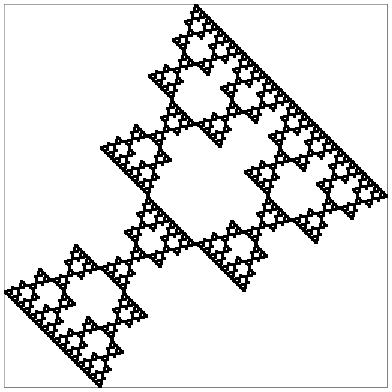
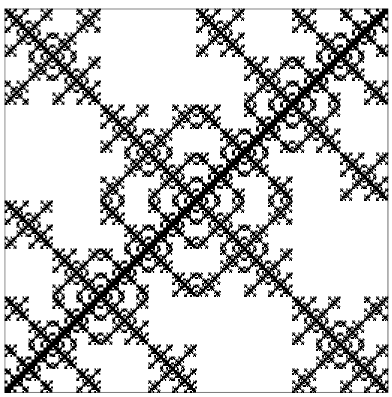
Practice Exam 6 Solutions

1. These IFS rules generate these fractals.

(a) 

r	s	θ	φ	e	f
.5	.5	180	180	.5	.5
.5	.5	0	0	.5	.25
.5	.5	0	0	.25	.5

(b) 

r	s	θ	φ	e	f
.5	.5	0	0	0	0
.5	.5	0	0	.5	.5
-.25	.25	0	0	.75	.25
-.25	.25	0	0	1	0
-.25	.25	0	0	.25	.75
-.25	.25	0	0	.5	.5

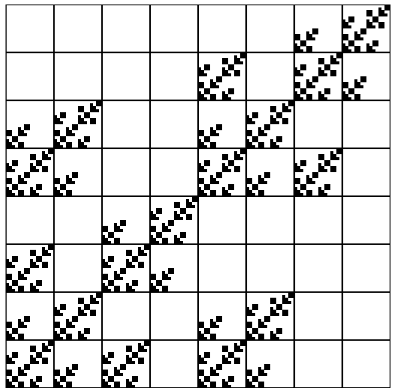
In the last four entries, each $r = -.25$ can be replaced by $r = .25$ if $\theta = \varphi = 90°$. Additionally, we may take $r = s = .25$ and $\theta = \varphi = -90°$, or $r = .25$, $s = -.25$, and $\theta = \varphi = 0$ if $(e, f) = (0, 1)$, $(.25, .75)$, $(.5, .5)$, and $(.75, .25)$.

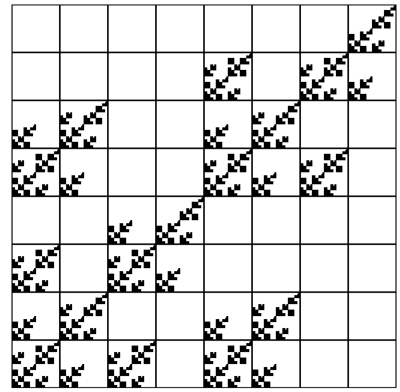
2.(a) This fractal consists of $N = 3$ pieces, each scaled by $r = 1/2$, so $d = \log(3)/\log(2)$.

(b) This fractal consists of 2 pieces scaled by 1/2 and 4 pieces scaled by 1/4. The Moran equation is $2 \cdot (1/2)^d + 4 \cdot (1/4)^d = 1$. Taking $x = (1/2)^d$, the Moran equation becomes $2x + 4x^2 = 1$. The positive solution is $x = (-1 + \sqrt{5})/4$ and so the dimension is $d = \log((-1 + \sqrt{5})/4)/\log(1/2)$.

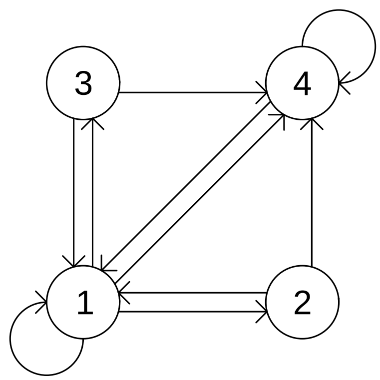
3. (a) For both (i) and (ii), the empty length 2 addresses are 22, 23, 24, 32, 33, and 34. In (i) the additional empty length 3 addresses are 122, 123, 124, 132, 133, 134, 422, 423, 424, 432, 433, and 434. All these contain forbidden pairs, so (i) is generated by forbidden pairs.

In (ii), the empty length 3 addresses are those of (i), and also 443. This does not contain a forbidden pair, so (ii) is not generated by forbidden pairs.

(i) 

(ii) 

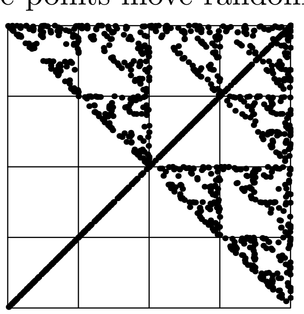
(b) The transition graph for (i) is

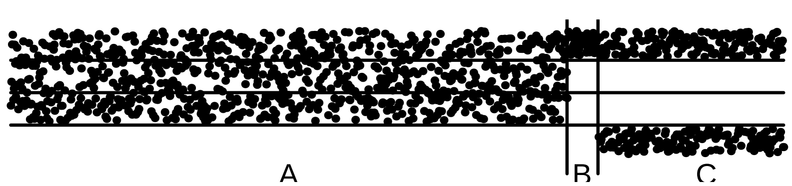


(c) In (i), 1 and 4 are romes, and there are transitions $1 \to 2$ and $1 \to 3$ from a rome to each non-rome, so (i) can be built by an IFS without memory. Because there are no loops among non-romes, only a finite collection of transformations are needed to generate this fractal by an IFS without memory.

r	s	θ	φ	e	f
.5	.5	0	0	0	0
.5	.5	0	0	.5	.25
.25	.25	0	0	.5	0
.25	.25	0	0	0	.5

4. The driven IFS consists of a gasket with corners 2, 3, and 4, and a line with corners 1 and 4. The driven IFS is clean, so the transition between these must be achieved through the common feature, the corner 4. The time series will have three regimes. In A, the points move randomly between bins 2, 3, and 4; in B the points stay in bin 4; and in C the points move randomly between bins 1 and 4.

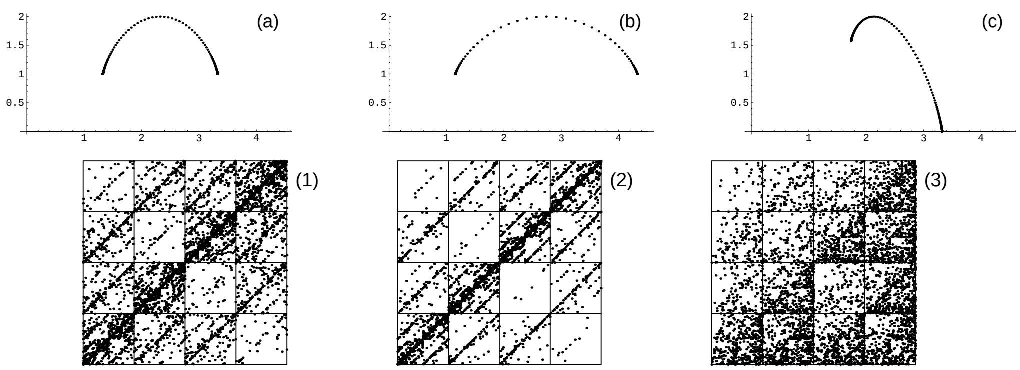




5. For (a) and (b), $f(\alpha_{\max}) = f(\alpha_{\min}) = 1$, so the minimum probability and maximum

probability occur on 1-dimensional sets, line segments, for example. Any two of the four transformations T_1, T_2, T_3, and T_4 generate line segments, so the $f(\alpha)$ curves of (a) and (b) can be the result of two of the T_i being applied with the same high probability, and the other two with the same low probability. In driven IFS (1) and (2), T_1 and T_4 are applied with about the same high probability, and T_2 and T_3 with about the same low probability. The $f(\alpha)$ curve (a) has the smaller α range, so the corresponding driven IFS has the smaller probability range. That is, (a) corresponds to (1). Similarly, (b) has a greater α range, coming from a driven IFS with larger range of probabilities. That is, (b) corresponds to (2), because squares 2 and 3 of (2) are much less filled than squares 2 and 3 of (1).

For (c), the maximum α, hence the minimum probability, occurs at a point, while the minimum α, hence the maximum probability, occurs on a gasket, or at least on a set with dimension that of the gasket. In driven IFS (3), the minimum probability occurs at corner 3, and the maximum on the 124 gasket. This supports the claim that (c) corresponds to (3).



6. First observe that the dimension formula for products gives

$$1 = d(A \times B) = d(A) + d(B) = \frac{\log(2)}{\log(3)} + \frac{\log(2)}{\log(1/r)}$$

Solving for $\log(1/r)$ we find

$$\log(1/r) = \frac{\log(2) \cdot \log(3)}{\log(3) - \log(2)}$$

Then

$$\frac{1}{r} = 10^{(\log(2)\cdot\log(3))/(\log(3)-\log(2))}$$

and so

$$r = 10^{-(\log(2)\cdot\log(3))/(\log(3)-\log(2))}$$

If you're interested, this is about 0.152882.