Math 116 Practice Final 3 Solutions

1. (a) Substitute $u = \sin(x)$, so $du = \cos(x)dx$. Then integrate by parts.

$$\begin{aligned}\int \cos(x)\sin(x)e^{\sin(x)}dx &= \int ue^u du \\ &= ue^u - e^u + C \\ &= \sin(x)e^{\sin(x)} - e^{\sin(x)} + C\end{aligned}$$

(b) Substitute $u = \tan(x)$, so $du = \sec^2(x)dx$. Then apply 28 from the integral table.

$$\begin{aligned}\int \frac{\sec^2(x)}{\sqrt{4-\tan^2(x)}}dx &= \int \frac{1}{\sqrt{4-u^2}}du \\ &= \sin^{-1}(u/2) + C \\ &= \sin^{-1}(\tan(x)/2) + C\end{aligned}$$

(c) Substitute $u = e^x$ so $du = e^x dx$. Then use partial fractions to show

$$\frac{1}{u^2-3u+2} = \frac{1}{u-2} - \frac{1}{u-1}$$

$$\begin{aligned}\int \frac{e^x}{e^{2x}-3e^x+2}dx &= \int \frac{1}{u^2-3u+2}du \\ &= \int \frac{1}{u-2}du - \int \frac{1}{u-1}du \\ &= \ln|u-2| - \ln|u-1| + C \\ &= \ln|e^x-2| - \ln|e^x-1| + C\end{aligned}$$

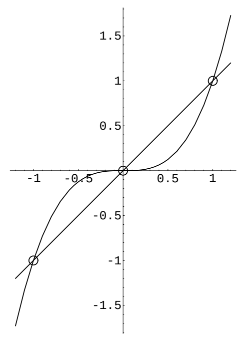
2. Apply the ratio test to find the radius of convergence

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(5x-1)^{n+1}/(n+1)3^{n+1}}{(5x-1)^n/n3^n}\right| &= \lim_{n\to\infty}\frac{n}{n+1}\frac{3^n}{3^{n+1}}|5x-1| \\ &= \frac{1}{3}|5x-1|\end{aligned}$$

Then the series converges for $(1/3)|5x-1| < 1$, that is, $-2/5 < x < 4/5$. From this we see the radius of convergence is $R = 3/5$.

To test the endpoints, substituting $x = -2/5$ and simplifying, the series becomes the aletrnating harmonic series, which converges. Substituting $x = 4/5$ and simplifying, the series becomes the harmonic series, which diverges. Then the interval of convergence is $[-2/5, 4/5)$.

3. (a) The x-nullcline is $y = x^3$; the y-nullcline is $y = x$.



(b) The fixed points are indicated on the graph. Their x-coordinates are the solutions of $x^3 = x$, that is, $0 = x - x^3 = x(1 - x^2)$, so $x = 0, \pm 1$ and the coordinates of the fixed points are $(-1, -1)$, $(0, 0)$, $(1, 1)$.

(c) To determine the stability of the fixed points, first compute the derivative matrix

$$D\vec{F}(x, y) = \begin{bmatrix} -3x^2 & 1 \\ -1 & 1 \end{bmatrix}$$

Then

$$D\vec{F}(-1, -1) = D\vec{F}(1, 1) = \begin{bmatrix} -3 & 1 \\ -1 & 1 \end{bmatrix}$$

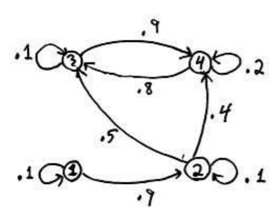
This has eigenvalues $-1 \pm \sqrt{3}$. One eigenvalue is positive, so these fixed points are unstable.

Next

$$D\vec{F}(0, 0) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

This has eigenvalues $(1 \pm \sqrt{3}i)/2$, Because the real part is positive, this fixed point is unstable.

4. (a) Labeling the states 1, 2, 3, and 4, the transition graph is



(b) Eventually, states 1 and 2 are emptied. The transitions between states 3 and 4 are governed by the matrix

$$\begin{bmatrix} .1 & .8 \\ .9 & .2 \end{bmatrix}$$

This is a stochastic, positive matrix, so the largest eigenvalue is 1, and the eventual distribution among states 3 and 4 is the unit eigenvector of the eigenvalue

1. To find this eigenvector, solve

$$\begin{bmatrix} .1 & .8 \\ .9 & .2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

The first equation is $.1x+.8y = x$; the second equation is redundant. Solving for y, $y = (9/8)x$. Then the unit eigenvector condition, $x + y = 1$, gives $x = 8/17$ and so $y = 9/17$. The eventual distribution is

$$[1, 2, 3, 4] = [0, 0, 8/17, 9/17]$$

5. First, compute r'

$$\begin{aligned} rr' &= xx' + yy' \\ &= x(x + 5y - x(x^2 + y^2)) + y(-x + y - y(x^2 + y^2)) \\ &= x^2 + y^2 + 4xy - (x^2 + y^2)^2 \\ &= r^2 + 4r^2 \cos(\theta)\sin(\theta) - r^4 \end{aligned}$$

Then

$$r' = r + 4r\cos(\theta)\sin(\theta) - r^3$$

Because

$$-1/2 \leq \cos(\theta)\sin(\theta) \leq 1/2$$

we see

$$-2r \leq 4r\cos(\theta)\sin(\theta) \leq 2r$$

and so

$$\begin{aligned} r - 2r - r^3 \leq r' \leq r + 2r - r^3 \\ -r - r^3 \leq r' \leq 3r - r^3 \end{aligned}$$

Then for $r = 2$, for example, $r' \leq -2$. That is, trajectories that enter the disc of radius 2 cannot leave.

Unfortunately, the lower bound, $-r - r^3$ orf r' is never positive, so we cannot construct an annular trappint region. To apply the Poincaré-Bendixson theorem, we must show the fixed point at the origin is unstable. At the origin, the derivative matrix is

$$\begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}$$

The eigenvalues are $1 \pm i\sqrt{5}$ so the origin is an unstable spiral and by the Poincaré-Bendixson theorem, the trapping region contains a limit cycle.

6. Write

$$x = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \cdots$$

Then

$$x' = a_0 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + \cdots$$

and

$$x + 2t - t^2 = a_0 + (a_1 + 2)t + (a_2 - 1)t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \cdots$$

Matching the coefficients of like powers of t,

	x'	$x + 2t - t^2$	
t^0	a_1	a_0	$a_1 = a_0 = 1$
t^1	$2a_2$	$a_1 + 2$	$a_2 = (1/2)(a_1 + 2) = 1/2 + 1$
t^2	$3a_3$	$a_2 - 1$	$a_3 = (1/3)(a_2 - 1) = 1/3!$
t^3	$4a_4$	a_3	$a_4 = (1/4)a_3 = 1/4!$
t^4	$5a_5$	a_4	$a_5 = (1/5)a_4 = 1/5!$
...	...	...	...

From this we see

$$\begin{aligned} x &= 1 + t + t^2 + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \\ &= e^t + t^2 \end{aligned}$$

Check:

$$\begin{aligned} x' &= (e^t + t^2)' = e^t + 2t \\ x + 2t - t^2 &= (e^t + t^2) + 2t - t^2 = e^t + 2t \\ x(0) &= e^0 + 0^2 = 1 \end{aligned}$$

7. (a) The derivative matrix is

$$\begin{bmatrix} -3x^2 + 2y^2 & 4xy \\ 4xy & 2x^2 - 18y^2 \end{bmatrix}$$

At $(0, 0)$, the eigenvalues of the derivative matrix are 0 and 0.
(b) Take $V(x, y) = x^2 + y^2$, positive definite. Then

$$\begin{aligned} V' &= \frac{\partial V}{\partial x} x' + \frac{\partial V}{\partial y} y' \\ &= 2x(-x^3 + 2xy^2) + 2y(2x^2 y - 6y^2) \\ &= -2x^4 + 8x^2 y^2 - 12y^4 \end{aligned}$$

Substituting $u = x^2$ and $v = y^2$, we find

$$V' = -4u^2 + 8uv - 12v^2$$

That is, $a = -2$, $b = 8$, and $c = -12$. Because $a < 0$ and $4ac - b^2 > 0$, V' is negative definite. Then by Liapunov's theorem, the origin is asymptotically stable.

8. The equations are

$$\begin{aligned} U' &= a - bU - cUV \\ I' &= cUV - dI \\ V' &= eI - fV \end{aligned}$$

For the U' equation, a is the birth rate, bU is the death rate, and cUV is the infection rate, leaving the U population and entering the I population.

For the I' equation, cUV is the rate of U cells entering the I population, dI the death rate.

For the V' equation, eI is the rate V is produced, fV is the death rate.