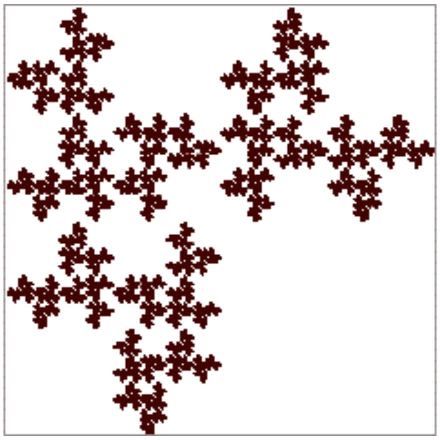
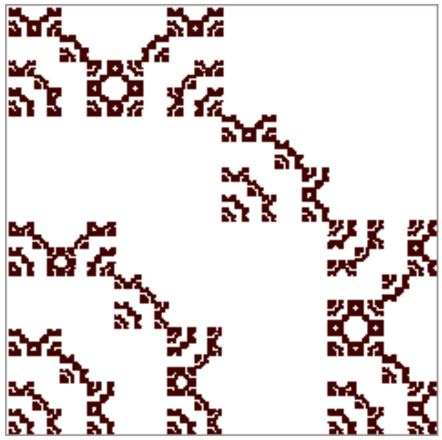
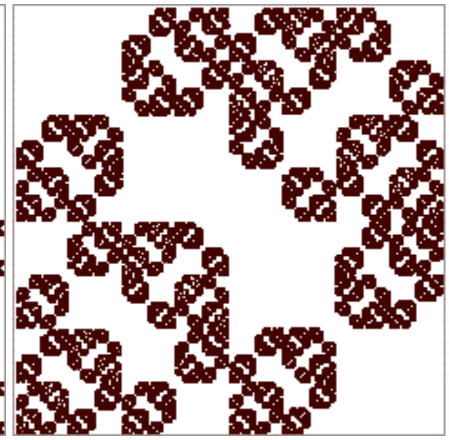
Practice Final 4 Answers

1. Here are IFS rules for these fractals.







R	S	θ	φ	E	F
-0.5	0.5	0	0	0.5	0.0
0.5	0.5	90	90	0.5	0.5
0.5	-0.5	0	0	0.5	1.0

R	S	θ	φ	E	F
0.5	0.5	0	0	0.0	0.0
0.25	0.25	0	0	0.75	0.0
0.25	-0.25	0	0	0.75	0.5
0.25	0.25	0	0	0.0	0.75
-0.25	0.25	0	0	0.5	0.75
0.25	0.25	0	0	0.5	0.5

R	S	θ	φ	E	F
0.5	0.5	0	0	0.0	0.0
-0.5	0.5	0	0	1.0	0.5
0.25	0.25	0	0	0.0	0.5
-0.25	-0.25	0	0	0.5	1.0
0.25	0.25	0	0	0.5	0.0
-0.25	-0.25	0	0	1.0	0.5

2. (a) The fractal consists of N = 3 pieces, each scaled by r = 1/2, so its dimension is log(3)/log(2).

(b) The fractal consists of 1 piece scaled by 1/2 and 5 pieces scaled by 1/4. The dimension is the solution, d, of the Moran equation

$$(1/2)^d + 5(1/4)^d = 1$$

Taking $x = (1/2)^d$, the Moran equation becomes the quadratic equation

$$x + 5x^2 = 1$$

The positive solution is $x = (-1 + \sqrt{21})/10$, so $d = \log((-1 + \sqrt{21})/10)/\log(1/2)$

(c) The fractal consists of 2 pieces scaled by 1/2 and 4 pieces scaled by 1/4. The dimension is the solution, d, of the Moran equation

$$2(1/2)^d + 4(1/4)^d = 1$$

Taking $x = (1/2)^d$, the Moran equation becomes the quadratic equation

$$2x + 4x^2 = 1$$

The positive solution is $x = (-1 + \sqrt{5})/4$, so $d = \log((-1 + \sqrt{5})/4)/\log(1/2)$

3. Because $\epsilon = 1/2^n$, we have $\epsilon \to 0$ when $n \to \infty$. Then the box-counting dimension is

$$d = \lim_{n \to \infty} \mathrm{Log}(2^n + 3^n + n)/\mathrm{Log}(1/(1/2^n))$$

$$= \lim_{n \to \infty} \mathrm{Log}((3^n)((2/3)^n + 1 + n/3^n)/\mathrm{Log}(2^n)$$

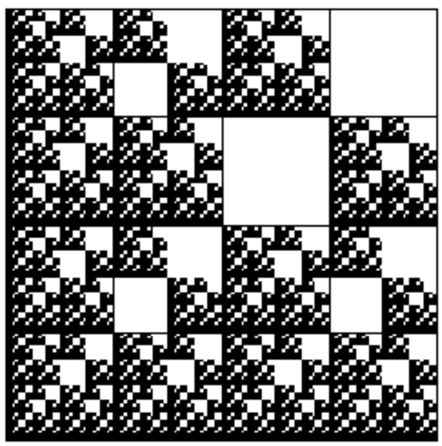
$$= \lim_{n \to \infty} (\mathrm{Log}(3^n) + \mathrm{Log}((2/3)^n + 1 + n/3^n))/\mathrm{Log}(2^n)$$

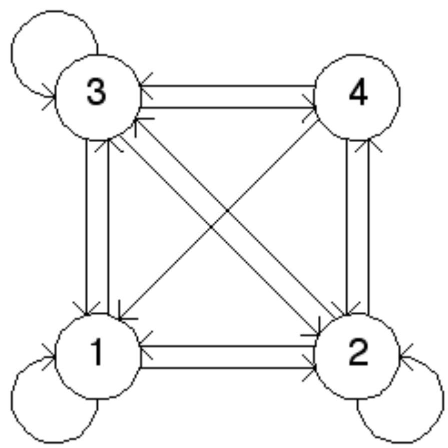
$$= \mathrm{Log}(3)/\mathrm{Log}(2) + \lim_{n \to \infty} \mathrm{Log}((2/3)^n + 1 + n/3^n)/\mathrm{Log}(2^n)$$

$$= \mathrm{Log}(3)/\mathrm{Log}(2)$$

because as $n \to \infty$, $(2/3)^n \to 0$ and $n/3^n \to 0$.

4. (a) The addresses 44 and 41 are empty, so the transition graph has all arrows except $4 \to 4$ and $4 \to 1$. The transition graph is shown on the right.





(b) From the transition graph, we see states 1, 2, and 3 are romes, there are no loops through non-romes (in this case, just state 4), and transitions from romes to all non-romes (in this case, $2 \to 4$ and $3 \to 4$). So this fractal can be generated by an IFS with a finite number of transformations.

R	S	θ	φ	E	F
0.5	0.5	0	0	0.0	0.0
0.5	0.5	0	0	0.5	0.0
0.5	0.5	0	0	0.0	0.5
0.25	0.25	0	0	0.5	0.75
0.25	0.25	0	0	0.75	0.5

where the last two transformations are the compositions T_4T_2 and T_4T_3.

(c) This fractal consists of 3 copies scaled by 1/2 and 2 copies scaled by 1/4, so the dimension, d, is the solution of the Moran equation

$$3(1/2)^d + 2(1/4)^d = 1$$

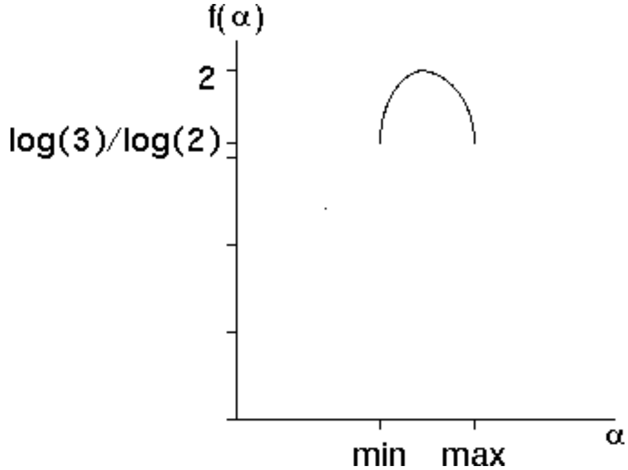
Taking $x = (1/2)^d$, the Moran equation becomes the quadratic equation

$$3x + 2x^2 = 1$$

The positive solution is $x = (-3 + \sqrt{17})/4$, so $d = \log((-3 + \sqrt{17})/4)/\log(1/2)$

5. (a) Both the minimum and maximum values of α occur on a Sierpinski gasket, so f(min(α)) = f(max(α)) =

log(3)/log(2), and the maximum value of the f(α) curve is 2. Here is a sketch.



(b) This multifractal cannot be generated by the four standard transformations, because T_1, T_2, and T_3 are applied with maximum probability, and T_2, T_3, and T_4 are applied with minimum probability.

6. (a) Measuring we see $dT_1 = dT_2 = dT_3 = dT_4 = 1/4$, and $dY_1 = dY_4 = 1/2$, and $|dY_2| = dY_3 = 1/4$. This generator is a multifractal because

$$\mathrm{Log}(dY_1)/\mathrm{Log}(dt_1) = \mathrm{Log}(1/4)/\mathrm{Log}(1/4) = 1$$

and

$$\mathrm{Log}(|dY_2|)/\mathrm{Log}(dt_1) = \mathrm{Log}(1/2)/\mathrm{Log}(1/4) = 1/2$$

(b) To find the trading time generators, first solve

$$2(1/2)^h + 2(1/4)^h = 1$$

With $x = (1/2)^h$ this becomes the quadratic equation

$$2x + 2x^2 = 1$$

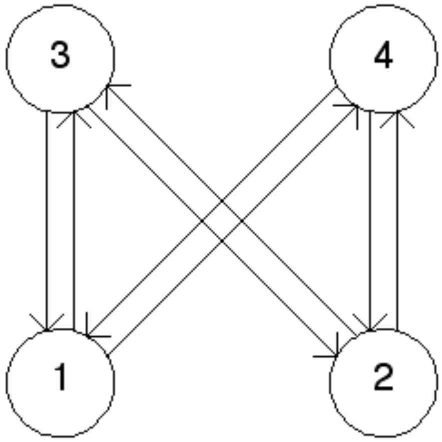
The positive solution is $x = (-1 + \sqrt{3})/2$, giving $h = \mathrm{Log}((-1 + \sqrt{3})/2)/\mathrm{Log}(1/2)$. The trading time generators are

$$dT_1 = dT_4 = (1/2)^h = (-1 + \sqrt{3})/2,$$

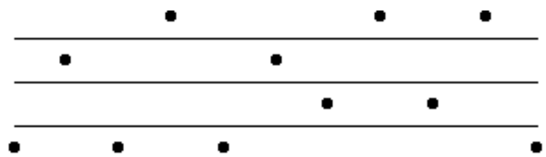
and

$$dT_2 = dT_3 = (1/4)^h = ((-1 + \sqrt{3})/2)^2$$

7. (a) The empty addresses are 11, 12, 21, 22, 33, 34, 43, and 44, so the transition graph is



(b) A time series must employ all the allowed transitions. A beginning is



8. Reading left to right, we find live cells are produced by these neighborhood configurations

(DDL), (DLD), (LLD)

and dead cells are produced by these neighborhood configurations

(DDD), (LDL), (LDD), (DLL)

So two CA could generate this pattern,
live cells given by (DDL), (DLD), (LLD)
and
live cells given by (DDL), (DLD), (LLD), (LLL).

9. The center of each 4-cycle component is a solution of $0 = F_c^4(0)$. Now

$$\begin{aligned} F_c^4(0) &= F_c^3(c) \\ &= F_c^2(c^2 + c) \\ &= F_c((c^2 + c)^2 + c) \\ &= F_c(c^4 + 2c^3 + c^2 + c) \\ &= (c^4 + 2c^3 + c^2 + c)^2 + c \\ &= c^8 + 4c^7 + 6c^6 + 6c^5 + 5c^4 + 2c^3 + c^2 + c \end{aligned}$$

This is an 8th degree polynomial, and has 8 roots. We know there are three 4-cycle discs, one attached to the 2-cycle disc, and two attached to the main cardioid, one in each of the upper and lower principal series. A 2-cycle also is a 4-cycle, so one of the remaining five is the 2-cycle disc. A fixed point is an 4-cycle, so the main cardioid also is a 4-cycle component. Consequently, three of the eight 4-cycle components must be cardioids of midgets.

Alternately, by Lavaurs' method, the 4-cycle components are determined by pairs of fractions of the form $k/(2^4 - 1) = k/15$. There are seven of these pairs, but one pair, 5/15 = 1/3 and 10/15 = 2/3, correspond to the 2-cycle disc. This leaves six 4-cycle components. As observed earlier in this answer, three of these are 4-cycle discs, leaving three to be cardioids of 4-cycle midgets.