Math 116 Practice Final 4 Solutions

1. (a) Evaluate the integral $\int \frac{x+1}{x^3-3x^2+2x}dx$.
First, observe

$$x^3-3x^2+2x = x(x^2-3x+2) = x(x-1)(x-2)$$

and so

$$\frac{x+1}{x^3-3x^2+2x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

This gives

$$x+1 = A(x-1)(x-2) + Bx(x-2) + Cx(x-1)$$

Then

$$\begin{aligned} x=1 &\quad \text{gives } B=-2 \\ x=2 &\quad \text{gives } C=3/2 \\ x=0 &\quad \text{gives } A=1/2 \end{aligned}$$

Then

$$\begin{aligned} \int \frac{x+1}{x^3-3x^2+2x}dx &= \frac{1}{2}\int \frac{1}{x}dx - 2\int \frac{1}{x-1}dx + \frac{3}{2}\int \frac{1}{x-2}dx \\ &= \frac{1}{2}\ln|x| - 2\ln|x-1| + \frac{3}{2}\ln|x-2| + C \end{aligned}$$

(b) $\int x^2\cos(x)dx$
Integrate by parts twice

$$\begin{aligned} \int x^2\cos(x)dx &= x^2\sin(x) - 2\int x\sin(x)dx \\ &= x^2\sin(x) - 2\left(-x\cos(x) - \int -\cos(x)dx\right) \\ &= x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C \end{aligned}$$

For the first integration by parts, take $u = x^2$ and $dv = \cos(x)dx$.
For the second integration by parts, take $u = x$ and $dv = \sin(x)dx$.
(c) $\int \frac{\cos(x)}{\sin(x)\sqrt{1+\sin^2(x)}}dx$
Substituting $u = \sin(x)$ so $du = \cos(x)dx$, we find

$$\int \frac{\cos(x)}{\sin(x)\sqrt{1+\sin^2(x)}}dx = \int \frac{du}{u\sqrt{1+u^2}}$$

$$= -\ln\left|\frac{\sqrt{1+u^2}+1}{u}\right| + C \quad \text{by 22, integral table}$$

$$= -\ln\left|\frac{\sqrt{1+\sin^2(x)}+1}{\sin(x)}\right| + C$$

2. Say E_1 represents all people under age 60 with colorectal cancer, E_2 those under 60 without colorectal cancer, and A those with an APC mutation. The data provided are

$$P(A|E_1) = 0.95 \quad P(A|E_2) = 0.5 \quad P(E_1) = 0.1 \quad \text{and so } P(E_2) = 0.9.$$

We are asked to find $P(E_1|A)$. Apply Bayes' theorem,

$$P(E_1|A) = \frac{P(A|E_1)\cdot P(E_1)}{P(A)}$$

To find $P(A)$, use the law of conditioned probabilities

$$P(A) = P(A|E_1)\cdot P(E_1) + P(A|E_2)\cdot P(E_2)$$

Substituting in the values gives

$$P(A) = 0.95\cdot 0.1 + 0.5\cdot 0.9 = 0.545$$

Then Bayes' theorem gives

$$P(E_1|A) = \frac{0.95\cdot 0.1}{0.545} = 0.174$$

3. Apply the ratio test with $a_n = \dfrac{2^n}{n^2+4}(x-2)^n$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{2^n}\frac{n^2+4}{(n+1)^2+4}|x-2| = 2|x-2|$$

Then convergence is given by $\lim_{n\to\infty}|a_{n+1}/a_n| < 1$, that is, $2|x-2| < 1$, and $|x-2| < 1/2$, so the radius of convergence is $R = 1/2$. To find the interval of convergence, first find the endpoints, then test each separately.

$$|x-2| < 1/2, \quad \text{so } -1/2 < x-2 < 1/2, \quad \text{so } 3/2 < x < 5/2$$

Test the endpoints to determine the interval of convergence.

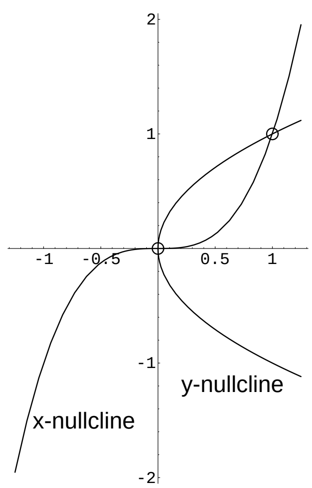
Substituting $x = 3/2$ in the series becomes $\sum(-1)^n/(n^2+4)$, which converges by the alternating series test.
Substituting $x = 5/2$ in the series becomes $\sum 1/(n^2+4)$, which converges by comparison with $\sum 1/n^2$.

Then the interval of convergence of $\sum_{n=1}^{\infty} \frac{2^n}{n^2+4}(x-2)^n$ is $[3/2, 5/2]$.

4. For the system

$$dx/dt = y - x^3$$
$$dy/dt = x - y^2$$

(a) the x-nullcline is the curve $y - x^3 = 0$, that is, $y = x^3$. The y-nullcline is the curve $x - y^2 = 0$, that is, $x = y^2$.



(b) The fixed points are circled in the diagram. To find their coordinates, solve $x = y^2 = (x^3)^2 = x^6$. That is, $0 = x^6 - x = x(x^5 - 1)$. The solutions are $x = 0$ and $x = 1$. (The equation $x^5 - 1 = 0$ has 5 solutions, but only 1 is real.) The fixed points are $(0,0)$ and $(1,1)$.
(c) To test the stability of these fixed points, first compute the derivative matrix

$$D\vec{F}(x,y) = \begin{bmatrix} -3x^2 & 1 \\ 1 & -2y \end{bmatrix}$$

Then we see

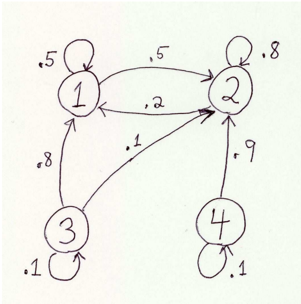
$$D\vec{F}(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad D\vec{F}(1,1) = \begin{bmatrix} -3 & 1 \\ 1 & -2 \end{bmatrix}$$

The eigenvalues of $D\vec{F}(0,0)$ are ± 1, so the origin is unstable because (at least) one eigenvalue is positive.

The eigenvalues of $D\vec{F}(1,1)$ are $(-5 \pm \sqrt{5})/2$, both negative, so the fixed point $(1,1)$ is asymptotically stable.

5. First draw the transition graph of the Markov process with this transition matrix

$$\begin{bmatrix} .5 & .2 & .8 & 0 \\ .5 & .8 & .1 & .9 \\ 0 & 0 & .1 & 0 \\ 0 & 0 & 0 & .1 \end{bmatrix}$$



From the graph we see that states 3 and 4 eventually are emptied, leaving the population to shift between states 1 and 2, governed by the matrix

$$\begin{bmatrix} .5 & .2 \\ .5 & .8 \end{bmatrix}$$

This is a stochastic matrix with all entries positive, so the larger eigenvalue is $\lambda = 1$. An eigenvector of $\lambda = 1$ is a solution of

$$\begin{bmatrix} .5 & .2 \\ .5 & .8 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 1 \cdot \begin{bmatrix} u \\ v \end{bmatrix}$$

That is, $u = (2/5)v$. The values of the distribution must sum to 1, so $1 = u + v = (2/5)v + v$. This gives

$$\text{fraction in state } 1 = u = 2/7$$
$$\text{fraction in state } 2 = v = 5/7$$

6. Recall $rr' = xx' + yy'$ and substitute in the expressions for x' and y' from the system

$$x' = 2x - 2y - x(x^2 + y^2)$$
$$y' = x + 2y - y(x^2 + y^2)$$

we obtain

$$\begin{aligned} rr' &= x(2x - 2y - x(x^2 + y^2)) + y(x + 2y - y(x^2 + y^2)) \\ &= 2(x^2 + y^2) - xy - (x^2 + y^2)^2 \\ &= 2r^2 - r^2 \cos(\theta) \sin(\theta) - r^4 \end{aligned}$$

where the last equality was obtained by the polar coordinate substitution $x = r\cos(\theta)$, $y = r\sin(\theta)$. This gives

$$r' = 2r - r\cos(\theta)\sin(\theta) - r^3$$

Now certainly $-1 \leq \cos(\theta)\sin(\theta) \leq 1$. In fact, recalling $\cos(\theta)\sin(\theta) = \sin(2\theta)/2$, we get the stronger bounds $-1/2 \leq \cos(\theta)\sin(\theta) \leq 1/2$. In fact, the weaker bounds suffice.

From $-1 \leq \cos(\theta)\sin(\theta) \leq 1$ we obtain $-r \leq r\cos(\theta)\sin(\theta) \leq r$. Adding $2r - r^3$ across the inequality,

$$r - r^3 \leq r' \leq 3r - r^3$$

At $r = 1/2$, the lower bound gives $3/8 \leq r'$; at $r = 2$ the upper bound gives $r' < -2$. Then the annulus $1/2 \leq r \leq 2$ is a trapping region. Because the origin is the only fixed point, it follows from the Poincaré-Bendixson theorem that this annulus contains a limit cycle.

7. Assuming $x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots$, the condition $x(0) = 1$ gives $a_0 = 1$. To solve the equation

$$x'(t) = tx(t) + t, \qquad x(0) = 1$$

we need series expressions for $x'(t)$ and for $tx(t) + t$. Differentiating term-by-term we find

$$x'(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + \cdots$$

and

$$tx(t) = a_0 t + a_1 t^2 + a_2 t^3 + a_3 t^4 + a_4 t^5 + \cdots$$

Then the series for $tx(t) + t$ is obtained from the sereis for $tx(t)$ by adding 1 to the coefficient of the t term in that series:

$$tx(t) + t = (a_0 + 1)t + a_1 t^2 + a_2 t^3 + a_3 t^4 + a_4 t^5 + \cdots$$

Next, equate the coeffcients of like powers of t in the series for $x'(t)$ and the series for $tx(t) + t$.

t^n	x'	$tx + t$	
t^0	a_1	0	$a_1 = 0$
t^1	$2a_2$	$1 + a_0$	$2a_2 = 1 + 1$ so $a_2 = 1$
t^2	$3a_3$	a_1	$3a_3 = a_1$ so $a_3 = 0$
t^3	$4a_4$	a_2	$4a_4 = a_2$ so $a_4 = 1/4$
t^4	$5a_5$	a_3	$5a_5 = a_3$ so $a_5 = 0$
t^5	$6a_6$	a_4	$6a_6 = a_4$ so $a_6 = 1/(6 \cdot 4)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$

We see all the odd subscript coefficients are 0, that is, $a_{2k+1} = 0$ for $k = 0, 1, 2, \dots$.

The even coefficients are a bit more complicated:

$$a_2 = 1, a_4 = 1/4, a_6 = 1/(6 \cdot 4), a_8 = 1/(8 \cdot 6 \cdot 4), \cdots$$

The denominators are the products of even numbers starting with 4, so factoring a 2 from each factor in the denominator, we have

$$\begin{aligned}
a_4 = a_{2\cdot 2} &= \frac{1}{2 \cdot 2} = \frac{1}{2^1 \cdot 2!} \\
a_6 = a_{2\cdot 3} &= \frac{1}{2^2 \cdot 3 \cdot 2} = \frac{1}{2^2 \cdot 3!} \\
a_8 = a_{2\cdot 4} &= \frac{1}{2^3 \cdot 4 \cdot 3 \cdot 2} = \frac{1}{2^3 \cdot 4!} \\
&\cdots
\end{aligned}$$

In fact, a_2 fits this pattern: $a_2 = a_{2\cdot 1} = 1/(2^0 \cdot 1!)$. That is, all the even subscript coefficients have the form

$$a_{2k} = \frac{1}{2^{k-1}k!} = \frac{2}{2^k k!}$$

for $k \geq 1$. Except we have seen that $a_0 = 1$, so to fit the pattern that includes the 2 in the numerator of a_{2k}, we must write $a_0 = 2 - 1$. Then

$$\begin{aligned}
x(t) &= a_0 + a_2 t^2 + a_4 t^4 + a_6 t^6 + a_8 t^8 + \cdots \\
&= (2-1) + 2\frac{t^2}{2} + 2\frac{t^4}{2^2 \cdot 2} + 2\frac{t^6}{2^3 \cdot 3!} + 2\frac{t^8}{2^4 \cdot 4!} + \cdots \\
&= 2\left(1 + \left(\frac{t^2}{2}\right) + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{t^2}{2}\right)^3 + \frac{1}{4!}\left(\frac{t^2}{2}\right)^4 + \cdots\right) - 1 \\
&= 2e^{t^2/2} - 1
\end{aligned}$$

To check this is correct, first observe

$$x(0) = 2e^0 - 1 = 1$$

Next, by the chain rule

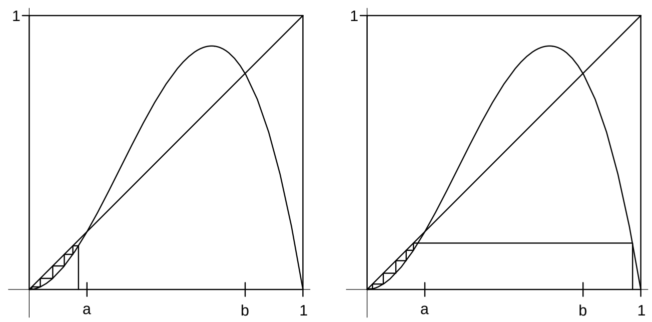
$$x' = 2e^{t^2/2} \cdot (t^2/2)' = 2e^{t^2/2} \cdot t$$

and

$$tx + t = t\left(2e^{t^2/2} - 1\right) + t = 2e^{t^2/2} \cdot t - t + t = 2e^{t^2/2} \cdot t$$

8. For the population equation

$$P_{n+1} = rP_n^2(1 - P_n)$$



with $r = 6$, the left side of the figure illustrates by graphical iteration that every $P_0 < a$ iterates to 0. These populations become extinct.
(b) The right side of the figure illustrates that P_0 near 1 iterates to a point less than a, thence to 0.