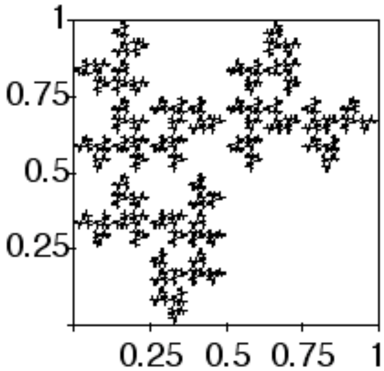
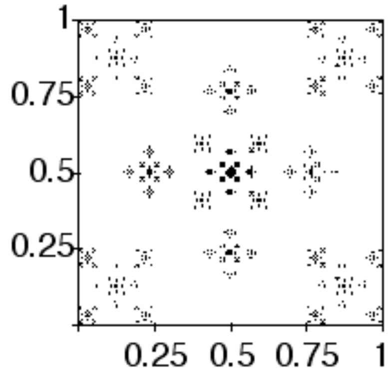
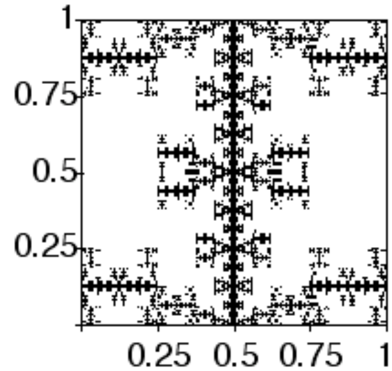
Practice Final 3 Answers

1. Here are the IFS rules for these fractals.







R	S	Theta	Phi	E	F
-0.5	0.5	0	0	0.5	0.0
0.5	0.5	90	90	0.5	0.5
0.5	-0.5	0	0	0.5	1.0

R	S	Theta	Phi	E	F
0.5	0.5	45	45	0.5	0.15
0.25	0.25	0	0	0.0	0.0
0.25	0.25	0	0	0.75	0.75
0.25	0.25	0	0	0.75	0.0
0.25	0.25	0	0	0.0	0.75

R	S	Theta	Phi	E	F
0.5	0.5	0	0	0.25	0.0
0.5	0.5	0	0	0.25	0.5
0.25	0.25	90	90	0.25	0.0
0.25	0.25	90	90	1.0	0.0
0.25	0.25	90	90	0.25	0.75
0.25	0.25	90	90	1.0	0.75

2. (a) The fractal consists of N = 3 pieces, each scaled by r = 1/2, so its dimension is log(3)/log(2).

(b) The fractal consists of 1 piece scaled by 1/2 and 4 pieces scaled by 1/4. The dimension is the solution, d, of the Moran equation

$$(1/2)^d + 4(1/4)^d = 1$$

Taking $x = (1/2)^d$, the Moran equation becomes the quadratic equation

$$x + 4x^2 = 1$$

The positive solution is $x = (-1 + \sqrt{17})/8$, so $d = \log((-1 + \sqrt{17})/8)/\log(1/2)$

(c) The fractal consists of 2 pieces scaled by 1/2 and 4 pieces scaled by 1/4. The dimension is the solution, d, of the Moran equation

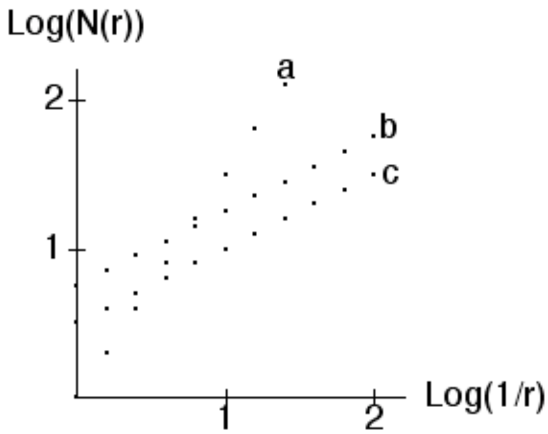
$$2(1/2)^d + 4(1/4)^d = 1$$

Taking $x = (1/2)^d$, the Moran equation becomes the quadratic equation

$$2x + 4x^2 = 1$$

The positive solution is $x = (-1 + \sqrt{5})/4$, so $d = \log((-1 + \sqrt{5})/4)/\log(1/2)$

3. If the points of the log-log plot fall along a straight line, the slope of that line is the box-counting dimension. Points of **a** lie along the line of steepest slope, so correspond to highest dimension. Points of **b** and **c** lie on parallel lines, so correspond to fractals of the same dimension, lower than that of **a**.



4. (a) The Cantor middle-thirds set has dimension log(2)/log(3), so by the intersection formula, the typical intersection of two Cantor sets in the plane is

log(2)/log(3) + log(2)/log(3) - 2, about -.74.

The typical intersection of two Cantor middle thirds sets in 3-dimensional space has dimension

log(2)/log(3) + log(2)/log(3) - 3, about -1.74.

In both cases, typically the Cantor sets miss one another.

(b) The Sierpinski gasket has dimension log(3)/log(2), so by the intersection formula, the typical intersection of two gaskets in the plane has dimension

log(3)/log(2) + log(3)/log(2) - 2, about 1.17.

The typical intersection of two gaskets in 3-dimensional space has dimension

log(3)/log(2) + log(3)/log(2) - 3, about .17.

The typical intersection of two gaskets in 4-dimensional space has dimension

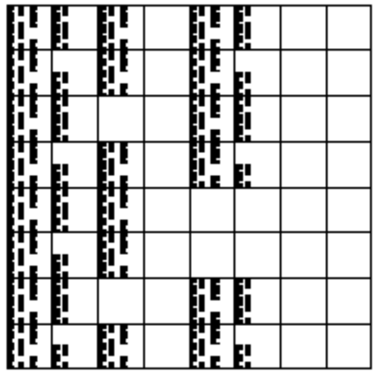
log(3)/log(2) + log(3)/log(2) - 4, about -.83.

(c) By the intersection formula, the typical intersection of two copies of A in E-dimensional space has dimension

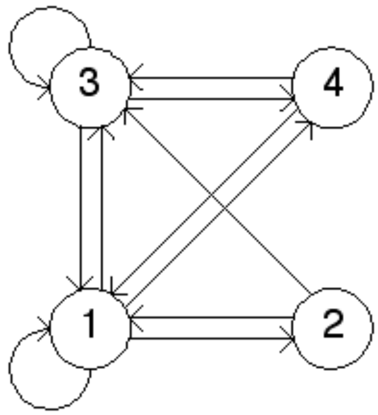
d + d - E

In order for the typical intersection to be empty this dimension must be negative, so E is the smallest integer greater than 2d.

5. (a) The forbidden pairs are 22, 23, 24, 42, and 44. The forbidden triples (not corresponding to subsquares of the forbidden pairs) are 122, 123, 124, 142, 144, 322, 323, 324, 342, and 344. Every forbidden triple contains a forbidden pair, so (at least to the level of triples) this IFS can be generated by forbidden pairs.



(b) The forbidden pairs exclude these transitions 2 → 2, 3 → 2, 4 → 2, 2 → 4, and 4 → 4. Consequently, the transition graph is



(c) Addresses 1 and 3 are romes, so are copies of the entire fractal scaled by 1/2. Addresses 21, 41, and 43 are the images of romes, hence are copies of the entire fractal scaled by 1/4. These five pieces constitute the whole fractal, so the dimension is given by the Moran equation

$$2(1/2)^d + 3(1/4)^d = 1$$

Taking $x = (1/2)^d$, the Moran equation becomes the quadratic equation

$$2x + 3x^2 = 1$$

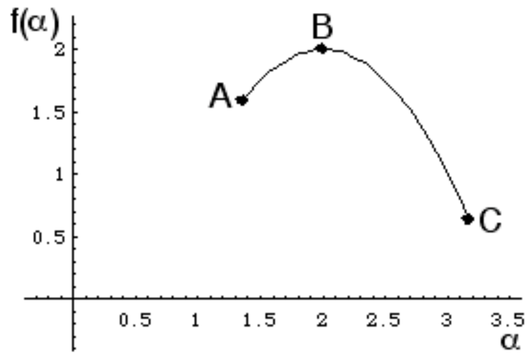
The positive solution is x = 1/3 and so

$$d = \log(1/3)/\log(1/2) = -\log(3)/(-\log(2)) = \log(3)/\log(2).$$

6. (a) The minimum value of α occurs on a gasket, of dimension log(3)/log(2). Consequently, the point on the f(α) curve above the minimum value of α is the dimension of this gasket. This is the point labeled A.

The maximum point on the f(α) curve gives 2, the dimension of the attractor. This is the point labeled B.

The maximum value of α occurs on a Cantor middle-thirds set, of dimension log(2)/log(3). Consequently, the point on the f(α) curve above the maximum value of α is the dimension of this Cantor set. This is the point labeled C.

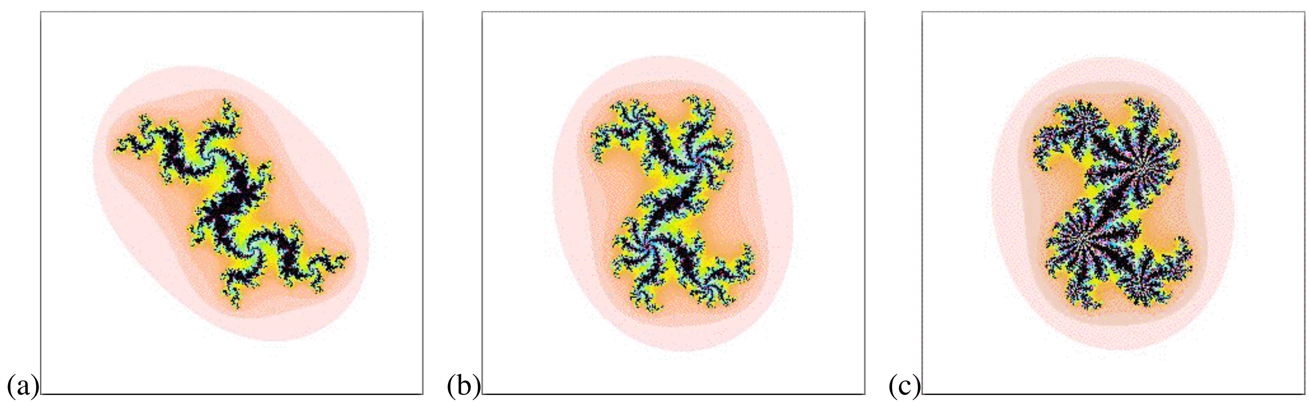


(b) Generating the gasket with an IFS requires 3 transformations; generating a Cantor set requires 2. Because the gasket and the Cantor set have different scalings, these five transformations must be distinct. (Note the simplest way to achieve an attractor of dimension 2 is to add a fourth transformation with scaling 1/2. This brings the total number of transformations up to 6.)

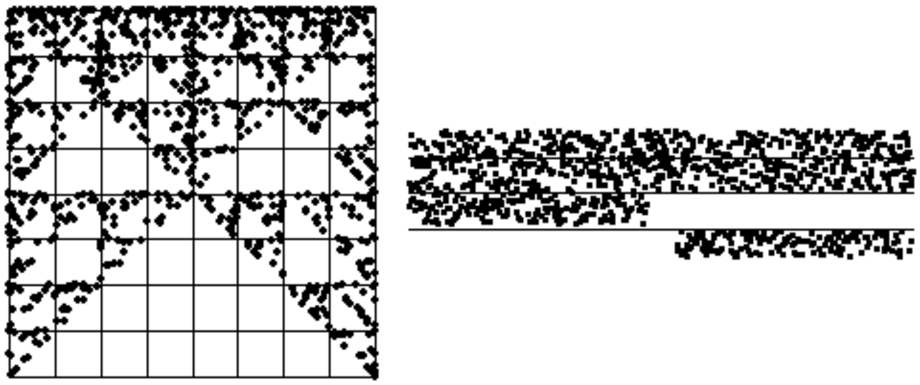
7. Julia set (a) has spokes with 3 branches and spokes with 5 branches, so could be the Julia set for a c in a 15-cycle, perhaps a 5-cycle disc attached to a 3-cycle disc.

In Julia set (b) we see spokes with 6 branches. Because 6 does not divide 15, this Julia set cannot correspond to a 15-cycle.

In Julia set (c) all the spokes appear to have 15 branches. This Julia set could correspond to a 15-cycle disc attached to the main cardioid.

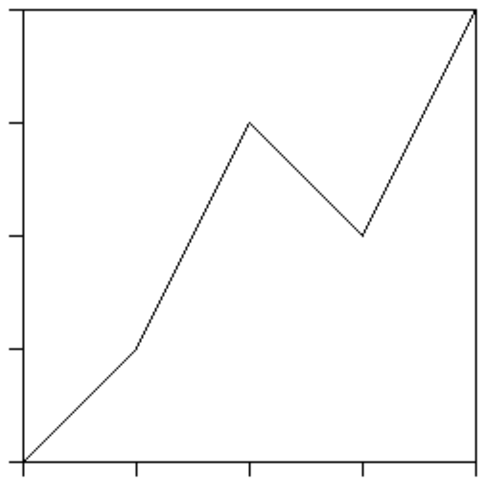


8. (a) The driven IFS consists of a gasket with corners 2, 3, and 4, and a gasket with corners 1, 3, and 4, and nothing else. This driven IFS can be produced by a time series divided into three regimes: one with points randomly scattered through bins 2, 3, and 4 (producing points on the right gasket), one with points randomly scattered in bins 3 and 4 (common to both gaskets, so going from the right gasket to the line produces points only in the right gasket, and going from the line to the left gasket produces points only in the left gasket), and the last with points randomly scattered in bins 1, 3, and 4 (producing points on the left gasket).



(b) This driven IFS is not produced by forbidden pairs alone. The forbidden pairs are 12 and 21, but the forbidden triples include 132, 142, 231, and 241. These are not forbidden as consequences of the forbidden pairs.

9. Observe $dt_1 = dt_2 = dt_3 = dt_4 = 1/4$, $dY_1 = 1/4$, $dY_2 = 1/2$, $dY_3 = -1/4$, and $dY_4 = 1/2$.



To find the trading time generator, first solve

$$|dY_1|^D + |dY_2|^D + |dY_3|^D + |dY_4|^D$$

That is, $2(1/2)^D + 2(1/4)^D = 1$. Taking $x = (1/2)^D$, this becomes the quadratic equation $2x + 2x^2 = 1$. The positive solution is $D = (-1 + \sqrt{3})/2$ and so

$$D = \log((-1 + \sqrt{3})/2)/\log(1/2)$$

The Trading Time generators are $dT_i = |dY_i|^D$

$dT_1 = (1/4)^{\log((-1 + \sqrt{3})/2)/\log(1/2)} = ((1/2)^2)^{\log((-1 + \sqrt{3})/2)/\log(1/2)} = ((-1 + \sqrt{3})/2)^2$

$dT_2 = (1/2)^{\log((-1 + \sqrt{3})/2)/\log(1/2)} = (-1 + \sqrt{3})/2$

$dT_3 = (1/4)^{\log((-1 + \sqrt{3})/2)/\log(1/2)} = ((-1 + \sqrt{3})/2)^2$

$dT_4 = (1/2)^{\log((-1 + \sqrt{3})/2)/\log(1/2)} = (-1 + \sqrt{3})/2$