Math 116 Midterm Practice 2 Solutions

1. (a) Substitute $u = x + 1$, so $du = dx$ and $x = u - 1$. Then

$$\begin{aligned}
\int \frac{x}{\sqrt{x+1}}dx &= \int \frac{u-1}{\sqrt{u}}du \\
&= \int u^{1/2} - u^{-1/2}du \\
&= \frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2} + C \\
&= \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + c
\end{aligned}$$

(b) Use integration by parts, with $u = \ln(x)$ and $dv = \sqrt{x}dx$, so $du = dx/x$ and $v = (2/3)x^{3/2}$. Then

$$\begin{aligned}
\int \sqrt{x}\ln(x)dx &= \frac{2}{3}x^{3/2}\ln(x) - \int \frac{2}{3}x^{3/2}\frac{1}{x}dx \\
&= \frac{2}{3}x^{3/2}\ln(x) - \frac{2}{3}\int x^{1/2}dx \\
&= \frac{2}{3}x^{3/2}\ln(x) - \frac{4}{9}x^{3/2} + c
\end{aligned}$$

(c) Substitute $u = \sin(x)$ so $du = \cos(x)dx$. Then

$$\begin{aligned}
\int \frac{\cos(x)}{\sin(x)\sqrt{9-\sin^2(x)}}dx &= \int \frac{du}{u\sqrt{9-u^2}} \\
&= -\frac{1}{3}\ln\left|\frac{\sqrt{9-u^2}+3}{u}\right| + C \quad \text{by rule 30} \\
&= -\frac{1}{3}\ln\left|\frac{\sqrt{9-\sin^2(x)}+3}{\sin(x)}\right| + C
\end{aligned}$$

2. Because $2 < b + \tau < 3$, the graph of ϕ_{n+1} has three branches, and the fixed point occurs where the middle branch intersects the diagonal. The equation of the middle branch is $\phi_{n+1} = b\phi_n + \tau - 1$, so the fixed point is the solution of

$$\phi_n = b\phi_n + \tau - 1$$

That is,

$$\phi_n = \frac{\tau - 1}{1 - b}$$

Because the slope of every branch of the graph is greater than 1, the fixed point is unstable.

3. For the differential equation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & a \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

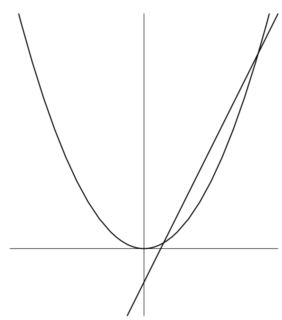
the trace and determinant are

$$\text{tr} = a + 1 \quad \text{and} \quad \det = 2a$$

so $\det = 2\text{tr} - 2$ for every differential equation in this family. The line $\det = 2\text{tr} - 2$ crosses the parabola $\det = \text{tr}^2/4$ at

$$\text{tr} = 4 \pm 2\sqrt{2}$$

From where the line crosses the tr-axis and the parabola $\det = \text{tr}^2/4$, we see the fixed point at the origin is
a saddle point for $\text{tr} < 1$, that is, $a < 0$,
an unstable node for $1 < \text{tr} \leq 4 - 2\sqrt{2}$, that is, $0 < a \leq 3 - 2\sqrt{2}$,
an unstable spiral for $4 - 2\sqrt{2} < \text{tr} < 4 + 2\sqrt{2}$, that is, $3 - 2\sqrt{2} < \text{tr} < 3 + 2\sqrt{2}$, and
an unstable node for $4 + 2\sqrt{2} < \text{tr}$, that is, $3 + 2\sqrt{2} < a$.



4. Suppose $N_r(A) = N_r(B) + 5$. Because both A and B have positive dimensions, $N_r(A) \to \infty$ and $N_r(B) \to \infty$ as $r \to 0$, so $N_r(B)$ is the larger of $N_r(B)$ and 5. Then

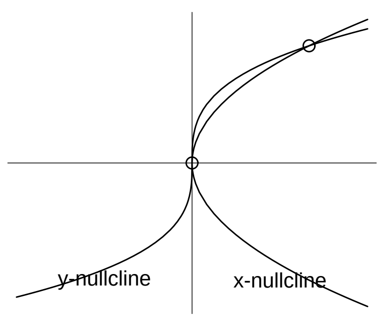
$$
\begin{aligned}
d(A) &= \lim_{r\to 0} \frac{\log(N_r(A))}{\log(1/r)} \\
&= \lim_{r\to 0} \frac{\log(N_r(B)+5)}{\log(1/r)} \\
&= \lim_{r\to 0} \frac{\log(N_r(B)(1+5/N_r(B)))}{\log(1/r)} \\
&= \lim_{r\to 0} \left(\frac{\log(N_r(B))}{\log(1/r)} + \frac{\log(1+5/N_r(B))}{\log(1/r)} \right) \\
&= d(B)
\end{aligned}
$$

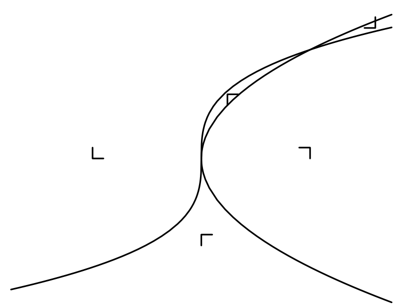
because $\log(1+5/N_r(B)) \le \log(5)$ and the denominator $\log(1/r)$ goes to ∞.

5. (a) The system

$$
\begin{aligned}
x' &= x - y^2 \\
y' &= x - y^3
\end{aligned}
$$

has x-nullcline $x = y^2$ and y-nullcline $x = y^3$, pictured on the left.





(b) The fixed points are $(0,0)$ and $(1,1)$.

(c) For (x,y) to the left of the x-nullcline, $x < y^2$, so $x' < 0$ and the vector field points left; for (x,y) to the right of the x-nullcline, $x > y^2$, so $x' > 0$ and the vector field points right. For (x,y) above the y-nullcline, $x < y^3$, so $y' < 0$ and the vector field points down; for (x,y) below the y-nullcline, $x > y^3$, so $y' > 0$ and the vector field points up. Combining this information, we have the directions shown on the right.