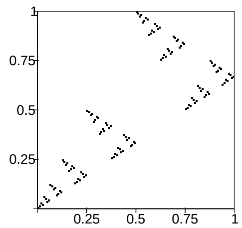
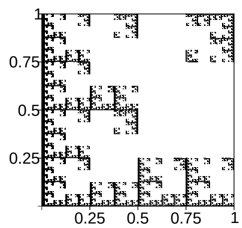
Final Exam Solutions

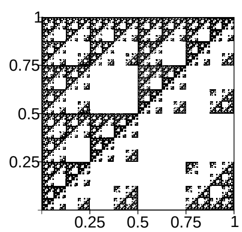
1. Fractals and their IFS generators.



r	s	θ	φ	e	f
.5	.5	0	0	0	0
.5	$-.5$	0	0	.5	1



r	s	θ	φ	e	f
.5	.5	0	0	0	0
.5	.5	0	0	0	.5
.25	.25	0	0	.5	0
.25	.25	0	0	.75	0
$-.25$	$-.25$	0	0	1	1



r	s	θ	φ	e	f
.5	.5	0	0	0	0
.5	.5	0	0	.5	.5
.25	.25	0	0	0	.5
.25	.25	0	0	0	.75
.25	.25	0	0	.25	.75
$-.25$	$-.25$	0	0	1	.25

2. (a) This fractal consists of $N = 2$ pieces, each scaled by a factor of $r = 1/2$, so the dimension is $d = \log(2)/\log(2) = 1$.

(b) This fractal consists of 5 pieces, with scaling factors $r_1 = r_2 = 0.5$, $r_3 = r_4 = r_5 = 0.25$. The Moran equation is

$$2 \cdot .5^d + 3 \cdot .25^d = 1$$

Taking $x = .5^d$ this becomes the quadratic equation $2x + 3x^2 = 1$. This gives $x = 1/3$ and so $d = \log(1/3)/\log(1/2) = \log(3)/\log(2)$.

(c) This fractal consists of 6 pieces, with scaling fractors $r_1 = r_2 = 0.5$, $r_3 = r_4 = r_5 = r_6 = 0.25$. The Moran equation is

$$2 \cdot .5^d + 4 \cdot .25^d = 1$$

Taking $x = .5^d$ this becomes the quadratic equation $2x + 4x^2 = 1$. This gives $x = (-1 + \sqrt{5})/4$ and so $d = \log((-1 + \sqrt{5})/4)/\log(1/2)$.

3. Suppose for each $n > 0$, the minimum number of boxes of side length $\epsilon = 1/2^n$ needed to cover a fractal A is

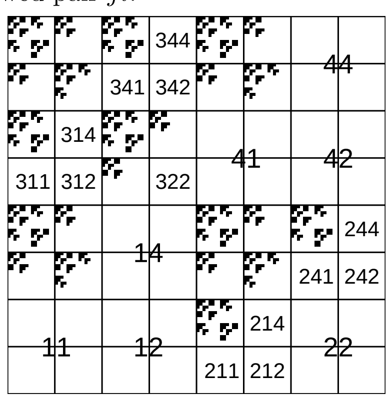
$$N(\epsilon) = 2^n + 3^n + 4^n$$

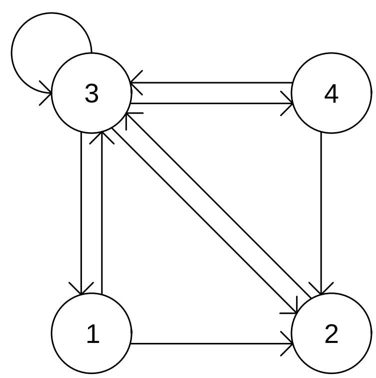
The box-counting dimension is

$$\begin{aligned}
d &= \lim_{\epsilon\to 0} \frac{\log(N(\epsilon))}{\log(1/\epsilon)} \\
&= \lim_{n\to\infty} \frac{\log(2^n + 3^n + 4^n)}{\log(2^n)} \\
&= \lim_{n\to\infty} \frac{\log(4^n((1/2)^n + (3/4)^n + 1))}{\log(2^n)} \\
&= \lim_{n\to\infty} \frac{\log(4^n) + \log((1/2)^n + (3/4)^n + 1))}{\log(2^n)} \\
&= \lim_{n\to\infty} \frac{n\log(4)}{n\log(2)} + \lim_{n\to\infty} \frac{\log((1/2)^n + (3/4)^n + 1)}{n\log(2)} \\
&= \frac{\log(4)}{\log(2)} \\
&= 2
\end{aligned}$$

because $\lim_{n\to\infty} n/\log(n) = 0$.

4. (a) On the left we see the IFS with memory. The forbidden pairs are 11, 12, 14, 22, 41, 42, and 44; the forbidden triples are 211, 212, 214, 241, 242, 244, 311, 312, 314, 322, 341, 342, and 344. Every forbidden pair contains a forbidden triple, so this IFS with memory can be generated by forbidden pairs. The transition graph is shown on the right: each arrow $i \to j$ corresponds to an allowed pair ji.





(b) Corner 3 is a rome, there is a path from the rome 3 to each non-rome, and there are no loops among non-romes, so this fractal can be generated by an IFS without memory. The transition graph gives the transformations and their compositions of this IFS:

$$T_3, T_1 \circ T_3, T_2 \circ T_3, T_4 \circ T_3, T_2 \circ T_1 \circ T_3, T_2 \circ T_4 \circ T_3$$

and so the IFS table is

r	s	θ	φ	e	f
.5	.5	0	0	0	.5
.25	.25	0	0	0	.25
.25	.25	0	0	.5	.25
.25	.25	0	0	.5	.75
.125	.125	0	0	.5	.125
.125	.125	0	0	.75	.375

(c) To find the dimension of this fractal, because one copy is scaled by 1/2, three copies are scaled by 1/4, and two by 1/8 the Moran equation becomes

$$(1/2)^d + 3 \cdot (1/4)^d + 2 \cdot (1/8)^d = 1$$

Taking $x = (1/2)^d$, the Moran equation becomes $x + 3x^2 + 2x^3 = 1$.

5. For this IFS

r	s	θ	φ	e	f	prob
.25	.25	0	0	0	0	0.05
.25	.25	0	0	.25	0	0.05
.25	.25	0	0	.25	.25	0.1
.25	.25	0	0	0	.25	0.1
.25	.25	0	0	.5	.5	0.1
.25	.25	0	0	.75	.75	0.2
.25	.25	0	0	.75	0	0.2
.25	.25	0	0	0	.75	0.2

The minimum and maximum values of α are the minimum and maximum values of $\log(p_i)/\log(r_i)$, so $\alpha_{\min} = \log(.2)/\log(.25)$ and $\alpha_{\max} = \log(.05)/\log(.25)$. The set on which the minimum α occurs consists of $N = 3$ pieces, each scaled by $r = .25$, so

$$f(\alpha_{\min}) = \log(3)/\log(1/.25) = \log(3)/\log(4).$$

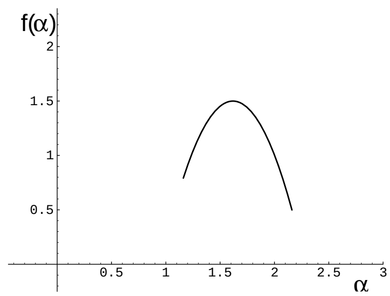
The set on which the maximum α occurs consists of $N = 2$ pieces, each scaled by $r = .25$, so

$$f(\alpha_{\max}) = \log(2)/\log(1/.25) = \log(2)/\log(4) = 1/2$$

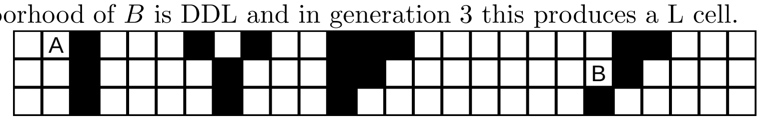
The maximum value of the $f(\alpha)$ curve is the dimension of the IFS attractor. This consists of $N = 8$ pieces scaled by $r = .25$, so

$$d = \log(8)/\log(1/.25) = \log(8)/\log(4) = 3/2$$

Combining this information, we have

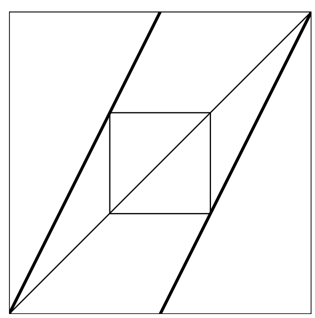


6. There is no such $N = 3$ CA. To see this, compare cells A and B. The neighborhood of A is DDL and in generation 2 this produces a D cell. The neighborhood of B is DDL and in generation 3 this produces a L cell.



7. By the multiplier rule, b must divide 18. The possibilities are
$18 = 9 \cdot b$ so $b = 2$,
$18 = 6 \cdot b$ so $b = 3$,
$18 = 3 \cdot b$ so $b = 6$, and
$18 = 2 \cdot b$ so $b = 9$.

8. (a) The graphical iteration plot is shown here.



(b) To find the coordinates of the 2-cycle, we solve $f(f(x)) = x$. For $x < 1/2$, we see $f(x) > 1/2$ and consequently $f(f(x)) = x$ becomes

$$2(2x) - 1 = x$$

This gives $x = 1/3$ and so the other point of the cycle is $f(x) = 2x = 2/3$.

9. Suppose C is the Cantor middle thirds set along the x-axis, I is the unit interval along the y-axis, and A is the part of $C \times I$ lying below the curve $y = x$ for (a), and $y = x^2$ for (b). In the figures, $C \times I$ is shown in gray, A in black, and say B is the region outlined by the box. Certainly

$$B \subset A \subset C \times I$$

and so

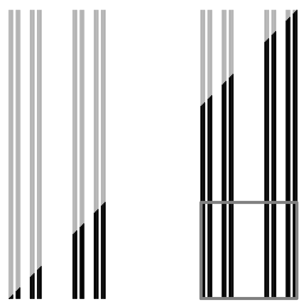
$$\dim(B) \leq \dim(A) \leq \dim(C \times I)$$

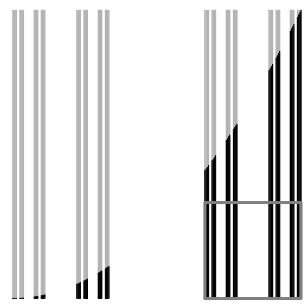
On the other hand, B is itself a product of a Cantor middle-thirds set and an interval, so

$$\dim(B) = \dim(C \times I) = \dim(C) + \dim(I)$$

applying the product rule for dimensions. Then we see

$$\dim(A) = \dim(C \times I) = \dim(C) + \dim(I) = \log(2)/\log(3) + 1$$


(a)

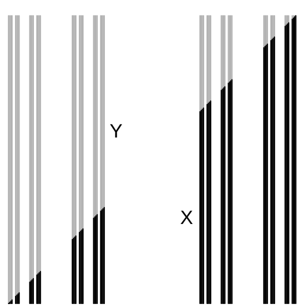

(b)

Here are two other approaches to part (a). In (c) we see the product $C \times I$ divided into two subsets X and Y, with Y the rotation of X by 180°, and consequently, X and Y have the same dimension. Now apply the union rule,

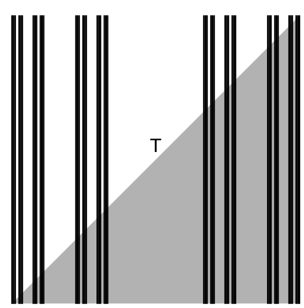
$$\dim(C \times I) = \max\{\dim(X), \dim(Y)\} = \dim(X)$$

In (d) we see the product $C \times I$ and a right isosceles triangle T. The set X is the intersection of T and $C \times I$. Noting that both the ambient space and the triangle T have dimension 2, we apply the intersection rule,

$$\dim(X) = \dim((C \times I) \cap T) = \dim(C \times I) + \dim(T) - 2 = \dim(C \times I)$$



(c)



(d)