Math 116 Practice Final 2 Solutions

1. (a)

$$\int \frac{x+1}{2x^2-3x-2}dx$$

Use partial fractions

$$\frac{x+1}{2x^2-3x-2} = \frac{x+1}{(2x+1)(x-2)} = \frac{A}{2x+1} + \frac{B}{x-2}$$

from which we find

$$x+1 = A(x-2) + B(2x+1)$$

Taking $x = 2$ gives $B = 3/5$; taking $x = -1/2$ gives $A = -1/5$, and so

$$\begin{aligned}\int \frac{x+1}{2x^2-3x-2}dx &= -\frac{1}{5}\int \frac{1}{2x+1}dx + \frac{3}{5}\int \frac{1}{x-2}dx \\ &= -\frac{1}{10}\ln|2x+1| + \frac{3}{5}\ln|x-2| + C\end{aligned}$$

(b)

$$\int \sin(x)\cos(x)\ln(\sin(x))dx$$

Substitute $w = \sin(x)$, so $dw = \cos(x)dx$ and

$$\int \sin(x)\cos(x)\ln(\sin(x))dx = \int w\ln(w)dw$$

Now integrate by parts.

$$\begin{aligned} u &= \ln(w) & dv &= wdw \\ du &= (1/w)dw & v &= w^2/2 \end{aligned}$$

Then

$$\begin{aligned}\int w\ln(w)dw &= (w^2/2)\ln(w) - (1/2)\int wdw \\ &= (w^2/2)\ln(w) - w^2/4 + C\end{aligned}$$

Substituting back in for w,

$$\int \sin(x)\cos(x)\ln(\sin(x))dx = (\sin(x)^2/2)\ln(\sin(x)) - \sin^2(x)/4 + C$$

(c)

$$\int \frac{1}{x\ln(x)\sqrt{1+(\ln(x))^2}}dx$$

Substitute $u = \ln(x)$, so $du = (1/x)dx$ and the integral becomes

$$\begin{aligned}
\int \frac{1}{x\ln(x)\sqrt{1+(\ln(x))^2}}dx &= \int \frac{du}{u\sqrt{1+u^2}} \\
&= -\ln\left|\frac{\sqrt{1+u^2}+1}{u}\right| + C \quad \text{rule 22} \\
&= -\ln\left|\frac{\sqrt{1+(\ln(x))^2}+1}{\ln(x)}\right| + C
\end{aligned}$$

2. To find the radius of convergence of $\sum \frac{n3^n}{n^2+1}(2x-1)^n$, use the ratio test

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{((n+1)3^{n+1}/((n+1)^2+1))(2x-1)^{n+1}}{(n3^n/(n^2+1))(2x-1)^n}\right| &< 1 \\
\lim_{n\to\infty}\frac{n+1}{n}\frac{3^{n+1}}{3^n}\frac{n^2+1}{n^2+2n+2}|2x-1| &< 1 \\
3|2x-1| &< 1 \\
|x-1/2| &< 1/6
\end{aligned}$$

so the radius of convergence is 1/6.

From $|x-1/2| < 1/6$, we see

$$\begin{aligned}
-1/6 < x - 1/2 < 1/6 \\
1/3 < x < 2/3
\end{aligned}$$

Substituting $x = 1/3$ into the series, we obtain

$$\begin{aligned}
\sum \frac{n3^n}{n^2+1}(2\cdot(1/3)-1)^n &= \sum \frac{n3^n}{n^2+1}(-1/3)^n \\
&= \sum \frac{n(-1)^n}{n^2+1}
\end{aligned}$$

which converges by the alternating series test.

Substituting $x = 2/3$ into the series, we obtain

$$\begin{aligned}
\sum \frac{n3^n}{n^2+1}(2\cdot(2/3)-1)^n &= \sum \frac{n3^n}{n^2+1}(1/3)^n \\
&= \sum \frac{n}{n^2+1}
\end{aligned}$$

The terms look like $1/n$, but because $n/(n^2+1) < 1/n$, we can't use the comparison test. Instead, use the limit comparison test,

$$\lim_{n\to\infty}\frac{n/(n^2+1)}{1/n} = \lim_{n\to\infty}\frac{n^2}{n^2+1} = 1$$

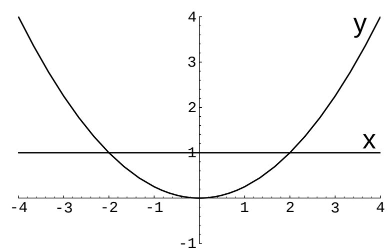
Because the harmonic series diverges, so does $\sum n/(n^2+1)$.

Combining these observations, the interval of convergence is $[1/3, 2/3)$.

3. For the differential equation

$$dx/dt = y - 1$$
$$dy/dt = y - x^2$$

(a) The x-nullcline is the horizontal line $y = 1$, the y-nullcline is the parabola $y = x^2$. The fixed points are the intersections of the nullclines, $(-1, 1)$ and $(1, 1)$.



(b) The derivative matrix is

$$D\vec{F} = \begin{bmatrix} 0 & 1 \\ -2x & 1 \end{bmatrix}$$

Evaluated at the fixed points,

$$D\vec{F}(-1, 1) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad D\vec{F}(1, 1) = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$$

The eigenvalues of $D\vec{F}(-1, 1)$ are 2 and -1, so this fixed point is unstable.

The eigenvalues of $D\vec{F}(1, 1)$ are $(1 \pm \sqrt{7}i)/2$, so this fixed point is unstable, too.

4. To show the system

$$x' = (3/2)x + y - 2x(x^2 + y^2)$$
$$y' = -x + y - 2x(x^2 + y^2)$$

has a limit cycle, we find a trapping region, show it contains no fixed points, and apply the Poincaré-Bendixson theorem.

$$\begin{aligned} rr' &= xx' + yy' \\ &= x((3/2)x + y - 2x(x^2 + y^2)) + y(-x + y - 2y(x^2 + y^2)) \\ &= x^2 + y^2 + (1/2)x^2 - 2(x^2 + y^2)^2 \\ &= r^2 + (1/2)r^2\cos^2(\theta) - 2r^4 \end{aligned}$$

From this, we see

$$r' = r + (1/2)r\cos^2(\theta) - 2r^3$$

From $0 \leq \cos^2(\theta) \leq 1$ we see $0 \leq (1/2)r\cos^2(\theta) \leq r/2$, and from this we find lower and upper bounds for r':

$$r - 2r^3 \leq r + (1/2)r\cos^2(\theta) - 2r^3 \leq r + (r/2) - 2r^3$$

That is,

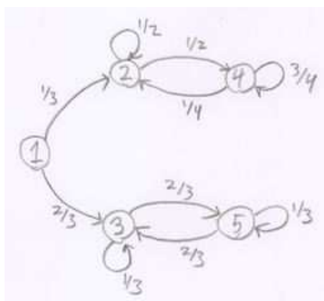
$$r - 2r^3 \le r' \le r + (r/2) - 2r^3$$

On the circle $r = 1$, $r + (r/2) - 2r^3 = -1/2 < 0$, so $r' < 0$. On the circle $r = 1/2$, $r - 2r^3 = 1/4 > 0$, and so $r' > 0$. That is, the region between these circles is a trapping region, and contains no fixed points. Then by the Poincaré-Bendixson theorem, this region contains a limit cycle for the system.

5. From the transition graph for this Markov process

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/4 & 0 \\ 2/3 & 0 & 1/3 & 0 & 2/3 \\ 0 & 1/2 & 0 & 3/4 & 0 \\ 0 & 0 & 2/3 & 0 & 1/3 \end{bmatrix}$$

we see that states 2 and 4 transform into states 2 and 4, and states 3 and 5 transform into states 3 and 5.



The transition matrix for states 2 and 4 is

$$\begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}$$

This is a positive stochastic matrix, so its largest eigenvalue is 1. The eigenvectors of $\lambda = 1$ satisfy

$$\begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

The two equations are redundant; we solve the first, $x/2 + y/4 = x$, obtaining $y = 2x$. Then the condition $x + y = 1$ gives $x = 1/3$ and $y = 2/3$.

The transition matrix for states 3 and 5 is

$$\begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$$

This is a positive stochastic matrix, so its largest eigenvalue is 1. The eigenvectors of $\lambda = 1$ satisfy

$$\begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

The two equations are redundant; we solve the first, $x/3 + 2y/3 = x$, obtaining $y = x$. Then the condition $x + y = 1$ gives $x = 1/2$ and $y = 1/2$.

So of the portion of the population that enters states 2 and 4, 1/3 winds up in state 2 and 2/3 in state 4. Of the portion of the population that enters states 3 and 5, 1/2 winds up in state 3 and 1/2 in state 5.

Initially, all the population is in state 1. From state 1, 1/3 goes into states 2 and 4, and 2/3 goes into states 3 and 5. Then the final distribution of the population is

$$\{1, 2, 3, 4, 5\} = \{0, 1/9, 1/3, 2/9, 1/3\}$$

6. To solve the differential equation $x' = 2x - t$, with $x(0) = 1$, write the series expansion for $x(t)$,

$$x = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 + \cdots$$

Then the series for x' is

$$x' = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + 6a_6 t^5 + 7a_7 t^6 + 8a_8 t^7 + \cdots$$

and the series for $2x - t$ is

$$2x - t = 2a_0 + (2a_1 - 1)t + 2a_2 t^2 + 2a_3 t^3 + 2a_4 t^4 + 2a_5 t^5 + 2a_6 t^6 + 2a_7 t^7 + \cdots$$

Note the condition $x(0) = 1$ gives $a_0 = 1$. Matching coefficients for like powers of t of the series for x' and $2x - t$ we find

t^0: $a_1 = 2a_0$, so $a_1 = 2a_0 = 2$.
t^1: $2a_2 = 2a_1 - 1$, so $a_2 = (2a_1 - 1)/2 = 3/2$.
t^2: $3a_3 = 2a_2$, so $a_3 = 2a_2/3 = 1$.
t^3: $4a_4 = 2a_3$, so $a_4 = 2a_3/4 = 2/4$.
t^4: $5a_5 = 2a_4$, so $a_5 = 2a_4/5 = 2^2/(5 \cdot 4)$.
t^5: $6a_6 = 2a_5$, so $a_6 = 2a_5/6 = 2^3/(6 \cdot 5 \cdot 4)$.
t^6: $7a_7 = 2a_6$, so $a_7 = 2a_6/7 = 2^4/(7 \cdot 6 \cdot 5 \cdot 4)$.
t^7: $8a_8 = 2a_7$, so $a_8 = 2a_7/8 = 2^5/(8 \cdot 7 \cdot 6 \cdot 5 \cdot 4)$.
and so on.

Starting at a_4, the denominators are parts of factorials, and the numerators are powers of 2. To turn these denominators into $n!$ and the numerators into 2^n, we multiply by $(3/4) \cdot (4/3)$, absorbing the second factor into the coefficients. So far, this gives

$$\begin{aligned}
x &= 1 + 2t + \frac{3}{2}t^2 + \frac{3}{4}\left(\frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} + \cdots\right) \\
&= 1 + 2t + \frac{3}{2}t^2 + \frac{3}{4}\left(1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} + \cdots\right) \\
&\quad - \frac{3}{4}\left(1 + 2t + \frac{(2t)^2}{2!}\right) \\
&= 1 + 2t + \frac{3}{2}t^2 + \frac{3}{4}e^{2t} - \frac{3}{4}\left(1 + 2t + \frac{(2t)^2}{2!}\right) \\
&= \frac{1}{4} + \frac{t}{2} + \frac{3}{4}e^{2t}
\end{aligned}$$

Check:

$$
\begin{aligned}
x' &= (1/4 + t/2 + 3e^{2t}/4)' = 1/2 + 3e^{2t}/2 \\
2x - t &= 2(1/4 + t/2 + 3e^{2t}/4) - t = 1/2 + 3e^{2t}/2 \\
x(0) &= 1/4 + 0/2 + 3e^{0}/4 = 1/4 + 3/4 = 1
\end{aligned}
$$

7. For the system

$$
\begin{aligned}
x' &= -x^3 - 4y - x^5 \\
y' &= -y^3 + 2x - y^5
\end{aligned}
$$

the derivative at the origin has eigenvalues 0 and 0, so provides no stability information. Try the positive definite $V = x^2 + y^2$ for a Liapunov function. Then

$$
\begin{aligned}
V' &= \frac{\partial V}{\partial x}x' + \frac{\partial V}{\partial y}y' \\
&= 2x(-x^3 - 4y - x^5) + 2y(-y^3 + 2x - y^5) \\
&= -2x^4 - 2x^6 - 2y^4 - 2y^6 - 4xy
\end{aligned}
$$

This function is not negative definite, because of the $-4xy$ term. To remove this term, use a different positive definite function$V = x^2 + 2y^2$. With this choice,

$$
\begin{aligned}
V' &= \frac{\partial V}{\partial x}x' + \frac{\partial V}{\partial y}y' \\
&= 2x(-x^3 - 4y - x^5) + 4y(-y^3 + 2x - y^5) \\
&= -2x^4 - 2x^6 - 4y^4 - 4y^6
\end{aligned}
$$

This is negative definite, so we deduce that the origin is asymptotically stable.

8. (a) For the SIS model

$$
\begin{aligned}
S' &= S + rI - mSI \\
I' &= I - rI + mSI
\end{aligned}
$$

The I-nullcline equation gives $0 = I(1 - r + mS)$, with nonzero solution $S = (r-1)/m$. Substituting this into the S-nullcline equation, $S + rI - mSI = 0$, gives $I = (1-r)/m$.

(b) The derivative matrix of this system is

$$
D\vec{F} = \begin{bmatrix} 1 - mI & r - mS \\ mI & 1 - r + mS \end{bmatrix}
$$

Evaluated at the fixed point $((r-1)/m, (1-r)/m)$, the derivative is

$$
D\vec{F}((r-1)/m, (1-r)/m) = \begin{bmatrix} r & 1 \\ 1 - r & 0 \end{bmatrix}
$$

The trace is $\mathrm{tr} = r$, the determinant is $\det = r - 1 = \mathrm{tr} - 1$. This line intersects the parabola $\det = \mathrm{tr}^2/4$ at $\mathrm{tr} = 2$. Then we see the nonzero fixed point is an unstable saddle point for $r = \mathrm{tr} < 1$ and an unstable node for $r = \mathrm{tr} > 1$.