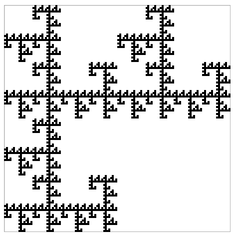
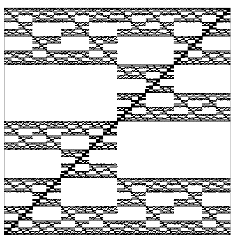
Math 190 Midterm Solutions

1. Here ar the IFS rules to generate these fractals.



r	s	θ	φ	e	f
0.5	0.5	90	90	0.5	0
0.5	0.5	90	90	1.0	0.5
0.5	0.5	90	90	0.5	0.5



r	s	θ	φ	e	f
0.5	0.5	0	0	0	0
0.5	0.5	0	0	0.5	0.5
0.25	−0.25	0	0	0.5	0.25
0.25	−0.25	0	0	0.75	0.25
0.25	−0.25	0	0	0	1.0
0.25	−0.25	0	0	0.25	1.0

or

r	s	θ	φ	e	f
0.5	0.5	0	0	0	0
0.5	0.5	0	0	0.5	0.5
−0.25	0.25	0	0	0.75	0
−0.25	0.25	0	0	1.0	0
−0.25	0.25	0	0	0.25	0.75
−0.25	0.25	0	0	0.5	0.75

2. (a) This fractal consists of 3 copies scaled by 1/2. Because all scalings are the same, the basic similarity dimensio formula can be used

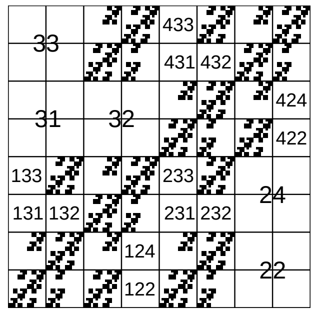
$$d = \frac{\log(N)}{\log(1/r)} = \frac{\log(3)}{\log(2)}$$

(b) This fractal consists of 2 copies scaled by 1/2 and 4 copies scaled by 1/4, so the Moran equation becomes

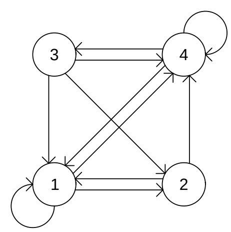
$$2 \cdot (1/2)^d + 4 \cdot (1/4)^d = 1$$

Substituting $x = (1/2)^d$, the Moran equation becomes $2x + 4x^2 = 1$. The positive root is $x = (-1 + \sqrt{5})/4$ and so $d = \log((-1+\sqrt{5})/4)/\log(1/2)$.

3. (a) For (i) the forbidden pairs are 22, 24, 31, 32, and 33. The forbidden triples that are not contained in these pairs are 122, 124, 131, 132, 133, 231, 232, 233, 422, 424, 431, 432, and 433. Each of these contains a forbidden pair, so to the level of forbidden triples, this fractal is generated by forbidden pairs.



(b) The transition graph is obtained by filling in each arrow corresponding to an allowed pair, taking care that the pair ij corresponds to the transition $j \to i$.



(c) First, note that 1 and 4 are romes; because in the transition graph four arrows go to vertex 1 and 4.

Next, there are paths from a rome to each non-rome: $4 \to 3$, $1 \to 2$, and also $4 \to 3 \to 2$.

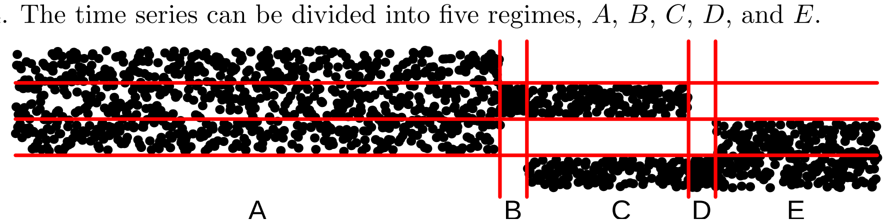
Finally, there are no loops among non-romes, because the only path through only non-romes is $3 \to 2$, not a loop.

Consequently, the fractal generated by this IFS with memory also can be generated by an IFS without memory.

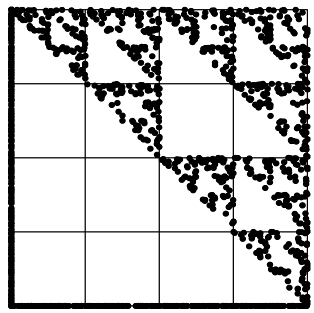
The allowed compositions, T_1, T_4, $T_2 \circ T_1$, $T_3 \circ T_4$, and $T_2 \circ T_3 \circ T_4$, determine the IFS rules

r	s	θ	φ	e	f
0.5	0.5	0	0	0	0
0.5	0.5	0	0	0.5	0.5
0.25	0.25	0	0	0.5	0
0.25	0.25	0	0	0.25	0.75
0.125	0.125	0	0	0.625	0.375

4. The time series can be divided into five regimes, A, B, C, D, and E.



Regime A generates points on the gasket with corners 2, 3, and 4. Regime B produces a sequence of points converging to corner 3. Regime C produces points on the line between corners 3 and 1. Regime D produces points converging to corner 1. Regime E produces points on the line between corners 1 and 2. So the driven IFS shows a gasket the line segment between $(0, 1)$ and $(0, 0)$, and the segment between $(0, 0)$ and $(1, 0)$.



5. Using the formula for the dimension of a product

$$\dim(A \times B) = \dim(A) + \dim(B)$$

we have

$$\dim(A) + \dim(B) = \frac{2}{3}$$

By the basic similarity dimension formula, we have

$$\dim(A) = \dim(B) = \frac{\log(2)}{\log(1/r)}$$

That is,

$$2\frac{\log(2)}{\log(1/r)} = \frac{2}{3}$$

Canceling a 2 from both sides and cross-multiplying,

$$3\log(2) = \log(1/r)$$

Using the exponent rule for logarithms,

$$\log(8) = \log(1/r)$$

Consequently, $8 = 1/r$ and so

$$r = \frac{1}{8}$$

6. From the IFS table

r	s	θ	φ	e	f	prob
0.5	0.5	0	0	0	0	0.45
0.5	0.5	0	0	0.5	0	0.45
0.25	0.25	0	0	0	0.75	0.05
0.25	0.25	0	0	0.75	0.75	0.05

we see $\alpha_{\min}$ and $\alpha_{\max}$ are the minimum and maximum values of $\log(p_i)/\log(r_i)$. These are

$$\log(.45)/\log(.5) \approx 1.152 \quad \log(.05)/\log(.25) \approx 2.161$$

so $\alpha_{\min} = \log(.45)/\log(.5)$ and $\alpha_{\max} = \log(.05)/\log(.55)$.

The value of $\alpha_{\min}$ occurs on the attractor of the IFS $\{T_1, T_2\}$, consisting of $N = 2$ pieces each scaled by $r = 1/2$, so of dimension $\log(2)/\log(2) = 1$. That is, $f(\alpha_{\min}) = 1$.

The value of $\alpha_{\max}$ occurs on the attractor of the IFS $\{T_3, T_4\}$, consisting of $N = 2$ pieces each scaled by $r = 1/4$, so of dimension $\log(2)/\log(4) = 1/2$. That is, $f(\alpha_{\max}) = 1/2$.

The maximum value of $f(\alpha)$ is the dimension of the attractor of the IFS, that is, the solution of the Moran equation

$$2 \cdot (1/2)^d + 2 \cdot (1/4)^d = 1$$

Taking $x = (1/2)^d$, the Moran equation becomes the quadratic equation $2x + 2x^2 = 1$, with positive solution $(-1 + \sqrt{3})/2$, Then the maximum value of $f(\alpha)$ is $\log((-1 + \sqrt{3})/2)/\log(1/2)$.

Combining this information, here is a sketch of the $f(\alpha)$ curve for this IFS with probabilities.