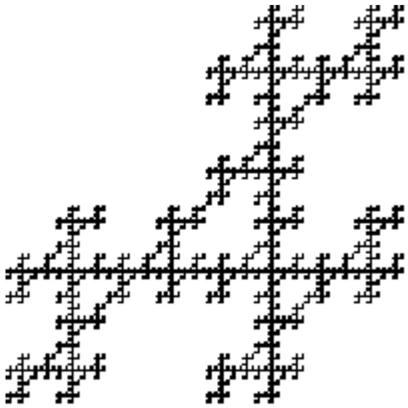
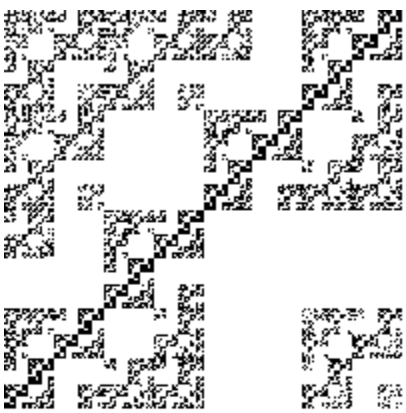
Practice Exam 4 Solutions

1. (a)



R	S	Theta	Phi	E	F
-0.5	0.5	-90	-90	0	0
-0.5	0.5	-90	-90	0.5	0.5
0.5	0.5	180	180	1.0	0.5

(b)



R	S	Theta	Phi	E	F
0.5	0.5	180	180	0.5	0.5
0.5	0.5	180	180	1.0	1.0
0.25	0.25	0	0	0	0.5
0.25	0.25	0	0	0	0.75
0.25	0.25	0	0	0.25	0.75
0.25	0.25	0	0	0.75	0

2. (a) There are three pieces, each scaled by 0.5. The dimension is given by the similarity dimension formula d = Log(N)/Log(1/r) = Log(3)/Log(2).

(b) $r_1 = r_2 = 0.5$, $r_3 = r_4 = r_5 = = r_6 = 0.25$. Because some of the r_i are different, we must use the Moran equation

$$2(.5^d) + 4(.25^d) = 1.$$

Taking $x = .5^d$, the Moran equation becomes the quadratic

$$2x + 4(x^2) = 1.$$

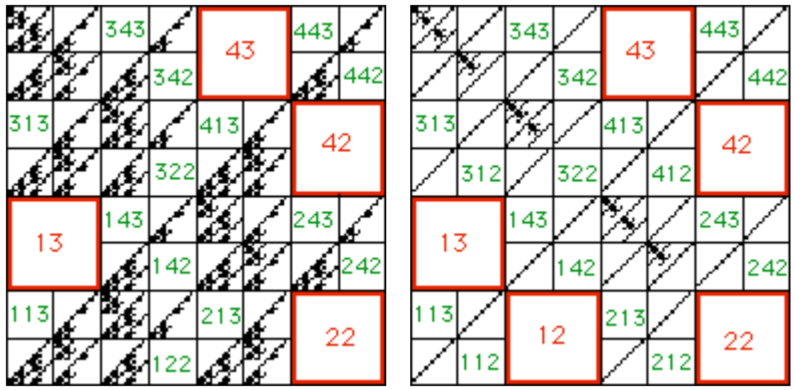
The positive solution given by the quadratic formula is

$$x = (-1 + \sqrt{5})/4$$

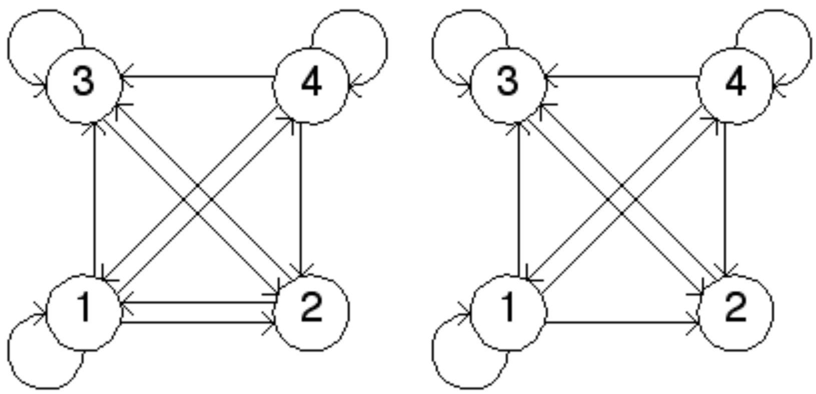
Then d = Log((-1 + √5)/4)/Log(1/2)

3. (a) For the left image we see the **forbidden pairs** are 13, 22, 42, and 43. The **forbidden triples** are 113, 122, 142, 143, 213, 242, 243, 313, 322, 342, 343, 413, 442, and 443. All the forbidden triples contian forbidden pairs, so we shall say this image is generated by forbidden pairs.

For the right image we see the **forbidden pairs** are 12, 13, 22, 42, and 43. The **forbidden triples** are 112, 113, 122, 142, 143, 212, 213, 242, 243, 312, 313, 322, 342, 343, 412, 413, 442, and 443. All the forbidden triples contain forbidden pairs, so we shall say this image is generated by forbidden pairs.



(b) Here are the graphs. Each arrow corresponds to an allowed pair.



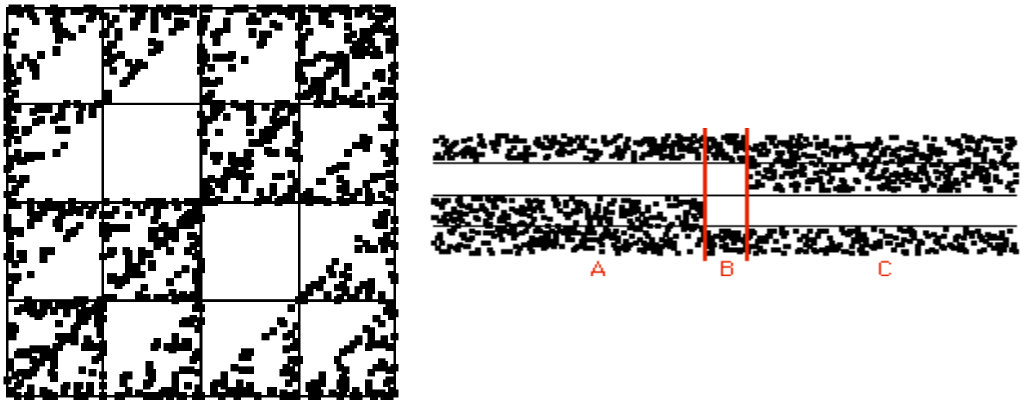
(c) For both, 3 is a rome. The left image has paths from the rome to every non-rome (3 → 2 → 1 → 4), so can be produced by an IFS without memory. The right image has no path from 3 to 1 or 4, so cannot be produced by an IFS without memory.

(d) The left has arbitrarily long paths through non-romes, the 1 → 1 loop, for example, so a memoryless IFS to produce that image requires infinitely many transformations.

4. The time series is divided into three regimes, A, B, and C. In A the data points are spread between bins 1, 2, and 4, apparently in all combinations, and so produce driven IFS points on the gasket with corners 1, 2, and 4.

In B the data points are spread between bins 1 and 4, and so produce driven IFS points on the line between corners 1 and 4.

In C, the data points are spread between bins 1, 3, and 4, apparently in all combinations, and so produce driven IFS points on the gasket with corners 1, 3, and 4.



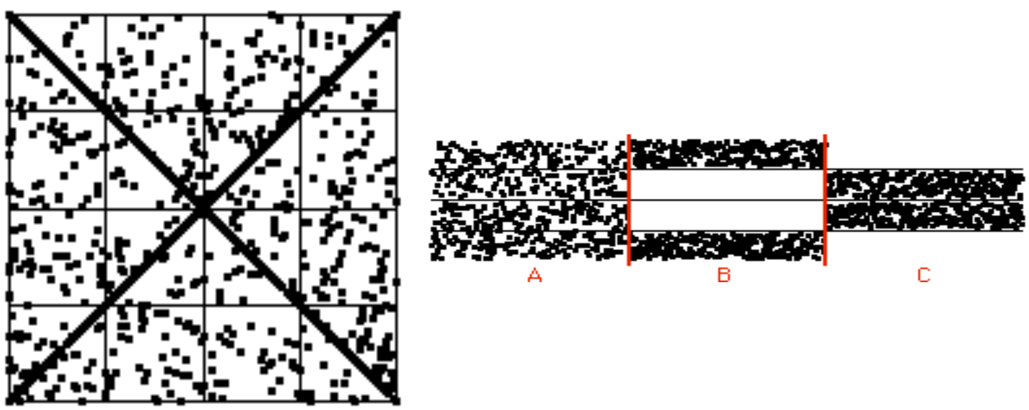
In the driven IFS we see lines between the 1-4 and 2-3 corners, and a background scattering of points. Three different features suggest three regimes.

In A the time series points are spread over all four bins, producing the background scattering of points in the driven IFS.

In B the time series points are spread between bins 1 and 4, producing driven IFS points on the line between corners 1 and 4.

In C the time series points are spread between bins 2 and 3, producing driven IFS points on the line between corners 2 and 3.

The order in which these three regimes occur is immaterial.



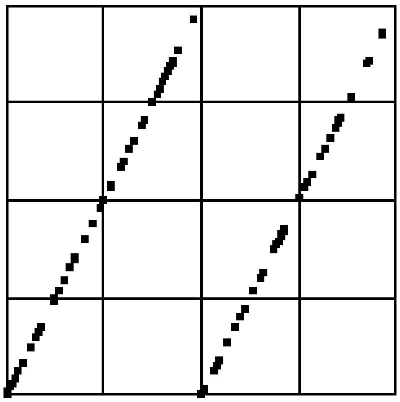
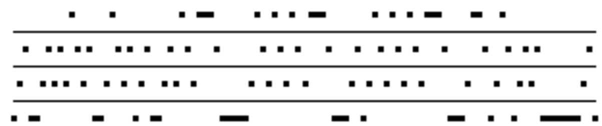
To generate points on these diagonal lines, first observe these addresses are occupied

11, 13, 21, 23, 32, 34, 42, 44

and consequently, these bin transitions are allowed

$$1 \to 1,\ 3 \to 1,\ 1 \to 2,\ 3 \to 2,\ 2 \to 3,\ 4 \to 3,\ 2 \to 4,\ 4 \to 4$$

These two lines are left unchanged under these combinations of transformations, so they are the attractor of this IFS with memory. Consequently, the time series must allow these combinations and no others. That is, a point in bin 1 must be followed by a point in either bin 1 or 2, a point in bin 2 must be followed by a point in either bin 3 or 4, a point in bin 3 must be followed by a point in either bin 1 or 2, and a point in bin 4 must be followed by a point in either bin 3 or 4.

5. (a) The set P is a product of a Cantor middle-thirds set C and a filled-in square S. Then by the product formula

$$\dim(C \times S) = \dim(C) + \dim(S)$$

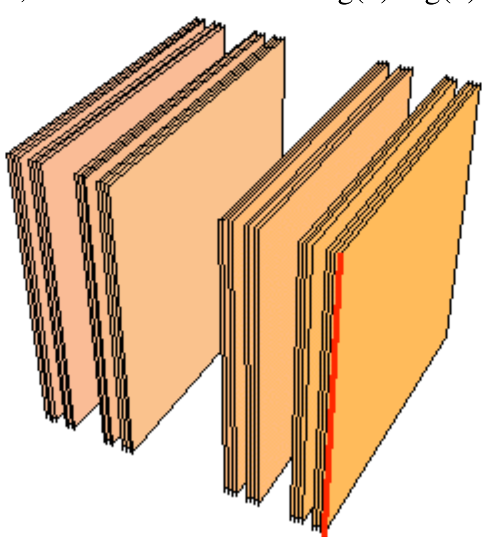
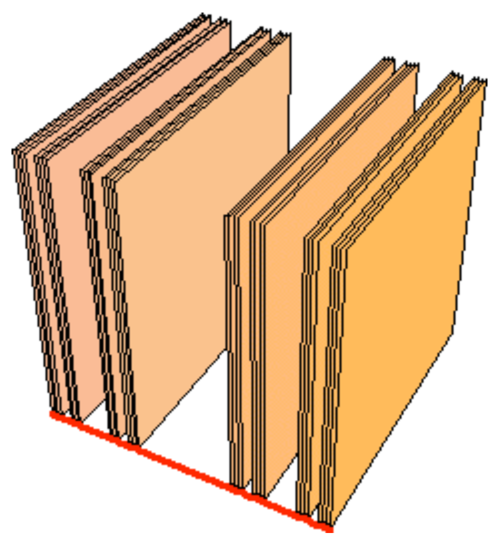
That is, $\dim(P) = \log(2)/\log(3) + 2$.

(b) To find the dimension of a typical intersection in 3-dimensional space of P and a line segment L, use the intersection formula

$$\dim(P \cap L) = \dim(P) + \dim(L) - 3 = \log(2)/\log(3) + 2 + 1 - 3 = \log(2)/\log(3)$$

(c) In the left picture the **line segment** is aligned with a side of the square, so $P \cap L = L$, which has dimension 1.

In the right picture the **line segment** passes through the lower left corners of all the finlled-in squares, so $P \cap L = C$, which has dimension $\log(2)/\log(3)$.

6. (a) $d(A) = \lim_{r \to 0} \text{Log}(N_A(r))/\text{Log}(1/r)$.

(b) One way to achieve $d(B) = (1/2)d(A)$ is to have $N_B(r)$ equal to the square root of $N_A(r)$. Then

$$\begin{aligned} d(A) &= \lim_{r \to 0} \text{Log}(N_B(r))/\text{Log}(1/r) \\ &= \lim_{r \to 0} \text{Log}(\text{sqrt}(N_A(r)))/\text{Log}(1/r) \end{aligned}$$

$= \lim_{r \to 0} \text{Log}((N_A(r))^{1/2})/\text{Log}(1/r)$

$= \lim_{r \to 0} (1/2)\ \text{Log}(N_A(r))/\text{Log}(1/r)$

$= (1/2)\lim_{r \to 0} \text{Log}(N_A(r))/\text{Log}(1/r)$

$= (1/2)d(A)$