Proposition 2 and the same kind of reasoning as above (in particular with the same x_1),

$$\begin{aligned} \int_{P_\rho(x_0, t_0)} |D^2 u - (D^2 u)_\rho|^2 &\leq C \int_{P_{2\rho}(x_0, 0)} |D^2 u|^2 \\ &\leq \rho^{2\mu} \left(\rho_1^{-2\mu} \int_{P_{\rho_1}(x_0, 0)} |D^2 u|^2 + M^2 \right) \\ &\leq C \rho^{2\mu} \left(\rho_1^{-2\mu} \int_{P_{2\rho_1}(x_1, 0)} |D^2 u|^2 + M^2 \right) \\ &\leq C \rho^{2\mu} \left(\int_{P_2(x_1, 0)} |D^2 u|^2 + M^2 \right), \end{aligned}$$

which is (39).

(b) $2\rho > \rho_1(x_0, 0)$

In this case we have

$$\begin{aligned} \int_{P_\rho(x_0, t_0)} |D^2u - (D^2u)_\rho|^2 &\leq C \int_{P_{2\rho}(x_0, 0)} |D^2u|^2 \leq C \int_{P_{4\rho}(x_1, 0)} |D^2u|^2 \\ &\leq C \rho^{2\mu} \left(\int_{P_4(x_1, 0)} |D^2u|^2 + M^2 \right), \end{aligned}$$

which finishes the proof. ■

Proposition 3 and Theorem 1 immediately imply global Hölder estimates for D^2u .

Theorem 2 *Let $u \in C^{2,1}(Q_2)$ be a classical solution of (21). Then D^2u and u_t are in $C^{\mu, \mu/2}(Q_1)$ and the following a priori estimate holds:*

$$[D^2u]_{\mu, Q_1} + [u_t]_{\mu, Q_1} \leq C \left(|D^2u|_{0, Q_2} + [f]_{\mu, Q_2} \right),$$

where C does not depend on T .

5 The general equation

In this section we consider the general parabolic system

$$\begin{cases} u_t^\alpha - A_{ij}^{\alpha\beta} D_{ij} u^\beta + B_i^{\alpha\beta} D_i u^\beta + C^{\alpha\beta} u^\beta &= f^\alpha & \text{in } \Omega_T \\ u &= 0 & \text{on } \Gamma_T, \end{cases} \quad (43)$$

with $A, B, C, f \in C^{\mu, \mu/2}$, A as in (21), and Ω a $C^{2+\mu}$ -domain (we are using the standard notation $\Omega_T \equiv \Omega \times [0, T]$ and $\Gamma_T \equiv \overline{\Omega_T} \setminus \{t = T\}$). Furthermore, we assume that f satisfies the compatibility condition $f = 0$ on $\partial\Omega \times \{t = 0\}$.

Under these conditions we have the following result.

Theorem 3 *Let $u \in C^{2,1}(\Omega_T)$ be a classical solution of problem (43). Then $D^2u, u_t \in C^{\mu, \mu/2}(\Omega_T)$, and*

$$[D^2u]_{\mu, \Omega_T} + [u_t]_{\mu, \Omega_T} \leq C \left(|u|_{0, \Omega_T} + [f]_{\mu, \Omega_T} \right),$$

where C depends on $n, \lambda, \Lambda, \mu, \Omega$, and the Hölder norms of the coefficients, but not on T .

Proof: For every point $p \in \partial\Omega$ there exist neighborhoods U and V of p and a $C^{2+\mu}$ diffeomorphism ϕ such that $\phi(U) = B_1^+$ and $\phi(V) = B_2^+$. Let $\Omega = \bigcup_{j=0}^N U_j$, where U_0 is

compactly contained in Ω and (U_j, V_j, ϕ_j) are as above for $j = 1, 2, \dots, N$. By changing coordinates and applying Theorem 2 we conclude that, for $j = 1, 2, \dots, N$,

$$\begin{aligned} [D^2u]_{\mu, U_j \times [0, T]} + [u_t]_{\mu, U_j \times [0, T]} &\leq C \left(|D^2u|_{0, \Omega_T} + |Du|_{0, \Omega_T} + [Du]_{\mu, \Omega_T} + \right. \\ &\quad \left. + [u]_{\mu, \Omega_T} + |u|_{0, \Omega_T} + [f]_{\mu, \Omega_T} \right). \end{aligned} \quad (44)$$

Note that we cannot directly apply Theorem 2 to $U_0 \times [0, T]$ because of the difference in the geometry. However, it is easily seen that estimates (23) and (27) of Proposition 2, for interior points and points on the lower boundary, respectively, together with the proofs of Proposition 3 and Theorem 1, imply

$$\begin{aligned} [D^2u]_{\mu, U_0 \times [0, T]} + [u_t]_{\mu, U_0 \times [0, T]} &\leq C \left(|D^2u|_{0, \Omega_T} + [f - BDu - Cu]_{\mu, \Omega_T} \right) \\ &\leq C \left(|D^2u|_{0, \Omega_T} + |Du|_{0, \Omega_T} + [Du]_{\mu, \Omega_T} + [u]_{\mu, \Omega_T} \right. \\ &\quad \left. + |u|_{0, \Omega_T} + [f]_{\mu, \Omega_T} \right). \end{aligned} \quad (45)$$

Clearly, inequalities (44) and (45) imply that $D^2u, u_t \in C^{\mu, \mu/2}(\Omega_T)$. Furthermore, it is easily seen that

$$\begin{aligned} [D^2u]_{\mu, \Omega_T} + [u_t]_{\mu, \Omega_T} &\leq C \left(\sum_{j=0}^N ([D^2u]_{\mu, U_j \times [0, T]} + \right. \\ &\quad \left. + [u_t]_{\mu, U_j \times [0, T]}) + |D^2u|_{0, \Omega_T} + |u_t|_{0, \Omega_T} \right). \end{aligned} \quad (46)$$

The theorem now follows by substituting (44) and (45) into (46), applying Lemma 3 and choosing ϵ sufficiently small. ■

The following elementary interpolation result was used in the proof of Theorem 3.

Lemma 3 *For all $u \in C^{2,1}(\Omega_T)$ such that $D^2u, u_t \in C^{\mu, \mu/2}(\Omega_T)$, and all $\epsilon > 0$, we have*

$$|D^2u|_{0, \Omega_T} + |Du|_{0, \Omega_T} + [Du]_{\mu, \Omega_T} + [u]_{\mu, \Omega_T} \leq \epsilon [D^2u]_{\mu, \Omega_T} + \epsilon [u_t]_{\mu, \Omega_T} + C(\epsilon) |u|_{0, \Omega_T}.$$

Proof: We shall be using the following notation:

$$\begin{aligned} \langle v \rangle_x^{(\mu)} &\equiv \sup \left\{ \frac{|v(x, t) - v(y, t)|}{|x - y|^\mu} \mid (x, t), (y, t) \in \Omega_T, |x - y| \leq 1 \right\} \\ \langle v \rangle_t^{(\nu)} &\equiv \sup \left\{ \frac{|v(x, t) - v(x, s)|}{|t - s|^\nu} \mid (x, t), (x, s) \in \Omega_T, |t - s| \leq 1 \right\}. \end{aligned}$$

Clearly, $[v]_{\mu, \Omega_T} \leq \langle v \rangle_x^{(\mu)} + \langle v \rangle_t^{(\mu/2)}$. By the interpolation lemma 6.35 in [7] we have (with t as a parameter)

$$\begin{aligned} |D^2 u|_{0, \Omega_T} + |Du|_{0, \Omega_T} + \langle Du \rangle_x^{(\mu)} + \langle u \rangle_x^{(\mu)} &\leq \epsilon \langle D^2 u \rangle_x^{(\mu)} + C(\epsilon) |u|_{0, \Omega_T} \\ &\leq \epsilon [D^2 u]_{\mu, \Omega} + C(\epsilon) |u|_{0, \Omega_T}. \end{aligned} \quad (47)$$

It remains to estimate $\langle Du \rangle_t^{(\mu/2)}$ and $\langle u \rangle_t^{(\mu/2)}$. For the latter term, one easily sees that

$$\langle u \rangle_t^{(\mu/2)} \leq \epsilon |u_t|_{0, \Omega_T} + C(\epsilon) |u|_{0, \Omega_T}. \quad (48)$$

To estimate the first term, we write $\Omega = \bigcup U_j$ as in the proof of the previous theorem. In particular, we may assume w.l.o.g. that the boundary is flat. Let $x \in \Omega$ and $0 < s - t \leq 1$. Set $x_i = x + h e_i$, where e_i is the i^{th} coordinate vector, $1 \leq i \leq n$, and $h > 0$ is yet to be chosen. Then

$$D_i u(\hat{x}_i, t) = h^{-1} (u(x_i, t) - u(x, t))$$

and

$$D_i u(\tilde{x}_i, s) = h^{-1} (u(x_i, s) - u(x, s)),$$

for suitable \hat{x}_i and $\tilde{x}_i \in \Omega$. Now we can write

$$\begin{aligned} |D_i u(x, t) - D_i u(x, s)| &\leq |D_i u(\hat{x}_i, t) - D_i u(\tilde{x}_i, s)| + |D_i u(x, t) - D_i u(\hat{x}_i, t)| + \\ &\quad + |D_i u(x, s) - D_i u(\tilde{x}_i, s)| \\ &\leq h^{-1} |u(x_i, t) - u(x_i, s)| + h^{-1} |u(x, t) - u(x, s)| + \\ &\quad + |D^2 u|_{0, \Omega_T} |\hat{x}_i - x| + |D^2 u|_{0, \Omega_T} |\tilde{x}_i - x| \\ &\leq 2h^{-1} |u_t|_{0, \Omega_T} |t - s| + 2 |D^2 u|_{0, \Omega_T} h. \end{aligned} \quad (49)$$

Let $h = 2\epsilon^{-1} |t - s|^{1-\mu/2}$. It then follows immediately from (49) that

$$|D_i u(x, t) - D_i u(x, s)| \leq |t - s|^{\mu/2} (\epsilon |u_t|_{0, \Omega_T} + |D^2 u|_{0, \Omega_T}),$$

provided $|t - s| \leq (\epsilon/4)^{1/(1-\mu)}$. If, on the other hand, $|t - s| > (\epsilon/4)^{1/(1-\mu)}$,

$$|D_i u(x, t) - D_i u(x, s)| \leq 2 |t - s|^{\mu/2} |Du|_{0, \Omega_T} (\epsilon/4)^{-\mu/2(1-\mu)}.$$

Consequently,

$$\langle Du \rangle_t^{(\mu/2)} \leq \epsilon |u_t|_{0, \Omega_T} + |D^2 u|_{0, \Omega_T} + C(\epsilon) |Du|_{0, \Omega_T}. \quad (50)$$

The lemma now follows from (47), (48) and (50). ■

6 BMO estimates

The Hölder estimates derived above are no longer true for $\mu = 0$. By analogy with the elliptic case, however, we expect that $D^2u, u_t \in \text{BMO}$ if $f \in L^\infty$. If we define BMO by using parabolic cylinders instead of Euclidian balls, this is indeed the case and shall be demonstrated below. By interpolating between BMO and L^2 via Stampacchia's theorem, we shall derive L^p -estimates for parabolic systems (see [2] and [1]).

Definition 3 *Let Q_1, Q_2 , and Γ be as in Definition 1, but with $T = 1$. Furthermore, we define*

$$Q_3 \equiv B_3^+ \times [0, 1], \hat{Q}_1 \equiv B_1(0) \times [0, 1], \hat{Q}_2 \equiv B_2(0) \times [0, 1], \hat{Q}_3 \equiv B_3(0) \times [0, 1].$$

In the following Proposition we derive BMO-estimates for weak solutions of the parabolic system

$$u_t^\alpha - A_{ij}^{\alpha\beta} D_{ij} u^\beta = f^\alpha, \quad (51)$$

where $A = \text{const.}$ is as in (5).

Proposition 4 *Suppose $f \in L^\infty(Q_2)$ and let u be a weak solution of (51) in Q_2 with $u = 0$ on Γ . Then, for any $(x_0, t_0) \in Q_1$ and any $0 < r \leq 1$,*

$$\int_{P_r(x_0, t_0)} |D^2u - (D^2u)_r|^2 \leq C \left(|f|_{0, Q_2}^2 + \int_{Q_2} |D^2u|^2 \right). \quad (52)$$

The same conclusion holds with \hat{Q}_1 and \hat{Q}_2 replacing Q_1 and Q_2 , respectively. In this case we only need to assume that $u = 0$ on $\{t = 0\}$.

Proof: We shall follow the proof of Proposition 2 closely. In particular, the same notation will be used throughout. Assume that f and u satisfy the hypotheses stated in Proposition 4. Let $(x_0, t_0) \in Q_1 \setminus \Gamma$ and $0 < \rho < r \leq \rho_0$. The calculations on p. 10 show that any solution w of (28) satisfies

$$\int_{P_r} |Dw|^2 \leq Cr^{n+2} |f|_{0, Q_2}^2.$$

Hence, we now have

$$\int_{P_\rho} |Dv - (Dv)_\rho|^2 \leq C \left(\frac{\rho}{r} \right)^{n+4} \int_{P_r} |Dv - (Dv)_r|^2 + Cr^{n+2} |f|_{0, Q_2}^2$$

instead of (30). In conjunction with Lemma 2 this implies

$$\int_{P_\rho} |D^2u - (D^2u)_\rho|^2 \leq C \int_{P_r} |D^2u - (D^2u)_r|^2 + C |f|_{0, Q_2}^2, \quad (53)$$

for $0 < \rho < r \leq \rho_0$.

Next let $(x_0, t_0) \in \Gamma_1$ and $0 < \rho < r \leq \rho_2$. It is easy to see that we now have

$$\sum_{i,j=1}^{n-1} \int_{P_\rho} |D_{ij}u|^2 \leq C \sum_{i,j=1}^{n-1} \int_{P_{\rho_2}} |D_{ij}u|^2 + C|f|_{0,Q_2}^2 \quad (54)$$

and

$$\sum_{i=1}^{n-1} \int_{P_\rho} |D_{in}u - (D_{in}u)_\rho|^2 \leq C \sum_{i=1}^{n-1} \int_{P_{\rho_2}} |D_{in}u - (D_{in}u)_{\rho_2}|^2 + C|f|_{0,Q_2}^2 \quad (55)$$

instead of (24) and (25). The missing term, $D_{nn}u$, can again be estimated via u_t . Here, however, it is unnecessary to introduce the auxiliary function ϕ as in (31). Namely, setting $v = u$ and $g = f$, the calculations on p. 11, with the same modifications as above, show that

$$\int_{P_\rho} |u_t|^2 \leq C \int_{P_{\rho_2}} |u_t|^2 + C|f|_{0,Q_2}^2.$$

By solving (51) for $D_{nn}u$, we conclude that

$$\begin{aligned} \int_{P_\rho} |D_{nn}u - (D_{nn}u)_\rho|^2 &\leq C \left\{ \int_{P_\rho} |u_t|^2 + \sum_{i+j < 2n} \int_{P_\rho} |D_{ij}u - (D_{ij}u)_\rho|^2 \right\} \\ &\leq C \left\{ \int_{P_{\rho_2}} |u_t|^2 + \int_{P_{\rho_2}} |D^2u|^2 + |f|_{0,Q_2}^2 \right\} \\ &\leq C \left\{ \int_{P_{\rho_2}} |D^2u|^2 + |f|_{0,Q_2}^2 \right\} \end{aligned} \quad (56)$$

for $0 < \rho < \rho_2$.

By similar arguments one can easily see, if $0 < \rho \leq \rho_i$, that u will satisfy

$$\int_{P_\rho(x_0,0)} |D^2u|^2 \leq C \left\{ \int_{P_{\rho_i}} |D^2u|^2 + |f|_{0,Q_2}^2 \right\} \quad (57)$$

for points $(x_0, 0)$ on the lower boundary or on the corner of Q_1 . Here $i = 1$ if $(x_0, 0) \in \Gamma_2$ and $i = 3$ if $(x_0, 0) \in \Gamma_3$. (52) now follows from estimates (53) through (57) by exactly the same procedure that was used in the proof of Proposition 3. The statements about \hat{Q}_1 and \hat{Q}_2 are implicit in the above. ■

7 L^p estimates

The following technical lemma will be needed in the proof of Theorem 4. It asserts that we can cover Ω_T by diffeomorphic images of parabolic cylinders with arbitrarily small

diameters and with overlap independent of the diameter. Moreover, we can control the distortion of the diffeomorphisms uniformly.

Lemma 4 *Let Ω be a bounded C^2 domain in R^n and let $T > 0$. Then for any $0 < \delta < 1$ there exist finitely many $P_i \subseteq \hat{P}_i \subseteq \bar{\Omega}_T$ and diffeomorphisms Ψ_i such that, with a fixed constant $C = C(n, \Omega)$,*

- i. $\bar{\Omega}_T = \bigcup_i P_i$
- ii. $\text{diam}(\hat{P}_i) < \delta$ for all i (the diameter being measured in the parabolic metric)
- iii. for all i , either $\Psi_i(P_i) \subseteq Q_1$ and $\Psi_i(\hat{P}_i) = Q_3$ if $P_i \cap (\partial\Omega \times [0, T]) \neq \emptyset$, or $\Psi_i(P_i) \subseteq \hat{Q}_1$ and $\Psi_i(\hat{P}_i) = \hat{Q}_3$ if $P_i \subseteq \Omega_T$
- iv. for all i , $\Psi_i(x, t) = (\psi_i(x), \lambda_i t)$, where λ_i is comparable to δ^{-2} and the following estimate holds:

$$\delta |D\Psi_i|_{0, \hat{P}_i} + \delta^{-1} |(D\Psi_i)^{-1}|_{0, \hat{P}_i} + \delta^2 |D^2\Psi_i|_{0, \hat{P}_i} \leq C$$

- v. $\sup_{x \in \Omega_T} \#\{i | x \in \hat{P}_i\} \leq C$ (i.e. the overlap of the \hat{P}_i 's does not depend on δ).

Proof: Let $\Omega = \bigcup_{j=1}^N U_j$ where U_j, V_j, ϕ_j are as in the proof of Theorem 3. Recall that $\phi_j(U_j) = B_1^+$ and $\phi_j(V_j) = B_2^+$. We can write B_1^+ as the union of pairwise disjoint, congruent dyadic cubes of arbitrarily small diameter, say 2^{-M} . Let C_1, C_2, \dots, C_m be those cubes with one face on $\{x_n = 0\}$. Set $U_{j,k} \equiv \phi_j^{-1}(\bar{C}_k)$ and $V_{j,k} \equiv \phi_j^{-1}(3.C_k)$, for all $1 \leq j \leq N$ and $1 \leq k \leq m$, where $3.C_k$ denotes the cube obtained from C_k through dilatation by a factor of three around the center of its face on $\{x_n = 0\}$. Clearly, we can choose M such that

$$\{x \in \Omega \mid \text{dist}(x, \partial\Omega) < \gamma\delta\} \subseteq \bigcup_{j,k} V_{j,k} \quad (58)$$

and $\text{diam}(V_{j,k}) < \delta$, where $\gamma = \gamma(n, \Omega) > 0$ is a suitable constant. Furthermore, any $x \in \Omega$ is contained in at most $3N$ of the $V_{j,k}$'s.

As for the interior of Ω , we let $V_0 \equiv \Omega \setminus \bigcup_{j,k} U_{j,k}$. V_0 is open, $\bar{V}_0 \subseteq \Omega$, and $\Omega = V_0 \cup \bigcup_{j,k} V_{j,k}$. V_0 is clearly contained in the union of congruent, dyadic subcubes E_l with disjoint interiors such that $\hat{E}_l \subseteq \Omega$. Here \hat{E}_l denotes the cube obtained from E_l by dilating it around its center by a factor of three. Furthermore, (58) shows that we may take $\text{diam}(E_l)$ to be bounded below by a constant multiple of δ . Moreover, the overlap of the \hat{E}_l is controlled by n . In summary, we can write

$$\begin{aligned} \Omega &= \bigcup_l E_l \cup \bigcup_{j,k} U_{j,k} \equiv \bigcup_i W_i \\ &= \bigcup_l \hat{E}_l \cup \bigcup_{j,k} V_{j,k} \equiv \bigcup_i \hat{W}_i. \end{aligned}$$

The sets on the right hand side have diameters comparable to δ and their overlap is controlled by n and Ω . If \hat{W}_i is one of the \hat{E}_l , we let ψ_i be a dilatation followed by a translation, and if W_i is one of the $V_{j,k}$, we let ψ_i be ϕ_j , followed by a dilatation and a translation.

Finally, define $P_i \equiv W_i \times ([0, T] \cap [m\delta^2, (m+1)\delta^2])$ and $\hat{P}_i \equiv \hat{W}_i \times ([0, T] \cap [(m-1)\delta^2, (m+2)\delta^2])$, where $m = 0, 1, 2, \dots, [T/\delta^2]$. It is clear that these sets, together with the Ψ_i given above, have the desired properties. ■

We shall now derive L^p -estimates for classical solutions of the general parabolic system (43). We assume in the following that $A \in C(\bar{\Omega}_T)$, that $B, C \in L^\infty(\Omega_T)$, and that $\partial\Omega \in C^2$. Unless otherwise indicated, constants will be allowed to depend only on $p, n, \lambda, \Lambda, \Omega, |B|_{0, \Omega_T} + |C|_{0, \Omega_T}$, and the modulus of continuity of A .

Theorem 4 *Let $u \in C^{2,1}(\Omega_T)$ be a classical solution of (43). Then, for $2 \leq p < \infty$,*

$$\|D^2 u\|_{L^p(\Omega_T)} + \|u_t\|_{L^p(\Omega_T)} \leq C_0 \left(\|f\|_{L^p(\Omega_T)} + \|u\|_{L^p(\Omega_T)} \right) \leq C_1(T) \|f\|_{L^p(\Omega_T)}.$$

Proof: Fix any $\delta \in (0, 1)$ and let P_i, \hat{P}_i , and Ψ_i be as in the previous lemma. Let (P, \hat{P}, Ψ) stand for any one of the (P_i, \hat{P}_i, Ψ_i) . We define v by the change of coordinates

$$u(x, t) = (v \circ \Psi)(x, t) = v(\Psi(x, t)) = v(\psi(x), \lambda t) = v(y, s).$$

It is easy to see that v will satisfy the equation

$$v_s - \hat{A} D^2 v + \hat{B} D v + \hat{C} v = \hat{f},$$

where $C_0^{-1} \lambda |\xi|^2 \leq (\hat{A} \xi, \xi) \leq C_0 \Lambda |\xi|^2$, $|\hat{B}|_{0, Q_3} + |\hat{C}|_{0, Q_3} \leq C_0(|B|_{0, \Omega_T} + |C|_{0, \Omega_T})$, and $|\hat{f}(y, s)| \leq C_0 \delta^2 |f(x, t)|$ for all $(x, t) \in \hat{P}$, with a constant $C_0 = C_0(n, \Omega)$. By (iii) of the previous lemma, the image of \hat{P} under Ψ will be either Q_3 or \hat{Q}_3 . For ease of notation, we shall make no distinction between those two cases. Let ζ be a smooth cutoff function such that $\zeta = 1$ on $\Psi(P) \subseteq Q_1$ and such that $w \equiv \zeta v = 0$ on ∂Q_3 , the parabolic boundary of Q_3 . Then w solves the system

$$\begin{cases} w_s - \hat{A}_0 D^2 w &= g & \text{in } Q_3 \\ w &= 0 & \text{on } \partial Q_3, \end{cases} \quad (59)$$

where

$$\begin{aligned} g &= \zeta(\hat{f} - \hat{B} D v - \hat{C} v) + \zeta(\hat{A} - \hat{A}_0) D^2 v + \zeta_s v - \\ &\quad - \hat{A}_0 (D \zeta \otimes D v + D v \otimes D \zeta) - \hat{A}_0 D^2 \zeta v \end{aligned} \quad (60)$$

and $\hat{A}_0 \equiv A(y_0, s_0)$ for an arbitrary but fixed $(y_0, s_0) \in Q_3$ ($D\zeta \otimes Dv$ stands for the matrix $(\zeta_{y^i} v_{y^j})_{i,j}$). A well known estimate asserts that

$$\int_{Q_2} |D^2 w|^2 \leq C \int_{Q_3} |g|^2 \quad (61)$$

for the weak solution w of (59) with an arbitrary $g \in L^2(Q_3)$. Furthermore, Proposition 4 in conjunction with (61) shows that

$$\int_{P_r(y_1, s_1)} |D^2 w - (D^2 w)_r|^2 \leq C |g|_{0, Q_3}^2, \quad (62)$$

for all $(y_1, s_1) \in Q_1$ and $0 < r \leq 1$. We now infer from (61), (62), and Theorem 5 that, for any $2 \leq p < \infty$,

$$\int_{Q_2} |D^2 w|^p \leq C \int_{Q_3} |g|^p. \quad (63)$$

Since $\zeta = 1$ on $\Psi(P) \subseteq Q_1$, we conclude that

$$\int_{\Psi(P)} |D^2 v|^p \leq C \int_{Q_3} |g|^p,$$

or, after transforming back into the (x, t) -variables and using (iv) of the previous lemma,

$$\int_P |(D^2 v) \circ \Psi|^p \leq C \int_{\hat{P}} |g \circ \Psi|^p. \quad (64)$$

Since $C^{-1} \delta^2 |D^2 u| - C |(Dv) \circ \Psi| \leq |(D^2 v) \circ \Psi| \leq C \delta^2 |D^2 u| + C |(Dv) \circ \Psi|$ everywhere in \hat{P} with some constant $C = C(n, \Omega)$, we obtain, by substituting the definition of g from (60) into (64)

$$\int_{P_j} |D^2 u|^p \leq C \sup_{\hat{P}_j} |A - A(x_j, t_j)|^p \int_{\hat{P}_j} |D^2 u|^p + C_\delta \int_{\hat{P}_j} (|f|^p + |Du|^p + |u|^p), \quad (65)$$

where $(x_j, t_j) \in P_j$ is arbitrary but fixed. Summing (65) over j yields

$$\begin{aligned} \int_{\Omega_T} |D^2 u|^p &\leq C \sup_j \sup_{\hat{P}_j} |A - A(x_j, t_j)|^p \int_{\Omega_T} |D^2 u|^p + \\ &+ C_\delta \int_{\Omega_T} (|f|^p + |Du|^p + |u|^p). \end{aligned} \quad (66)$$

Note that property (v) was used here in an essential way. Now choose $\delta > 0$ sufficiently small such that $C_0 \sup_j \sup_{\hat{P}_j} |A - A(x_j, t_j)|^p = \frac{1}{2}$ (recall that $\text{diam}(\hat{P}_i) < \delta$ for all i). With this choice of δ ,

$$\begin{aligned} \int_{\Omega_T} |D^2 u|^p &\leq C \int_{\Omega_T} (|f|^p + |Du|^p + |u|^p) \\ &\leq C \int_{\Omega_T} (|f|^p + \epsilon |D^2 u|^p + C(\epsilon) |u|^p), \end{aligned}$$

where we have used a standard interpolation inequality in Sobolev spaces (see [7], Theorem 7.28). We conclude that

$$\int_{\Omega_T} (|u_t|^p + |D^2 u|^p) \leq C \int_{\Omega_T} (|f|^p + |u|^p). \quad (67)$$

The theorem now follows easily from (67) and Gronwall's inequality. ■

8 Stampacchia's interpolation theorem

Stampacchia's interpolation theorem asserts that a sublinear operator which is bounded from L^∞ to BMO and from L^{p_0} to itself, for some finite $p_0 > 1$, is bounded as an operator from L^p to itself for p in the range (p_0, ∞) . As pointed out in [1], §3, the original proof in [10] carries over to the parabolic case simply by replacing Euclidean balls with parabolic cylinders. However, to keep this note as self-contained and transparent as possible, we shall deduce the interpolation theorem from an elegant argument given in [11], chapter IV, §3, which is based on the concept of the "sharp-function" from Hardy space theory (see also [5]). Finally, we want to remark that the case considered here is a special case of the general setting of nonisotropic dilations, in which many results of Hardy space theory hold (for this see [11], chapter IV, §6.21, and the references therein).

Definition 4 Let $\Delta_0 = \{Q_1\}$ and suppose that Δ_k has been defined up to some $k \geq 0$. Then Δ_{k+1} is the set of all cylinders which are obtained from the ones in Δ_k by bisecting their sides and dividing their height into four equal parts. Finally, we let $\Delta = \bigcup_k \Delta_k$. We shall refer to the elements of Δ as dyadic cylinders. For any $f \in L^1(Q_1)$ and $x \in Q_1$, we define

$$Mf(x) \equiv \sup_{x \in Q_1 \in \Delta} \int_P |f|$$

and

$$f^\sharp(x) \equiv \sup_{x \in P \in \Delta} \int_P |f - (f)_{Q_1}|.$$

We shall refer to $\{f \in L^1(Q_1) \mid \|f^\sharp\|_{L^\infty(Q_1)} < \infty\}$ as (parabolic, dyadic) BMO, and we let $\|f\|_{BMO} \equiv \|f^\sharp\|_{L^\infty}$.

Proposition 5 For all $f \in L^1(Q_1)$ and all $\lambda > 0$,

$$i. \quad |\{Q_1 \mid Mf > \lambda\}| \leq \frac{1}{\lambda} \int_{Q_1} |f|$$

$$ii. \quad \|Mf\|_{L^p(Q_1)} \leq C(p, n) \|f\|_{L^p(Q_1)} \text{ for all } 1 < p \leq \infty.$$

Proof: Let $\lambda_0 \equiv f_{Q_1} |f|$. If $\lambda \leq \lambda_0$, (i) is obvious. Hence, we may assume that $\lambda > \lambda_0$. This implies that for every $x \in \{Q_1 \mid Mf > \lambda\}$, there exists a unique maximal $P = P(x) \in \Delta$ such that $x \in P$ and $f_P |f| > \lambda$. Moreover, no two such maximal cubes can intersect. Hence,

$$\{Q_1 \mid Mf > \lambda\} = \bigcup_j P_j$$

with $P_j = P(x_j)$ as above. Thus

$$|\{Q_1 \mid Mf > \lambda\}| = \sum_j |P_j| < \frac{1}{\lambda} \sum_j \int_{P_j} |f| \leq \frac{1}{\lambda} \int_{Q_1} |f|.$$

Because $\|Mf\|_{L^\infty(Q_1)} \leq \|f\|_{L^\infty(Q_1)}$, (ii) follows from (i) and Marcinkiewicz's interpolation theorem. ■

Proposition 6 *Let $f \in L^1(Q_1)$. Then for all $\lambda \geq \lambda_0$ and all $\gamma > 0$,*

$$|\{Q_1 \mid Mf > 2\lambda, f^\# \leq \gamma\lambda\}| \leq 2^{n+2} \gamma |\{Q_1 \mid Mf > \lambda\}|. \quad (68)$$

Proof: As in the previous proof, we can write

$$\{Q_1 \mid Mf > \lambda\} = \bigcup_j P_j.$$

Suppose $x \in P_j$ and $Mf(x) > 2\lambda$. By the choice of P_j , $f_P |f| \leq \lambda$ for any $P \in \Delta$, $P \supset P_j$. Hence $M(f\chi_{P_j})(x) > 2\lambda$. Let $\tilde{P}_j \in \Delta$ be the parent cylinder of P_j , which exists because $P_j \neq Q_1$. Then

$$M[(f - (f)_{\tilde{P}_j})\chi_{P_j}](x) > 2\lambda - |(f)_{\tilde{P}_j}| \geq \lambda.$$

We thus obtain

$$\begin{aligned} |\{P_j \mid Mf > 2\lambda, f^\# \leq \gamma\lambda\}| &\leq |\{Q_1 \mid M[(f - (f)_{\tilde{P}_j})\chi_{P_j}] > \lambda, f^\# \leq \gamma\lambda\}| \\ &\leq \frac{|\tilde{P}_j|}{\lambda} \int_{\tilde{P}_j} |f - (f)_{\tilde{P}_j}| \\ &\leq \lambda^{-1} |\tilde{P}_j| \inf_{y \in \tilde{P}_j} f^\#(y) \leq \gamma 2^{n+2} |P_j|, \end{aligned} \quad (69)$$

where we assumed that the set on the left hand side is nonempty. Summing (69) over j yields

$$|\{Q_1 \mid Mf > 2\lambda, f^\# \leq \gamma\lambda\}| \leq \gamma 2^{n+2} |\{Q_1 \mid Mf > \lambda\}|. \quad \blacksquare$$

Proposition 7 *Let $f \in L^1(Q_1)$. Then for any $1 \leq p < \infty$,*

$$\int_{Q_1} |Mf|^p \leq C \left\{ \int_{Q_1} |f^\#|^p + \left(\int_{Q_1} |f| \right)^p \right\},$$

where $C = C(n, p)$.

Proof: Fix $A > \lambda_0$ and compute:

$$\begin{aligned} \int_{\lambda_0}^A |\{Q_1 \mid Mf > 2\lambda\}| p \lambda^{p-1} d\lambda &\leq \int_{\lambda_0}^A |\{Q_1 \mid Mf > 2\lambda, f^\# \leq \gamma\lambda\}| p \lambda^{p-1} d\lambda \\ &\quad + \int_{\lambda_0}^A |\{Q_1 \mid f^\# > \gamma\lambda\}| p \lambda^{p-1} d\lambda \\ &\leq 2^{n+2} \gamma \int_{\lambda_0}^A |\{Q_1 \mid Mf > \lambda\}| p \lambda^{p-1} d\lambda + \\ &\quad + \gamma^{-p} \int_{Q_1} |f^\#|^p \\ &\leq 2^{n+2+p} \gamma \int_{\lambda_0}^A |\{Q_1 \mid Mf > 2\lambda\}| p \lambda^{p-1} d\lambda + \\ &\quad + \gamma^{-p} \int_{Q_1} |f^\#|^p + 2^{2+n+p} \gamma \lambda_0^p |Q_1|. \end{aligned} \tag{70}$$

Choosing $\gamma = 2^{-3-n-p}$, we infer from (70) that

$$\int_{\lambda_0}^A |\{Q_1 \mid Mf > 2\lambda\}| p \lambda^{p-1} d\lambda \leq \lambda_0^p |Q_1| + C(p, n) \int_{Q_1} |f^\#|^p.$$

Letting $A \rightarrow \infty$ finishes the proof. ■

Stampacchia's interpolation result now follows easily from Propositions 5 and 7.

Theorem 5 *Let S be a subadditive operator from $L^\infty(X) + L^{p_0}(X)$, where (X, μ) is any finite measure space and $p_0 > 1$ is fixed, into the measurable functions on Q_1 . Suppose that*

$$\begin{aligned} \|Sf\|_{BMO(Q_1)} &\leq C_1 \|f\|_{L^\infty(X)} \quad \text{and} \\ \|Sf\|_{L^{p_0}(Q_1)} &\leq C_2 \|f\|_{L^{p_0}(X)}. \end{aligned}$$

Then S is bounded on L^p for $p_0 \leq p < \infty$ and

$$\|Sf\|_{L^p(Q_1)} \leq C(C_1, C_2, p, p_0, \mu(X)) \|f\|_{L^p(X)}$$

for all $f \in L^p(Q_1)$.

Proof: Define $\tilde{S}f \equiv (Sf)^\sharp$ for all $f \in L^\infty(X) + L^{p_0}(X)$. Then $\|\tilde{S}f\|_{L^\infty(Q_1)} \leq C_1 \|f\|_{L^\infty(X)}$ and

$$\|\tilde{S}f\|_{L^{p_0}(Q_1)} \leq 2\|M(Sf)\|_{L^{p_0}(Q_1)} \leq C(p_0, n)\|Sf\|_{L^{p_0}(Q_1)} \leq C(p_0, n)C_2\|f\|_{L^{p_0}(X)}.$$

Thus, by Marcinkiewicz's interpolation theorem,

$$\|\tilde{S}f\|_{L^p(Q_1)} \leq C(p_0, n, p, C_1, C_2)\|f\|_{L^p(X)}$$

for all $f \in L^p(X)$ and $p_0 \leq p < \infty$. From Proposition 7 we finally infer that

$$\begin{aligned} \int_{Q_1} |Sf|^p &\leq \int_{Q_1} |M(Sf)|^p \leq C \left\{ \int_{Q_1} |\tilde{S}f|^p + \left(\int_{Q_1} |Sf| \right)^p \right\} \\ &\leq C|Q_1|^{-1} \left\{ \int_X |f|^p d\mu + \left(\int_X |f|^{p_0} d\mu \right)^{\frac{p}{p_0}} \right\} \\ &\leq C|Q_1|^{-1} \int_X |f|^p d\mu. \blacksquare \end{aligned}$$

ACKNOWLEDGEMENT: The estimates in this note were needed in research on a nonlinear parabolic system, which I conducted under the guidance of Prof. L.C. Evans. I am grateful to him for introducing me to Campanato's method as well as for many fruitful conversations. I also want to thank Prof. Tom Wolff for the opportunity to discuss some of the material in [11] with him.

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