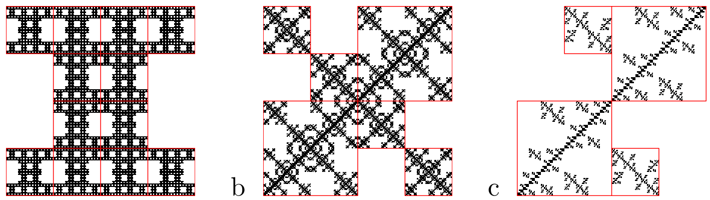
Fourth homework set solutions



a b c

1. (a) As indicated by the red boxes, we see fractal (a) is composed of $N = 12$ pieces, each scaled by $r = 1/4$, so the similarity dimension of this fractal is

$$d_s = \frac{\log(12)}{\log(4)} = 1 + \frac{\log(3)}{\log(4)} \approx 1.79248.$$

(b) As indicated by the red boxes, this fractal is composed of 2 pieces of size 1/2 and 4 pieces of size 1/4. By the Moran equation, the dimension d satisfies

$$2 \cdot (1/2)^d + 4 \cdot (1/4)^d = 1$$

Writing $x = (1/2)^d$, we see that $(1/4)^d = ((1/2)^2)^d = ((1/2)^d)^2 = x^2$, so the Moran equation can be rewitten

$$2x + 4x^2 = 1$$

Applying the quadratic formula, the positive solution is $x = (-1 + \sqrt{5})/4$ and so the dimension is

$$d_s = \frac{\log((-1 + \sqrt{5})/4)}{\log(1/2)} \approx 1.69424.$$

(c) As indicated by the red boxes, this fractal is composed of 2 pieces of size 1/2 and 2 pieces of size 1/4. By the Moran equation, the dimension d satisfies

$$2 \cdot (1/2)^d + 2 \cdot (1/4)^d = 1$$

Writing $x = (1/2)^d$, we see that $(1/4)^d = ((1/2)^2)^d = ((1/2)^d)^2 = x^2$, so the Moran equation can be rewitten

$$2x + 2x^2 = 1$$

Applying the quadratic formula, the positive solution is $x = (-1 + \sqrt{3})/2$ and so the dimension is

$$d_s = \frac{\log((-1 + \sqrt{3})/2)}{\log(1/2)} \approx 1.44998.$$

2. Suppose the xy-plane contains a fractal A consisting of $N = 4$ pieces, each scaled by a factor r, and suppose the z-axis contains a fractal B consisting of $N = 2$ pieces, each scaled by the same factor r.

(a) By the similarity dimension formula we see $d_s(A) = \log(4)/\log(1/r)$ and $d_s(B) = \log(2)/\log(1/r)$.

(b) By the product formula, $d_s(A \times B) = d_s(A) + d_s(B) = \log(4)/\log(1/r) + \log(2)/\log(1/r)$.

(c) The equation $d_s(A \times B) = 2$ becomes

$$\begin{aligned}
\frac{\log(4)}{\log(1/r)} + \frac{\log(2)}{\log(1/r)} &= 2 \\
\log(4) + \log(2) &= 2 \cdot \log(1/r) \\
\log(8) &= \log(1/r^2) \\
8 &= 1/r^2 \\
r^2 &= 1/8 \\
r &= 1/\sqrt{8} \approx 0.353553
\end{aligned}$$

3. (a) The Moran equation becomes

$$\begin{aligned}
(2/3)^d + (((2/3)^2)^d + ((2/3)^3)^d + \ldots = 1 \\
(2/3)^d + (((2/3)^d)^2 + ((2/3)^d)^3 + \ldots = 1
\end{aligned}$$

Writing $x = (2/3)^d$, we obtain

$$x + x^2 + x^3 + \cdots = 1$$

(b) Using the fact that for all x with $|x| < 1$,

$$x + x^2 + x^3 + \cdots = \frac{x}{1-x}$$

we obtain

$$\frac{x}{1-x} = 1$$

That is,

$$1 = 2x = 2(2/3)^d$$

Solving for d,

$$d = \frac{\log(1/2)}{\log(2/3)} \approx 1.70951$$