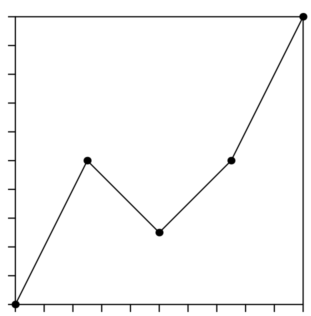
Seventh homework set solutions

1. (a) Recall $dt_1 = dt_2 = dt_3 = dt_4 = 1/4$, and measure $dY_1 = 0.5$, $dY_2 = -0.25$, $dY_3 = 0.25$, and $dY_4 = 0.5$. To show this is a multifractal, observe

$$\frac{\log(|dY_1|)}{\log(dt_1)} = \frac{\log(1/2)}{\log(1/4)} = \frac{1}{2}$$
$$\frac{\log(|dY_2|)}{\log(dt_2)} = \frac{\log(1/4)}{\log(1/4)} = 1$$
$$\frac{\log(|dY_3|)}{\log(dt_3)} = \frac{\log(1/4)}{\log(1/4)} = 1$$
$$\frac{\log(|dY_4|)}{\log(dt_4)} = \frac{\log(1/2)}{\log(1/4)} = \frac{1}{2}$$



(b) To find the trading time generators, first find D satisfying

$$(dY_1)^D + |dY_2|^D + (dY_3)^D + (dY_4)^D = 1$$

That is,

$$(1/2)^D + (1/4)^D + (1/4)^D + (1/2)^D = 1$$

Taking $x = (1/2)^D$, this becomes the quadratic equation

$$2x + 2x^2 = 1$$

The positive solution is $x = (-1 + \sqrt{3})/2$. Then $D = \log(x)/\log(1/2)$ and so the trading time generators are

$$dT_1 = (dY_1)^D = (1/2)^D = (1/2)^{\log((-1+\sqrt{3})/2)/\log(1/2)}$$
$$dT_2 = |dY_2|^D = (1/4)^D = (1/4)^{\log((-1+\sqrt{3})/2)/\log(1/2)}$$
$$dT_3 = (dY_3)^D = (1/4)^D = (1/4)^{\log((-1+\sqrt{3})/2)/\log(1/2)}$$
$$dT_4 = (dY_4)^D = (1/2)^D = (1/2)^{\log((-1+\sqrt{3})/2)/\log(1/2)}$$

These answers are fine, but they can be simplifird substantially.

$$\begin{aligned}\log(dT_1) &= \log\big((1/2)^{\log(x)/\log(1/2)}\big)\\ &= \frac{\log(x)}{\log(1/2)}\log(1/2)\\ &= \log(x)\end{aligned}$$

Because $\log(dT_1) = \log(x)$, we have

$$dT_1 = dT_4 = x = \frac{-1+\sqrt{3}}{2}$$

Similarly,

$$\begin{aligned}\log(dT_2) &= \log\big((1/4)^{\log(x)/\log(1/2)}\big)\\ &= \frac{\log(x)}{\log(1/2)}\log(1/4)\\ &= \frac{\log(x)}{\log(1/2)}\log((1/2)^2)\\ &= \frac{\log(x)}{\log(1/2)}2\log(1/2)\\ &= 2\log(x)\\ &= \log(x^2)\end{aligned}$$

Because $\log(dT_2) = \log(x^2)$, we have

$$dT_2 = dT_3 = x^2 = \left(\frac{-1+\sqrt{3}}{2}\right)^2$$

2. To be a unifractal we must have

$$\frac{\log(|dY_1|)}{\log(dt_1)} = \frac{\log(|dY_2|)}{\log(dt_2)} = \frac{\log(|dY_3|)}{\log(dt_3)} = \frac{\log(|dY_4|)}{\log(dt_4)}$$

That is,

$$\frac{\log(|1/2|)}{\log(a)} = \frac{\log(|1/2|)}{\log(1/2-a)} = \frac{\log(|1/2|)}{\log(1/4)} = \frac{\log(|1/2|)}{\log(1/2)}$$

Because all the numerators are the same, all the denominators must be the same. That is,

$$a = 1/2 - a = 1/4 = 1/4$$

That is, $a = 1/4$. With this value of a, the cartoon is a unifractal.