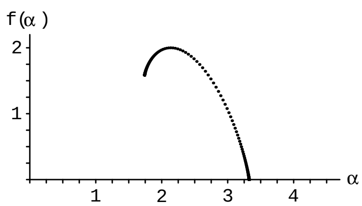
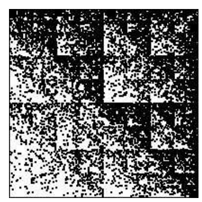
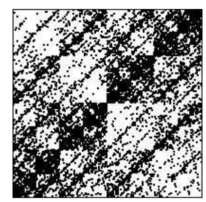
Fifth homework set solutions

1. From the $f(\alpha)$ curve we see $f(\alpha_{\max}) = 0$, so the maximum value of $\alpha$ occurs on a set of dimension 0, a point, for instance. Also, $f(\alpha_{\min})$ looks like about $\log(3)/\log(2) \approx 1.6$, so the minimum value of $\alpha$ occurs on a gasket. In (a) we see addresses 2, 3, and 4 are about equally filled, and more filled than address 1, suggesting $\text{prob}_1 < \text{prob}_2 = \text{prob}_3 = \text{prob}_4$. In (b) we see addresses 1 and 4 are about equally filled, and addresses 2 and 3 are about equally filled, and less than 1 and 4. This suggests $\text{prob}_2 = \text{prob}_3 < \text{prob}_1 = \text{prob}_4$. Recalling that so far as the $r$ values are the same, the maximum $\alpha$ corresponds to the minimum probability and the minimum $\alpha$ corresponds to the maximum probability, we see the multifractal (a) is better described by this $f(\alpha)$ curve




(a)

(b)


2. Consider the multifractal generated by this IFS.

| $r$ | $s$ | $\theta$ | $\varphi$ | $e$ | $f$ | prob |
|---|---|---|---|---|---|---|
| 1/3 | 1/3 | 0 | 0 | 0 | 0 | 3/20 |
| 1/3 | 1/3 | 0 | 0 | 1/3 | 0 | 3/20 |
| 1/3 | 1/3 | 0 | 0 | 2/3 | 0 | 3/20 |
| 1/3 | 1/3 | 0 | 0 | 0 | 1/3 | 3/20 |
| 1/3 | 1/3 | 0 | 0 | 1/3 | 1/3 | 3/20 |
| 1/3 | 1/3 | 0 | 0 | 2/3 | 1/3 | 3/20 |
| 1/3 | 1/3 | 0 | 0 | 0 | 2/3 | 1/30 |
| 1/3 | 1/3 | 0 | 0 | 1/3 | 2/3 | 1/30 |
| 1/3 | 1/3 | 0 | 0 | 2/3 | 2/3 | 1/30 |

(a) This IFS generates a fractal composed of $N = 9$ pieces, each scaled by $r = 1/3$, so having dimension $\log(9)/\log(3) = 2$. This is the maximum value of $f(\alpha)$.

(b) Because all the scaling factors are the same, 1/3, the maximum $\alpha$ corresponds to the minimum probability. This occurs for the fractal generated by the last three transformations, a fractal having dimension $\log(3)/\log(3) = 1$. That is, $f(\alpha_{\max}) = 1$.

(c) Similarly, the minimum value of $\alpha$ occurs for the fractal generated by the first six transformations, a fractal with dimension $\log(6)/\log(3)$. That is, $f(\alpha_{\min}) = \log(6)/\log(3)$.

3. (a) The maximum dimension occurs if at every stage $r_1 = r_2 = 1/2$. If this happens, the result is a fractal made of $N = 2$ pieces, each scaled by $r = 1/2$,

and so of dimension $\log(2)/\log(2) = 1$. The minimum dimension occurs if at every stage $r_1 = r_2 = 1/4$. If this happens, the result is a fractal made of $N = 2$ pieces, each scaled by $r = 1/4$, and so of dimension $\log(2)/\log(4) = 1/2$.

(b) In order to have the maximum dimension, at each stage we must have $r_1 = 1/2$ and $r_2 = 1/2$. The first has probability 1/2, the second has probability 1/4, so at each stage $r_1 = r_2 = 1/2$ occurs with probability $(1/2) \cdot (1/4) = 1/8$. Assuming the choices from stage to stage are independent of one another, the probability of finding $r_1 = r_2 = 1/2$ in $n$ stages is $(1/8)^n$. As $n \to \infty$, $(1/8)^n \to 0$, do the probability of obtaining a fractal of dimension 1 from this constrction is 0.

In order to have the minimum dimension, at each stage we must have $r_1 = 1/4$ and $r_2 = 1/4$. The first has probability 1/2, the second has probability 3/4, so at each stage $r_1 = r_2 = 1/4$ occurs with probability $(1/4) \cdot (3/4) = 3/16$. Assuming the choices from stage to stage are independent of one another, the probability of finding $r_1 = r_2 = 1/4$ in $n$ stages is $(3/16)^n$. As $n \to \infty$, $(3/16)^n \to 0$, do the probability of obtaining a fractal of dimension 1/2 from this constrction is 0.

(c) Because $r_1 = 1/2$ with probability 1/2 and $r_1 = 1/4$ with probability 1/2, the expected value of $r_1^d$ is $(1/2) \cdot (1/2)^d + (1/2) \cdot (1/4)^d$. Because $r_2 = 1/2$ with probability 1/4 and $r_2 = 1/4$ with probability 3/4, the expected value of $r_2^d$ is $(1/4) \cdot (1/2)^d + (3/4) \cdot (1/4)^d$. The the randomized Moran equation

$$\mathbb{E}(r_1^d) + \mathbb{E}(r_2^d) = 1$$

becomes

$$(1/2) \cdot (1/2)^d + (1/2) \cdot (1/4)^d + (1/4) \cdot (1/2)^d + (3/4) \cdot (1/4)^d = 1$$

Taking $x = (1/2)^d$, so $x^2 = ((1/2)^d)^2 = ((1/2)^2)^d = (1/4)^d$, the randomized Moran equation is

$$(1/2)x + (1/2)x^2 + (1/4)x + (3/4)x^2 = 1$$

The positive solution is $x = (-3 + \sqrt{89})/10$ and so the expected value of the dimension is

$$d = \frac{\log((-3 + \sqrt{89})/10)}{\log(1/2)} \approx 0.694242.$$