Math 116 Midterm Solutions

1. (a) Using the partial fractions expansion,

$$\frac{x}{x^2-4x-5} = \frac{x}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x+1}$$

we obtain

$$x = A(x+1) + B(x-5)$$

Taking $x=-1$ gives $B=1/6$.
Taking $x=5$ gives $A=5/6$.

Then the integral becomes

$$\begin{aligned}\int \frac{x}{x^2-4x-5}dx &= (5/6)\int \frac{1}{x-5}dx + (1/6)\int \frac{1}{x+1}dx \\ &= (5/6)\ln|x-5| + (1/6)\ln|x+1| + C\end{aligned}$$

(b) Substitute $w=\sin(x)$ so $dw=\cos(x)dx$ and we see

$$\int \sin(x)\cos(x)e^{\sin(x)}dx = \int we^w dw$$

Now integrate by parts, taking $u=w$ and $dv=e^w dw$, so $du=dw$ and $v=e^w$:

$$\begin{aligned}\int we^w dw &= we^w - \int e^w dw \\ &= we^w - e^w + C\end{aligned}$$

Consequently,

$$\int \sin(x)\cos(x)e^{\sin(x)}dx = \sin(x)e^{\sin(x)} - e^{\sin(x)} + C$$

(c) The only sensible substitution is $u=\tan(x)$, so $du=\sec^2(x)dx$. Then instead of dx in the numerator, we need $\sec^2(x)dx$. So multiply the numerator and the denominator by $\sec^2(x)$:

$$\int \frac{dx}{\cos(x)\sin(x)\sqrt{1-\tan^2(x)}} = \int \frac{\sec^2(x)dx}{\cos(x)\sin(x)\sec^2(x)\sqrt{1-\tan^2(x)}}$$

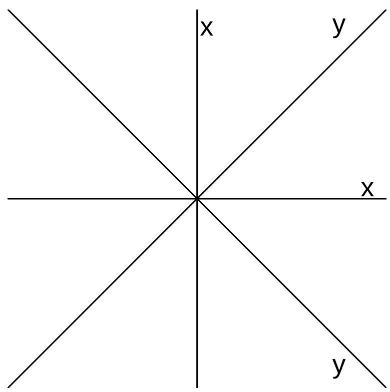
Now

$$\begin{aligned}\cos(x)\sin(x)\sec^2(x) &= \cos(x)\sin(x)\frac{1}{\cos^2(x)} \\ &= \frac{\sin(x)}{\cos(x)} \\ &= \tan(x)\end{aligned}$$

Consequently,

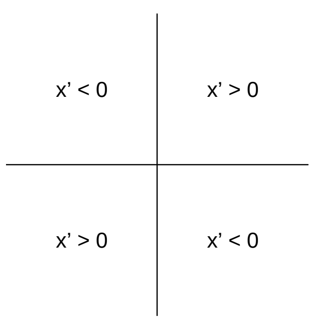
$$
\begin{aligned}
\int \frac{dx}{\cos(x)\sin(x)\sqrt{1-\tan^2(x)}} &= \int \frac{\sec^2(x)dx}{\tan(x)\sqrt{1-\tan^2(x)}} \\
&= \int \frac{du}{u\sqrt{1-u^2}} \qquad \text{sunstituting } u = \tan(x) \\
&= -\ln\left|\frac{\sqrt{1-u^2}+1}{u}\right| + C \qquad \text{30 in the table} \\
&= -\ln\left|\frac{\sqrt{1-\tan^2(x)}+1}{\tan(x)}\right| + C
\end{aligned}
$$

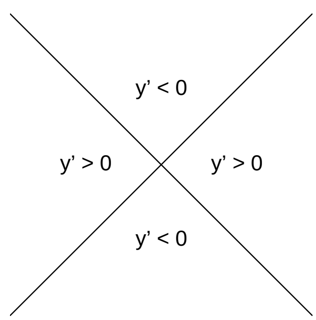
2. (a) The x-nullcline is $xy = 0$, that is, $x = 0$ and $y = 0$. The y-nullcline is $x^2 = y^2$, that is $y = \pm x$.



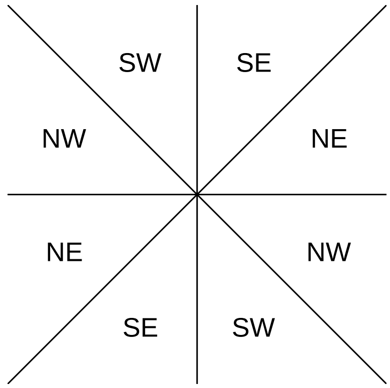
(b) First, consider the plane divided into regions by the x-nullcline. Because $x' = xy$, we see $x' > 0$ in the first and third quadrants, and $x' < 0$ in the second and fourth quadrants.

Next, consider the plane divided into regions by the y-nullcline. Because $y' = x^2 - y^2$, we can determine the sign of y' by testing at some points. At $(1, 0)$, $y' > 0$, At $(0, 1)$, $y' < 0$. At $(-1, 0)$, $y' > 0$. At $(0, -1)$, $y' < 0$.

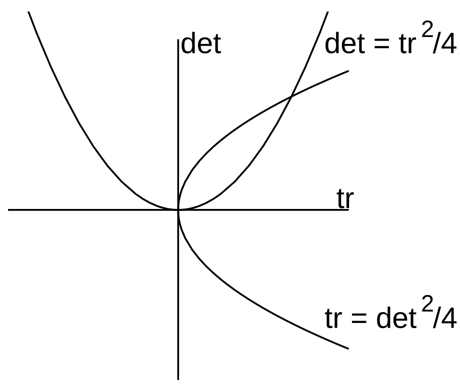




Combining these, we find



3. The trace of the matrix is $\mathrm{tr} = a^2/4$ and the determinant is $\det = a$. Consequently, for every value of a, tr and det are related by $\mathrm{tr} = \det^2/4$. In the trace-determinant plane, this is a parabola opening to the right.



The parabolas $\det = \mathrm{tr}^2/4$ and $\mathrm{tr} = \det^2/4$ intersect where

$$\det = \mathrm{tr}^2/4 = (\det^2/4)^2/4 = \det^4/4^3$$

That is,

$$0 = \det(1 - \det^3/4^3)$$

so $\det = 0$ and $\det = 4$.

Then the fixed point at the origin is

- a saddle point for $\det < 0$, that is, for $a < 0$,
- an unstable spiral for $0 < \det < 4$, that is, for $0 < a < 4$, and
- an unstable node for $4 \leq \det$, that is, for $4 \leq a$

4. (a) Use the box-counting dimension formula:

$$
\begin{aligned}
d = \lim_{n\to\infty} \frac{\log(2^n + 3^n)}{\log(1/(1/2^n))} &= \lim_{n\to\infty} \frac{\log(3^n(2^n/3^n + 1)}{\log(2^n)} \\
&= \lim_{n\to\infty} \left(\frac{n\log(3)}{n\log(2)} + \frac{\log(2^n/3^n + 1)}{n\log(2)} \right) \\
&= \frac{\log(3)}{\log(2)}
\end{aligned}
$$

(b) In order to have a box-counting dimension twice that of the shape in (a), observe

$$
2\frac{\log(3)}{\log(2)} = \frac{2\log(3)}{\log(2)} = \frac{\log(3^2)}{\log(2)} = \frac{\log(9)}{\log(2)}
$$

In the problem parameters, $N(1/2^n) = 2^n + 3^n + K^n$, this can be achieved by taking $K = 9$. For then

$$
\begin{aligned}
d = \lim_{n\to\infty} \frac{\log(2^n + 3^n + 9^n)}{\log(1/(1/2^n))} &= \lim_{n\to\infty} \frac{\log(9^n(2^n/9^n + 3^n/9^n + 1)}{\log(2^n)} \\
&= \lim_{n\to\infty} \left(\frac{n\log(9)}{n\log(2)} + \frac{\log(2^n/9^n + 3^n/9^n + 1)}{n\log(2)} \right) \\
&= \frac{\log(9)}{\log(2)}
\end{aligned}
$$