Math 116 Practice Midterm 1 Solutions

1. (a)

$$\int \frac{x+1}{x^2+3x-10}dx$$

Use partial fractions

$$\frac{x+1}{x^2+3x-10} = \frac{x+1}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2}$$

from which we find

$$x+1 = A(x-2) + B(x+5)$$

Taking $x = 2$ gives $B = 3/7$; taking $x = -5$ gives $A = 4/7$, and so

$$\begin{aligned}\int \frac{x+1}{x^2+3x-10}dx &= \frac{4}{7}\int \frac{1}{x+5}dx + \frac{3}{7}\int \frac{1}{x-2}dx \\ &= \frac{4}{7}\ln|x+5| + \frac{3}{7}\ln|x-2| + C\end{aligned}$$

(b)

$$\int e^{2x}\cos(e^x)dx$$

Substitute $w = e^x$, so $dw = e^x dx$ and

$$\int e^{2x}\cos(e^x)dx = \int (e^x)^2\cos(e^x)dx = \int w\cos(w)dw$$

Now integrate by parts.

$$\begin{aligned} u &= w & dv &= \cos(w)dw \\ du &= dw & v &= \sin(w)\end{aligned}$$

Then

$$\begin{aligned}\int w\cos(w)dw &= w\sin(w) - \int \sin(w)dw \\ &= w\sin(w) + \cos(w) + C\end{aligned}$$

Substituting back in for w,

$$\int e^{2x}\cos(e^x)dx = e^x\sin(e^x) + \cos(e^x) + C$$

(c)

$$\int \frac{x}{\sqrt{1+x^4}}dx$$

Substitute $u = x^2$, so $du = 2xdx$ and the integral becomes

$$\begin{aligned}\int \frac{x}{\sqrt{1+x^4}}dx &= \frac{1}{2}\int \frac{du}{\sqrt{1+u^2}} \\ &= \frac{1}{2}\ln\left|u+\sqrt{1+u^2}\right| + C \quad \text{rule 20} \\ &= \frac{1}{2}\ln\left|x^2+\sqrt{1+x^4}\right| + C\end{aligned}$$

2. Because ϕ_0 lies in the left interval and ϕ_1 in the right,

$$\phi_1 = f(\phi_0) = b\phi_0 + \tau$$

and

$$\phi_0 = f(\phi_1) = b\phi_1 + \tau - 2$$

Combining these

$$\phi_0 = b(b\phi_0 + \tau) + \tau - 2 = b^2\phi_0 + b\tau + \tau - 2$$

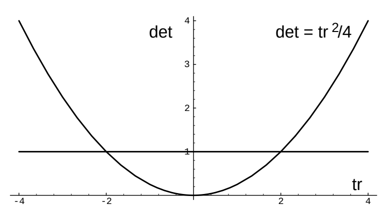
Solving for ϕ_0

$$\phi_0 = \frac{b\tau + \tau - 2}{1 - b^2}$$

3. For the differential equation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1+a & 1 \\ a & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

observe that $\text{tr} = 1+a+1 = 2+a$ and $\det = (1+a)\cdot 1 - a\cdot 1 = 1$. Then in the trace-determinant plane, the path of these matrices is the horizontal line at $\det = 1$.



The line $\det = 1$ crosses the parabola $\det = \text{tr}^2/4$ at $\text{tr}^2/4 = 1$, that is, $\text{tr} = \pm 2$. From examining the trace-determinant plane, we see the origin is

an asymptotically stable node for $\text{tr} < -2$, that is, for $a < -4$,

an asymptotically stable spiral for $-2 < \text{tr} < 0$, that is, for $-4 < a < -2$,

a center for $\text{tr} = 0$, that is, for $a = -2$,

an unstable spiral for $0 < \text{tr} < 2$, that is, for $-2 < a < 0$, and

an unstable node for $2 < \text{tr}$, that is, for $0 < a$.

4. The differential equation

$$\frac{dx}{dt} = x^3 - kx^2 = g(x)$$

(a) has fixed points where $g(x) = 0$, that is,

$$0 = x^3 - kx^2 = x^2(x - k)$$

so at $x = 0$ and $x = k$.

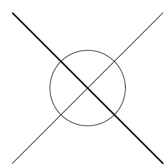
(b) To test stability of the fixed points, evaluate $dg/dx = 3x^2 - 2xk$ at the fixed points. At $x = k$, $dg/dx = k^2 > 0$, so this fixed point is unstable.

At $x = 0$, $dg/dx = 0$ so we must use another approach. At this fixed point, the second derivative $d^2g/dx^2 = -2k < 0$. This shows the graph of g is concave down at this point, and consequently this fixed point is neither stable nor unstable.

5. For the system

$$\begin{aligned} x' &= (y - x)(x^2 + y^2 - 1) \\ y' &= y + x \end{aligned}$$

(a) the x-nullcline is the line $y = x$ and the circle $x^2 + y^2 = 1$. The y-nullcline is the line $y = -x$. The y-nullcline is the darker line.



(b) Above the y-nullcline, $y + x > 0$ and so the vectors point up. Below the y-nullcline, $y + x < 0$ and so the vectors point down.

To the right of $y = x$, $y - x < 0$; to the left, $y - x > 0$. Inside the circle $x^2 + y^2 = 1$, $x^2 + y^2 - 1 < 0$; outside, $x^2 + y^2 - 1 > 0$. Combining these observations, we place the direction arrows in the regions of the graph.