Math 116 Practice Midterm 4 Solutions

1. (a) Using a partial fractions expansion, we obtain

$$\begin{aligned}\int \frac{x}{x^2+4x-5}dx &= \int \frac{x}{(x+5)(x-1)}dx \\ &= \frac{1}{6}\int \frac{1}{x-1}dx + \frac{5}{6}\int \frac{1}{x+5}dx \\ &= \frac{1}{6}\ln|x-1| + \frac{5}{6}\ln|x+5| + C\end{aligned}$$

1. (b) Substituting $u = \tan(x)$ so $du = \sec^2(x)dx$, we see

$$\begin{aligned}\int e^{\tan(x)}\sec^2(x)dx &= \int e^u du \\ &= e^u + C = e^{\tan(x)} + C\end{aligned}$$

1. (c) Substituting $u = \tan(x)$ so $du = \sec^2(x)dx = (1/\cos^2(x))dx$, we see

$$\begin{aligned}\int e^{\tan(x)}\frac{\sin(x)}{\cos^3(x)}dx &= \int e^u u du \\ &= ue^u - e^u + C \qquad \text{integration by parts} \\ &= \tan(x)e^{\tan(x)} - e^{\tan(x)} + C\end{aligned}$$

2. The Cantor middle-thirds set along the x-axis of A suggests covering with boxes of side length $1/3^n$. For each n, we need 2^n boxes of side length $1/3^n$ to cover the Cantor set. Each of these boxes is the base of a column of 3^n boxes to cover the line segment above every point of the Cantor set. Then $N(1/3^n) = 2^n \cdot 3^n = 6^n$, and we see the box-counting dimension of A is

$$\begin{aligned}d(A) &= \lim_{n\to\infty}\frac{\log(N(1/3^n))}{\log(1/(1/3^n))} \\ &= \lim_{n\to\infty}\frac{\log(6^n)}{\log(3^n)} \\ &= \lim_{n\to\infty}\frac{n\log(6)}{n\log(3)} \\ &= \frac{\log(6)}{\log(3)} = 1 + \frac{\log(2)}{\log(3)}\end{aligned}$$

The Cantor middle-halves set along the x-axis of B suggests covering with boxes of side length $1/4^n$. For each n, we need 2^n boxes of side length $1/4^n$ to cover the Cantor set. Each of these boxes is the base of a column of $2 \cdot 4^n$ boxes to cover the line segment above every point of the Cantor set. Then

$N(1/4^n) = 2^n \cdot 2 \cdot 4^n = 2 \cdot 8^n$, and we see the box-counting dimension of B is

$$\begin{aligned} d(B) &= \lim_{n\to\infty} \frac{\log(N(1/4^n))}{\log(1/(1/4^n))} \\ &= \lim_{n\to\infty} \frac{\log(2 \cdot 8^n)}{\log(4^n)} \\ &= \lim_{n\to\infty} \frac{\log(2) + n\log(8)}{n\log(4)} \\ &= \lim_{n\to\infty} \left(\frac{\log(2)}{n\log(4)} + \frac{n\log(8)}{n\log(4)} \right) \\ &= \frac{\log(8)}{\log(4)} = \frac{3}{2} \end{aligned}$$

Because $\log(2)/\log(3) > 1/2$, A had the higher box-counting dimension.

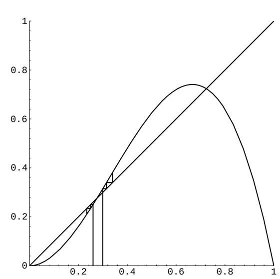
3. (a) The fixed points are the solutions of $f(x) = x$, that is,

$$f(x) - x = r(x^2 - x^3) - x = x(-1 + rx - rx^2)$$

From left to right, the fixed points occur at $x = 0$, $x = (r - \sqrt{r^2 - 4r})/2r$, and $x = (r + \sqrt{r^2 - 4r})/2r$.

(b) At the middle fixed point x_*, the graph of f crosses from below the diagonal to above the diagonal. Consequently, at x_* the derivative of f satisfies $f'(x_*) > 1$, so this fixed point is unstable.

Alternately, because at x_* the graph of f crosses from below the diagonal to above the diagonal, points near x_* move away from x_* under graphical iteration.



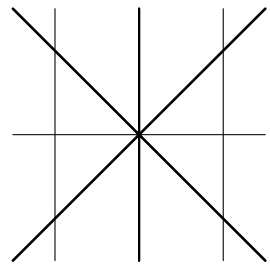
4. The x-nullcline is given by

$$0 = x^3 - xy^2 = x(x^2 - y^2)$$

That is, the lines $x = 0$ and $y = \pm x$. These are the heavier lines in the figure below.

The y-nullcline is given by

$$0 = yx^2 - y = y(x^2 - 1)$$

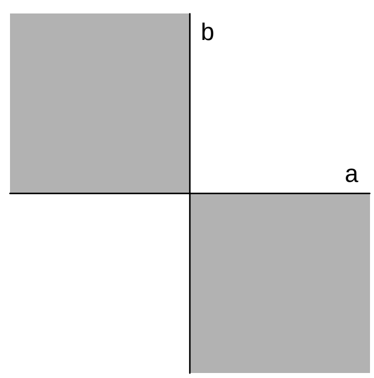


That is, the lines $y = 0$ and $x = \pm 1$. These are the lighter lines in the figure.
(b) The fixed points are the intersections of the x- and y-nullclines. That is, $(1, 1)$, $(1, -1)$, $(-1, -1)$, $(-1, 1)$, and $(0, 0)$.

5. The trace and determinant of this system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

are $\mathrm{tr} = 2$ and $\det = 1 - ab$. In the trace-determinant plane, the vertical line $\mathrm{tr} = 2$ intersects the parabola $\det = \mathrm{tr}^2/4$ at $\det = 1$, so the origin is an unstable spiral for $\det > 1$. Substituting in $\det = 1 - ab$, we see the origin is an unstable spiral when $ab < 0$. This describes quadrants 2 and 3 in the ab-plane: $a < 0$ and $b > 0$, and $a > 0$ and $b < 0$.



Alternately, the eigenvalues of this system are $1 \pm \sqrt{ab}$. In order for the origin to be a spiral, the eigenvalues must be complex, that is, $ab < 0$. Then the real part of these eigenvalues is 1, so the spirals are unstable. This gives the region $a < 0$ and $b > 0$, together with the region $a > 0$ and $b < 0$.