Math 116 Practice Midterm 3 Solutions

1 (a) Using a partial fractions expansion

$$\frac{x}{x^2-2x-8} = \frac{x}{(x+2)(x-4)} = \frac{A}{x+2} + \frac{B}{x-4}$$

gives $A = 1/3$ and $B = 2/3$. Then

$$\begin{aligned}\int \frac{x}{x^2-2x-8}dx &= \frac{1}{3}\int \frac{1}{x+2}dx + \frac{2}{3}\int \frac{1}{x-4}dx \\ &= \frac{1}{3}\ln|x+2| + \frac{2}{3}\ln|x-4| + C\end{aligned}$$

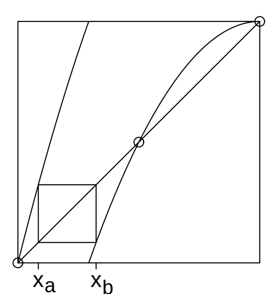
(b) Substitute $u = \sin(x)$ so $du = \cos(x)dx$. Then

$$\begin{aligned}\int e^{\sin(x)}\sin(x)\cos(x)dx &= \int e^u u du \\ &= ue^u - e^u + C \quad \text{integration by parts} \\ &= \sin(x)e^{\sin(x)} - e^{\sin(x)} + C\end{aligned}$$

(c) Substitute $u = \ln(x)$ so $du = (1/x)dx$. Then

$$\begin{aligned}\int \frac{1}{x\sqrt{1-(\ln(x))^2}}dx &= \int \frac{du}{\sqrt{1-u^2}} \\ &= \arcsin(u) + C \quad \text{28 from the integral table} \\ &= \arcsin(\ln(x)) + C\end{aligned}$$

2 (a) The fixed points are indicated by the circles. At both the left and middle fixed points, the slope of the tangent line is greater than 1, so these fixed points are unstable. At the right fixed point, the absolute value of slope of the tangent line is less than 1, so this fixed point is stable.



(b) Because $x_b = f(x_a)$ uses the left branch of f,

$$x_b = 2x_a(2 - x_a)$$

Because $x_a = f(x_b)$ uses the right branch of f,

$$x_a = 2x_b(2 - x_b) - 1$$

3. (a) The x-nullcline is

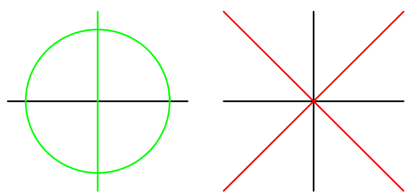
$$0 = x^3 + xy^2 - x = x(x^2 + y^2 - 1)$$

that is, $x = 0$ (the y-axis) and $x^2 + y^2 = 1$ (the circle of radius 1 and center $(0, 0)$), pictured on the left.

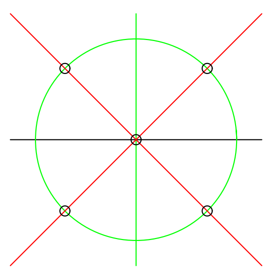
The y-nullcline is

$$0 = x^2 - y^2 = (x + y)(x - y)$$

that is, the lines $y = x$ and $y = -x$, pictured on the right.



(b) The fixed points are the points of intersection of the nullclines, circled here.



The fixed points are $(0, 0)$, $(1/\sqrt{2}, 1/\sqrt{2})$, $(-1/\sqrt{2}, 1/\sqrt{2})$, $(-1/\sqrt{2}, -1/\sqrt{2})$, and $(1/\sqrt{2}, -1/\sqrt{2})$.

4. At the fixed point, the eigenvalues of the system are

$$\frac{a + 1 \pm \sqrt{a^2 - 2a - 3}}{2}$$

The origin is a spiral when the eigenvalues are complex, that is, when $a^2 - 2a - 3 < 0$. The graph of $a^2 - 2a - 3$ is a parabola opening upward and crossing the horizontal axis at $0 = a^2 - 2a - 3 = (a - 3)(a + 1)$, that is, at $a = -1$ and $a = 3$.

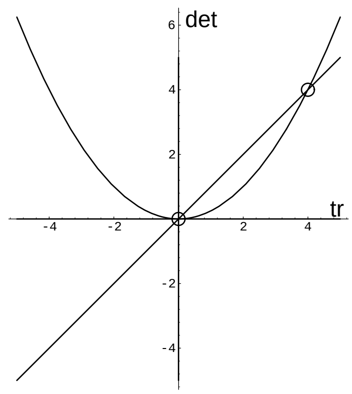
Then we see the origin is a spiral for $-1 < a < 3$. So long as $a > -1$, $(a+1)/2$, the real part of the eigenvalues, is positive and so the spiral is unstable.

Summarizing, the origin is an unstable spiral for $-1 < a < 3$.

Alternately, for this matrix we see

$$\text{tr} = a + 1 \quad \text{and} \quad \det = a + 1$$

That is, $\det = \text{tr}$. In the trace-determinant plane, this is



Unstable spirals occur above the parabola $\det = \text{tr}^2/4$ and in the first quadrant. The intersection of the line $\det = \text{tr}$ and the parabola $\det = \text{tr}^2/4$ occurs at

$$\begin{aligned} \text{tr} &= \text{tr}^2/4 \\ 4\text{tr} &= \text{tr}^2 \\ 0 &= \text{tr}^2 - 4\text{tr} \\ 0 &= \text{tr}(\text{tr} - 4) \end{aligned}$$

That is, $\text{tr} = 0$ and $\text{tr} = 4$. The origin is an unstable spiral for $0 < \text{tr} < 4$. Because $\text{tr} = a + 1$, we see the origin is an unstable spiral for $-1 < a < 3$.