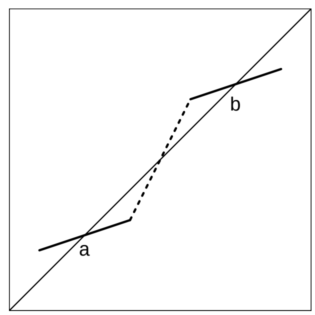
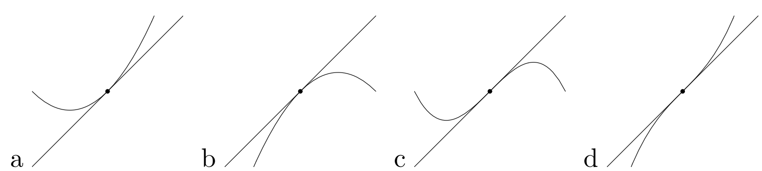
Eighth homework set solutions

1. Both fixed points cannot be stable. Recall that a fixed point is stable if values on both sides of the fixed point iterate toward the fixed point. One way this can happen is at the fixed point, the tangent line of the function has slope greater than -1 and less than $+1$ So to the immediate left of the fixed point, the graph of the function lies above the diagonal line, and to the immediate right, below the diagonal. As we see in this sketch, if a and b both are stable, then either there must be an unstable fixed point between a and b, or the graph must make a jump between $x = a$ and $x = b$. Neither of these is consistent with the conditions of the problem.



There is another possibility: a stable fixed point can occur with a slope of $+1$. This can happen in four ways, sketched here. By graphical iteration we see that only (c) is stable, and in this case the graph crosses the diagonal from above on the left to below on the right, so the preceding argument applies here as well.



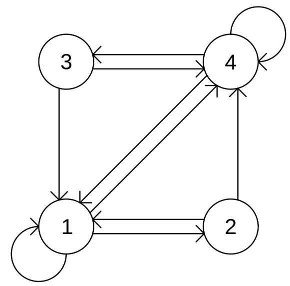
2. (a) From the graph we see the bin to bin transitions are

$$1 \to 1, 1 \to 2, 1 \to 4, 2 \to 4, 2 \to 1, 3 \to 1, 3 \to 4, 4 \to 4, 4 \to 3, 4 \to 1$$

(b) From the allowed transitions, we see the transition graph is

(c) Inspecting the transition graph, we see that 1 and 4 are romes. These give two copies of the fractal, scaled by 1/2. The other two transitions, $1 \to 2$ and $4 \to 3$, produce copies of the fractal scaled by 1/4. This is all. Then the dimension of the driven IFS fractal can be found by the Moran equation

$$2(1/2)^d + 2(1/4)^d = 1$$



Taking $x = (1/2)^d$, we have

$$2x + 2x^2 = 1$$

The positive solution is $x = (-1+\sqrt{3})/2$, and so the dimension is

$$d = \log((-1+\sqrt{3})/2)/\log(1/2).$$

3. If $x - y = 1/128$, then

$$\begin{aligned}
f(x) - f(y) &= 2 \cdot (1/128) = 1/64 \\
f^2(x) - f^2(y) &= 2 \cdot (1/64) = 1/32 \\
f^3(x) - f^3(y) &= 2 \cdot (1/32) = 1/16 \\
f^4(x) - f^4(y) &= 2 \cdot (1/16) = 1/8 \\
f^5(x) - f^5(y) &= 2 \cdot (1/8) = 1/4 \\
f^6(x) - f^6(y) &= 2 \cdot (1/4) = 1/2 \\
f^7(x) - f^7(y) &= 2 \cdot (1/2) = 1 \\
f^8(x) - f^8(y) &= 2 \cdot 1 = 2 \\
f^9(x) - f^9(y) &= 2 \cdot 2 = 4 \\
f^{10}(x) - f^{10}(y) &= 2 \cdot 4 = 8 \\
f^{11}(x) - f^{11}(y) &= 2 \cdot 8 = 16 \\
f^{12}(x) - f^{12}(y) &= 2 \cdot 16 = 32 \\
f^{13}(x) - f^{13}(y) &= 2 \cdot 32 = 64 \\
f^{14}(x) - f^{14}(y) &= 2 \cdot 64 = 128 \\
f^{15}(x) - f^{15}(y) &= 2 \cdot 128 = 256 \\
f^{16}(x) - f^{16}(y) &= 2 \cdot 256 = 512 \\
f^{17}(x) - f^{17}(y) &= 2 \cdot 512 = 1024
\end{aligned}$$

So $n = 17$ is the smallest value that works.