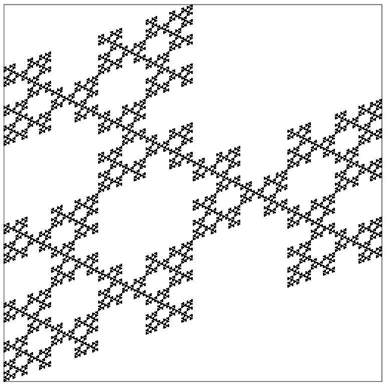
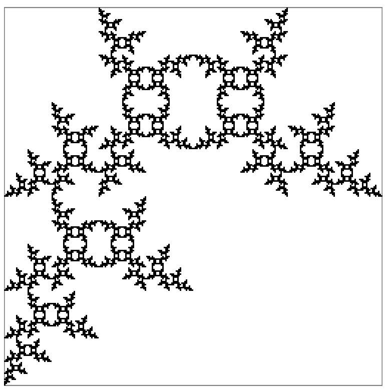
Math 190 Midterm Solutions

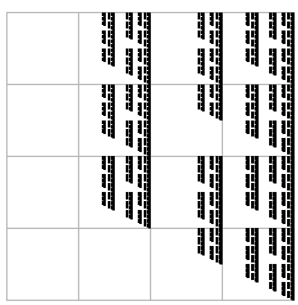
1 (a) 

r	s	θ	φ	e	f
0.5	0.5	0	0	0	0
-0.5	-0.5	0	0	0.5	1.0
-0.5	-0.5	0	0	1.0	0.75

(b) 

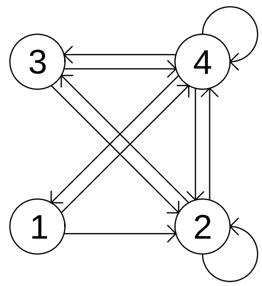
r	s	θ	φ	e	f
0.5	0.5	0	0	0	0
0.5	0.5	90	90	1	0.5
-0.5	0.5	-90	-90	0	0.5

2. (a) From the IFS we see that the occupied length-2 addresses are 14, 21, 22, 23, 24, 32, 34, 41, 42, 43, and 44.



Consequently, the transition graph is

(b) From inspecting the transition graph we see that 2 and 4 are romes, there are paths to each non-rome from some rome, $4 \to 1$, $2 \to 3$, and $4to3$, and there are no loops among non-romes. In fact, there is no path through only non-romes. Consequently, this IFS with memory can be generated by an IFS without memory.



(c) We see the fractal consists of two pieces scaled by $r = 1/2$ and three copies scaled by $r = 1/4$. We can deduce the palcement of the pieces by inspecting the fractal. The IFS table is

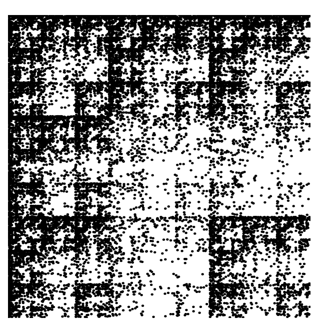
r	s	θ	φ	e	f
0.5	0.5	0	0	.5	0
0.5	0.5	0	0	0.5	0.5
0.25	0.25	0	0	0.25	0.25
0.25	0.25	0	0	0.25	0.5
0.25	0.25	0	0	0.25	0.75

3. The dimension of the Cantor middle-halves set C is $\log(2)/\log(4) = 1/2$. Then by the product rule

(a) $d(C \times C) = 1/2 + 1/2 = 1$.
(b) $d(C \times C \times C) = 1/2 + 1/2 + 1/2 = 3/2$.
(c) $d(C \times C \times C \times C) = 1/2 + 1/2 + 1/2 + 1/2 = 2$.
(d) $d(C \times C \times C \times C \times C) = 1/2 + 1/2 + 1/2 + 1/2 + 1/2 = 5/2$.
(e) $d(C \times C \times C \times C \times C \times C) = 1/2 + 1/2 + 1/2 + 1/2 + 1/2 + 1/2 = 3$.

Consequently, (a) and (b) have dimension less than 2 and (d) and (e) have dimension greater than 2.

4. We see the driven IFS image consists of 9 pieces each scaled by $r = 1/2$. Three have about the same low probability, one has an intermediate probability, and five have about the same high probability.



(a) Because all the scaling factors are the same, the minimum value of α corresponds to the maximum probability and the maximum value of α corresponds to the minimum probability.

The maximum probability occurs on $N = 5$ pieces scaled by $r = 1/3$, so

$$f(\min\alpha) = \frac{\log(5)}{\log(3)}$$

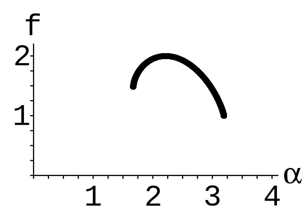
The minimum probability occurs on $N = 3$ pieces scaled by $r = 1/3$, so

$$f(\max\alpha) = \frac{\log(3)}{\log(3)} = 1$$

(b) The maximum value of the $f(\alpha)$ curve is the dimension of the IFS, which consists of $N = 9$ pieces scaled by $r = 1/3$. Consequently, the maximum value of the $f(\alpha)$ curve is

$$\frac{\log(9)}{\log(3)} = \frac{\log(3^2)}{\log(3)} = \frac{2\log(3)}{\log(3)} = 2$$

(c) Assembling this information we can sketch the $f(\alpha)$ curve.



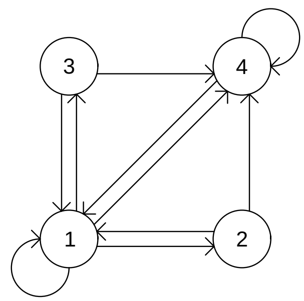
5 (a) From the transition graph we see that 1 and 4 are romes, so these give one copy each of the whole fractal scaled by $r = 1/2$. Each non-rome has a single arrow going to it: $1 \to 2$ and $1 \to 3$. Each of these arrows corresponds to a copy of the fractal scaled by $r = 1/4$. We have accounted for all the paths to non-romes, so these are all the pieces of the fractal. The Moran equation is

$$2 \cdot (1/2)^d + 2 \cdot (1/4)^d = 1$$

Taking $x = (1/2)^d$, the Moran equation becomes the quadratic equation $2x^2 + 2x - 1 = 0$. The positive solution for x is $(-1 + \sqrt{3})/2$ and so the dimension of the fractal generated by this transition graph is

$$d = \frac{\log((-1 + \sqrt{3})/2)}{\log(1/2)}$$

(b) From the transition graph we see that 1, 2, and 3 are romes, so give three copies of the fractal scaled by $r = 1/2$. The arrow $1 \to 4$ gives a copy scaled by $1/4$, but the each time we apply the arrow $4 \to 4$ we produce another copy



scaled by an additional factor of 1/2. So this fractal consists of three copies scaled by 1/2, and one copy scaled by each of 1/4, 1/8, 1/16, and so on, forever. Taking $x = (1/2)^d$, the Moran equation becomes

$$3x + x^2 + x^3 + x^4 + x^5 + \cdots = 1$$

That is,

$$3x + x^2(1 + x + x^2 + x^3 + \cdots) = 1$$

So long as $|x| < 1$, the bracketed terms sum to $1/(1-x)$ and the Moran equation can be rewritten

$$3x + \frac{x^2}{1-x} = 1$$

This simplifies to the quadratic equation $2x^2 - 4x + 1 = 0$. This has solutions

$$x = \frac{2+\sqrt{2}}{2} \text{ and } x = \frac{2-\sqrt{2}}{2}$$

Both are positive, but recall that in order to sum the geometriuc series we must have $|x| < 1$. Consequently, the appropriate solution is $x = (2-\sqrt{2})/2$ and so the dimension of the fractal generated by this transition graph is

$$d = \frac{\log((2-\sqrt{2})/2)}{\log(1/2)}$$