Math 116 Practice Final 1 Solutions

1. (a) Substitute $u = \tan(t)$, so $du = \sec^2(t)dt$. Then

$$\begin{aligned}\int \sec^2(t)\cos(\tan(t))dt &= \int \cos(u)du \\ &= \sin(u) + C \\ &= \sin(\tan(t)) + C\end{aligned}$$

(b) Substitute $w = t^2$, so $dw = 2tdt$. Then

$$\int t^3 \sin(t^2)dt = \frac{1}{2}\int w\sin(w)dw$$

Now integrate by parts.

$$\begin{aligned} u &= w & dv &= \sin(w)dw \\ du &= dw & v &= -\cos(w)\end{aligned}$$

Then

$$\begin{aligned}\int w\sin(w)dw &= -w\cos(w) - \int -\cos(w)dw \\ &= -w\cos(w) + \sin(w) + C\end{aligned}$$

Substituting back in for w,

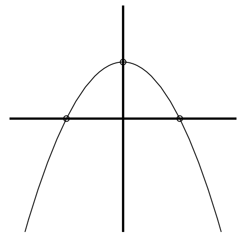
$$\int t^3\sin(t^2)dt = \frac{1}{2}(-t^2\cos(t^2) + \sin(t^2)) + C$$

(c) Substitute $u = t^2$, so $du = 2tdt$ and

$$\begin{aligned}\int \frac{\sqrt{t^4-1}}{t^2}2tdt &= \int \frac{\sqrt{u^2-1}}{u}du \\ &= \sqrt{u^2-1} + \cos^{-1}(1/u) + C \quad \text{by rule 34} \\ &= \sqrt{t^4-1} + \cos^{-1}(1/t^2) + C\end{aligned}$$

2. (a) The x-nullcline is the parabola $y = 1 - x^2$; the y-nullcline is the pair of lines $x = 0$ and $y = 0$.
(b) The fixed points are $(-1, 0)$, $(0, 1)$, and $(1, 0)$. To test their stability, use the derivative matrix

$$D\vec{F} = \begin{bmatrix} 2x & 1 \\ y & x \end{bmatrix}$$



Then we find

$$D\vec{F}(-1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \text{ eigenvalues are } -2 \text{ and } -1$$
$$D\vec{F}(0,1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ eigenvalues are } -1 \text{ and } 1$$
$$D\vec{F}(1,0) = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \text{ eigenvalues are } 2 \text{ and } 1$$

From the eigenvalues, we see that $(-1,0)$ is an asymptotically stable node, $(0,1)$ is a saddle point, and $(1,0)$ is an unstable node,

3. The transition matrix is stochastic, and the transition graph has a path from every state to every state, so by the Perron-Frobenius theorem, the largest eigenvalue is 1. The eigenvector equation is

$$\begin{bmatrix} .8 & .2 & 0 \\ .2 & .5 & .1 \\ 0 & .3 & .9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This gives these equations

$$\begin{aligned} .8x + .2y &= x \\ .2x + .5y + .1z &= y \\ .3y + .9z &= z \end{aligned}$$

The first equation gives $x = y$, the third gives $z = 3y$, and the second is redundant. Because $x + y + z = 1$, we see the eventual distribution is $1/5$ in A, $1/5$ in B, and $3/5$ in C,

4. Substituting x' and y' into $rr' = xx' + yy'$ and simplifying gives

$$\begin{aligned} rr' &= x(x + y - (x^2/4) - x(x^2+y^2)) + y(-x + y - y(x^2+y^2)) \\ &= x^2 + y^2 - (x^3)/4 - (x^2+y^2)^2 \\ &= r^2 - (r^3\cos^3(\theta))/4 - r^4 \end{aligned}$$

and so

$$r' = r - (r^2/4)\cos^3(\theta) - r^3$$

Because $-1 \le \cos^3(\theta) \le 1$,

$$-(r^2/4) \le -(r^2/4)\cos^3(\theta) \le r^2/4$$

and so

$$r - r^2/4 - r^3 \le r - (r^2/4)\cos^3(\theta) - r^3 \le r + r^2/4 - r^3$$

When $r = 1/2$, $r - r^2/4 - r^3 = 5/16$, so $r' > 0$ on the circle of radius 1/2. When $r = 2$, $r + r^2/4 - r^3 = -5$, so $r' < 0$ on the circle of radius 2. The annulus between these circles is a trapping region not containing a fixed point, so by the Poincaré-Bendixson theorem, this annulus must contain a limit cycle.

5. (a) Suppose x has a power series expansion

$$x = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \cdots$$

Note that $x(0) = 1$ implies that $a_0 = 1$. Then the series for x' and for $x - t + t^2$ are

$$\begin{aligned} x' &= a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + \cdots \\ x - t + t^2 &= a_0 + (a_1 - 1)t + (a_2 + 1)t^2 + a_3 t^3 + a_4 t^4 + \cdots \end{aligned}$$

Matching coefficients of like powers of t gives

	x'	$x - t + t^2$	
t^0	a_1	a_0	so $a_1 = a_0 = 1$
t^1	$2a_2$	$a_1 - 1$	so $a_2 = (a_1 - 1)/2 = 0$
t^2	$3a_3$	$a_2 + 1$	so $a_3 = (a_2 + 1)/3 = 1/3$
t^3	$4a_4$	a_3	so $a_4 = a_3/4 = 1/(4 \cdot 3)$
t^4	$5a_5$	a_4	so $a_5 = a_4/5 = 1/(5 \cdot 4 \cdot 3)$
		$\cdots$	

Then we have

$$x = 1 + t + \frac{t^3}{3} + \frac{t^4}{4 \cdot 3} + \frac{t^5}{5 \cdot 4 \cdot 3} + \cdots$$

(b) To turn as much as we can of this into an exponential, observe

$$\begin{aligned} x &= 1 + t + 2\left(\frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots\right) \\ &= 1 + t + 2\left(1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots\right) - 2(1 + t + t^2/2) \\ &= -1 - t - t^2 + 2e^t \end{aligned}$$

Checking,

$$\begin{aligned} x' &= (-1 - t - t^2 + 2e^t)' = -1 - 2t + 2e^t \\ x - t + t^2 &= (-1 - t - t^2 + 2e^t) - t + t^2 = -1 - 2t + 2e^t \\ x(0) &= -1 + 0 + 0 + 2 = 1 \end{aligned}$$

6. (a) For large n, $1/\sqrt{n^3 - n^2}$ looks much like $1/n^{3/2}$. Because $\sum 1/n^{3/2}$ is a p-series with $p = 3/2 > 1$, this series converges. Because $1/\sqrt{n^3 - n^2} > 1/n^{3/2}$, we cannot use the comparison test. So use the limit comparison test.

$$\begin{aligned} \lim_{n\to\infty} \frac{1/\sqrt{n^3 - n^2}}{1/n^{3/2}} &= \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3 - n^2}} \\ &= \lim_{n\to\infty} \frac{n^{3/2}}{n^{3/2}\sqrt{1 - (1/n)}} = 1 \end{aligned}$$

So both series converge by the limit comparison test.

(b) Because the series contains terms of the form $1/3^n$, we'll replace $1/3^n$ with x^n and see if we recognize a pattern. Then

$$1 + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3^2} + 4 \cdot \frac{1}{3^3} + 5 \cdot \frac{1}{3^4} + \cdots = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

Recalling that $1 + x + x^2 + x^3 + x^4 + \cdots = 1/(1 - x)$ for $|x| < 1$, we see

$$\begin{aligned} 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots &= (1 + x + x^2 + x^3 + x^4 + \ldots)' \\ &= \left(\frac{1}{1 - x}\right)' \\ &= \frac{1}{(1 - x)^2} \end{aligned}$$

Consequently,

$$\begin{aligned} 1 + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3^2} + 4 \cdot \frac{1}{3^3} + 5 \cdot \frac{1}{3^4} + \cdots &= \frac{1}{(1 - 1/3)^2} \\ &= \frac{9}{4} \end{aligned}$$

7. (a) The box-counting dimension is the slope of the log-log plot. The graph B has the higher slope, therefore the larger dimesnion.

(b) Using the definition of box-counting dimension, $d(A) = 2d(B)$ becomes

$$\begin{aligned} \lim_{r\to 0} \frac{\log(N_r(A))}{\log(1/r)} &= 2 \lim_{r\to 0} \frac{\log(N_r(B))}{\log(1/r)} \\ &= \lim_{r\to 0} 2\frac{\log(N_r(B))}{\log(1/r)} \\ &= \lim_{r\to 0} \frac{2\log(N_r(B))}{\log(1/r)} \\ &= \lim_{r\to 0} \frac{\log(N_r(B)^2)}{\log(1/r)} \end{aligned}$$

One way to obtain this result is by requiring that $N_r(A) = N_r(B)^2$.

8. (a) Because $2 < b + \tau < 3$, the graph has three branches

$$\phi_{i+1} = b\phi_i + \tau, \;\; \phi_{i+1} = b\phi_i + \tau - 1, \;\text{ and }\; \phi_{i+1} = b\phi_i + \tau - 2$$

Given the placement of the 2-cycle points, they are related by

$$\begin{aligned} \phi_2 &= b\phi_1 + \tau \\ \phi_1 &= b\phi_2 + \tau - 2 \end{aligned}$$

Solving for ϕ_1 and ϕ_2 we obtain

$$\begin{aligned} \phi_1 &= \frac{b\tau + \tau - 2}{1 - b^2} \\ \phi_2 &= \frac{b\tau - 2b + \tau}{1 - b^2} \end{aligned}$$

(b) The point A is where the left branch $\phi_{i+1} = b\phi_i + \tau$ crosses $\phi_{i+1} = 1$, so

$$A = \frac{1 - \tau}{b}$$

The point B is where the middle branch $\phi_{i+1} = b\phi_i + \tau - 1$ crosses $\phi_{i+1} = 1$, so

$$B = \frac{2 - \tau}{b}$$