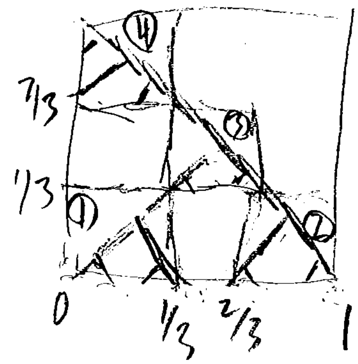
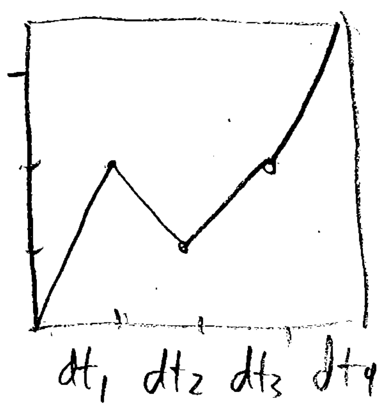
8 # 1 (a)



	r	s	θ	φ	e	f
1	−1/3	1/3	0	0	1/3	0
2	1/3	1/3	0	0	2/3	0
3	1/3	1/3	0	0	1/3	1/3
4	1/3	1/3	0	0	0	2/3

4 # 6



$dt_1 = dt_2 = dt_3 = dt_4 = \frac{1}{4}$

$dY_1 = 1/2 \quad dY_2 = -\frac{1}{4} \quad dY_3 = \frac{1}{4}$

$dY_4 = 1/2$

To test unifractal or multifractal, compute Holder exponents:

$$H_1 = \frac{\log dY_1}{\log dt_1} = \frac{\log \frac{1}{2}}{\log 1/4} = \frac{\log \frac{1}{2}}{\log((\frac{1}{2})^2)} = \frac{\log \frac{1}{2}}{2 \cdot \log \frac{1}{2}} = \frac{1}{2}$$

$$H_2 = \frac{\log |dY_2|}{\log dt_2} = \frac{\log \frac{1}{4}}{\log \frac{1}{4}} = 1$$

different, so the generator is multifractal

Trading time generators:

$$(dY_1)^D + |dY_2|^D + (dY_3)^D + (dY_4)^D = 1$$

$$\frac{1}{2}^D + \frac{1}{4}^D + \frac{1}{4}^D + \frac{1}{2}^D = 1$$

$$2\left(\frac{1}{2}\right)^D + 2\left(\frac{1}{4}\right)^D = 1$$

Take $x = \left(\frac{1}{2}\right)^D$

Then $x^2 = \left(\left(\frac{1}{2}\right)^D\right)^2$

$= \left(\left(\frac{1}{2}\right)^2\right)^D = \left(\frac{1}{4}\right)^D$

$$2x + 2x^2 = 1$$

$$2x^2 + 2x - 1 = 0 \qquad a = 2,\ b = 2,\ c = -1$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} = \frac{-2 \pm \sqrt{12}}{4}$$

$$= \frac{-2 \pm \sqrt{4}\sqrt{3}}{4} = \frac{-2 \pm 2\sqrt{3}}{4} = \frac{-1 \pm \sqrt{3}}{2}$$

Take $x = \frac{-1 + \sqrt{3}}{2}$

Then $\frac{-1+\sqrt{3}}{2} = \left(\frac{1}{2}\right)^D$

$$\log\left(\frac{-1+\sqrt{3}}{2}\right) = \log\left(\left(\frac{1}{2}\right)^D\right)$$

$$\log\left(\frac{-1+\sqrt{3}}{2}\right) = D \log\left(\frac{1}{2}\right)$$

$$D = \frac{\log\left(\frac{-1+\sqrt{3}}{2}\right)}{\log(1/2)}$$

$$dT_1 = dr_1^{\,D} = \left(\frac{1}{2}\right)^D$$

$$dT_2 = |dr_2|^D = \left(\frac{1}{4}\right)^D$$

$$dT_3 = dr_3^{\,D} = \left(\frac{1}{4}\right)^D$$

$$dT_4 = dr_4^{\,D} = \left(\frac{1}{2}\right)^D$$

$$dT_1 = \frac{1}{2}^{\log\left(\frac{-1+\sqrt{3}}{2}\right)/\log(1/2)}$$

To simplify,

$$\log dT_1 = \log\left(\frac{1}{2}^{\log\left(\frac{-1+\sqrt{3}}{2}\right)/\log(1/2)}\right)$$

$$= \frac{\log\left(\frac{-1+\sqrt{3}}{2}\right)}{\cancel{\log\left(\frac{1}{2}\right)}} \cdot \cancel{\log\left(\frac{1}{2}\right)}$$

$$\log dT_1 = \log\left(\frac{-1+\sqrt{3}}{2}\right)$$

So $dT_1 = \frac{-1+\sqrt{3}}{2}$

4 #9 Number of 4-cycle cardioids in the Mandelbrot set.

(i) Use the Lavaurs fractions to count the number of 4-cycle discs and cardioids.

(ii) Count the number of 4-cycle discs (principal series, Farey seq, multiplier rule)

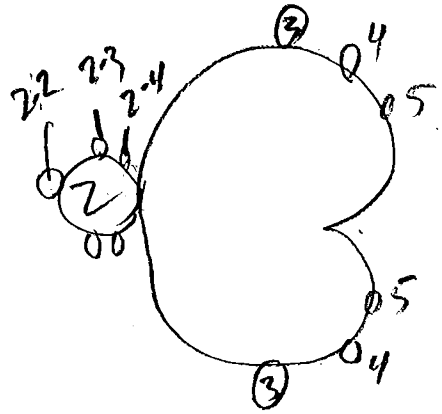
(iii) subtract.

For 4-cycle components, the Lavaurs fractions have denominator $2^4 - 1 = \cancel{32}\,16 - 1 = \cancel{31}\,15$

Lavaurs fractions

$\frac{1}{15}, \frac{2}{15}, \ldots, \frac{14}{15}$

combined in pairs, 7 pairs, so 7 4-cycle discs and cardioids. How many discs?



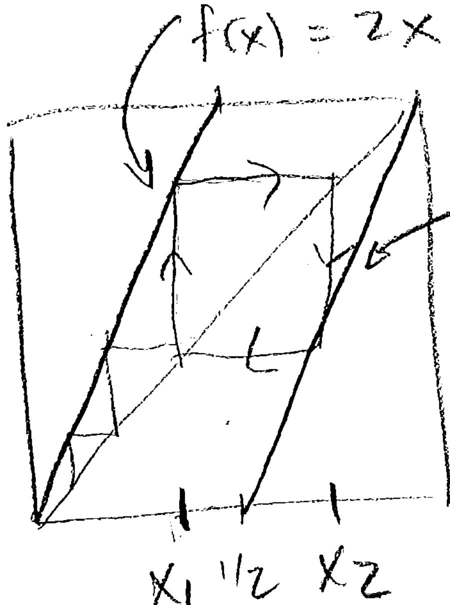
2 from the principal series

0 from the Farey sequence

1 from the multiplier rule

$7 - (2+0+1) = 4$ 4-cycle cardioids.

6#8



graphical iteration shows one 2-cycle point is $<1/2$, the other is $>1/2$

Say $x_1 < 1/2$, $x_2 > 1/2$

$$\boxed{f(x_1) = x_2 \quad \text{and} \quad f(x_2) = x_1}$$

Because $x_1 < 1/2$, $f(x_1) = 2x_1$

Because $x_2 > 1/2$, $f(x_2) = 2x_2 - 1$

$$\left.\begin{aligned} 2x_1 &= f(x_1) = x_2 \\ 2x_2 - 1 &= f(x_2) = x_1 \end{aligned}\right\}$$

$2x_2 - 1 = x_1$

$2(2x_1) - 1 = x_1$

$4x_1 - 1 = x_1$

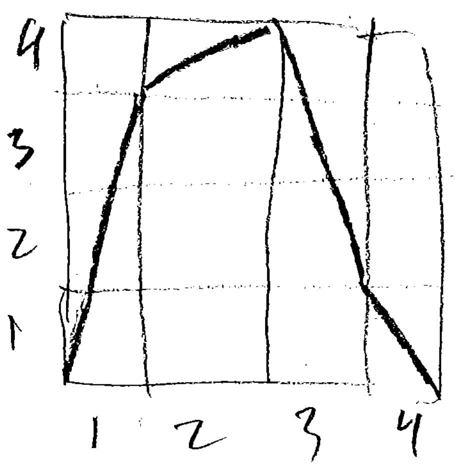
$3x_1 - 1 = 0$

$3x_1 = 1$

$x_1 = 1/3$

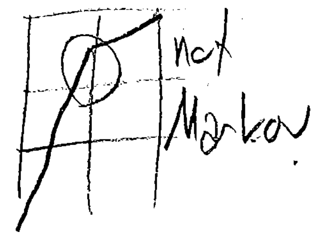
$x_2 = 2x_1 = 2/3$

7#8



Above each bin on the x-axis, every bin on the y-axis that is entered, is crossed completely

Markov



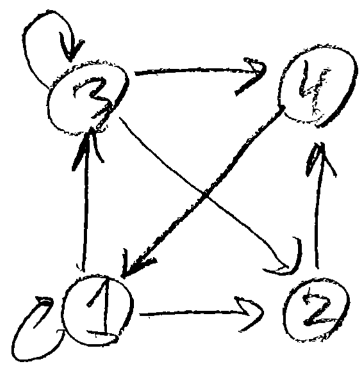
not Markov.

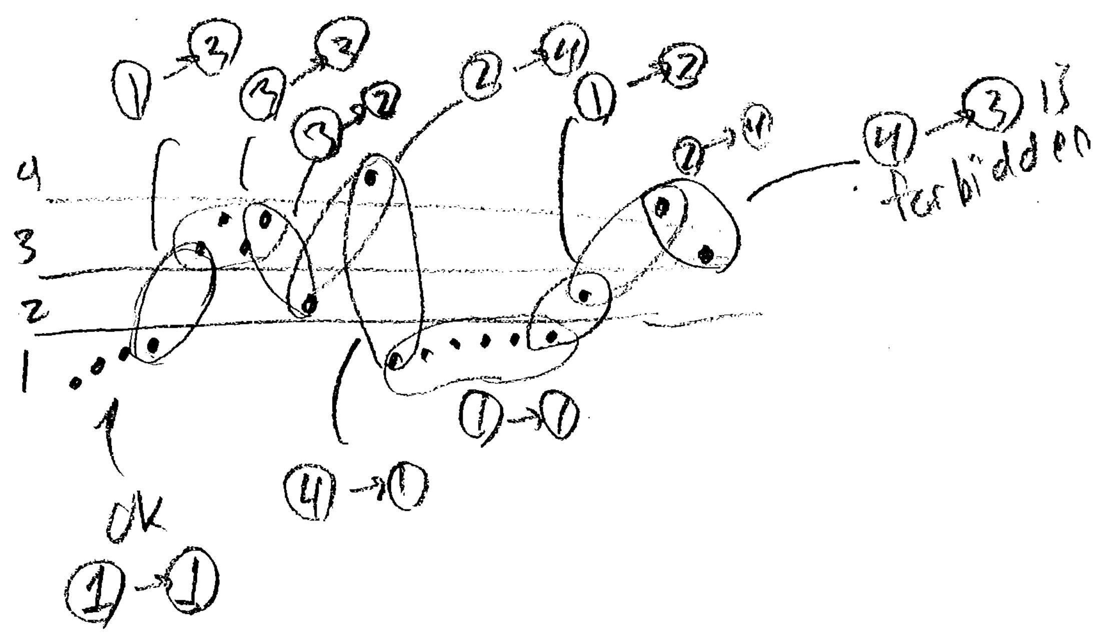
1 → 1, 2, 3

2 → 4

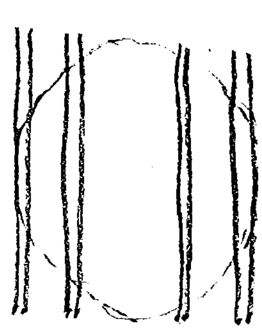
3 → 2, 3, 4

4 → 1





7#9



A = part of C×I inside the circle.

Find dim(A).

⑦

A is part of $C \times I$, so by monotonicity, $d(A) \le d(C \times I)$

shrunken copy of $C \times I$

B

$\dim(B) = \dim(C \times I)$

B is part of A, so by monotonicity again,

$d(B) \le d(A)$

$d(C \times I) = d(B) \le d(A) \le d(C \times I)$

Then $d(A) = d(C \times I) = d(C) + d(I)$

$= \dfrac{\log 2}{\log 3} + 1$

4#3 $\quad r = 1/2^n$ = box side length

$N(\frac{1}{2^n}) = 2^n + 3^n + n$

$$d = \lim_{1/2^n \to 0} \frac{\log(N(1/2^n))}{\log(1/(1/2^n))} = \lim_{1/2^n \to 0} \frac{\log(2^n + 3^n + n)}{\log(2^n)}$$

For the log of a sum, factor out the largest term

$n=1 \quad 2^1, 3^1, 1 \quad 2, 3, 1$

$n=2 \quad 2^2, 3^2, 2 \quad 4, 9, 2$

$n=3 \quad 2^3, 3^3, 3 \quad 8, 27, 3$

3^n is the largest, so

$$2^n + 3^n + n = 3^n\left(\frac{2^n}{3^n} + 1 + \frac{n}{3^n}\right)$$

$$d = \lim_{n\to\infty} \frac{\log\left(3^n\left(\frac{2^n}{3^n} + 1 + \frac{n}{3^n}\right)\right)}{\log(2^n)}$$

$$= \lim_{n\to\infty} \frac{\log(3^n) + \log\left(\frac{2^n}{3^n} + 1 + \frac{n}{3^n}\right)}{\log(2^n)}$$

As $n\to\infty$, $\frac{2^n}{3^n} \to 0$ $\quad \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$

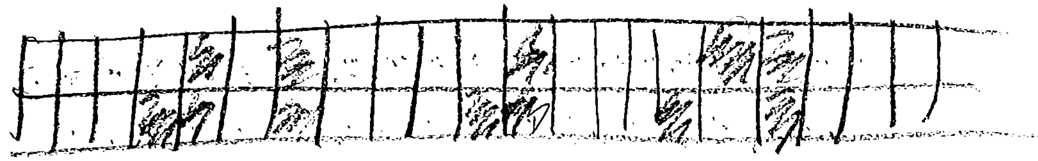
$\frac{n}{3^n} \to 0$ because $\frac{n}{3^n} < \frac{2^n}{3^n}$

As $n\to\infty$ $\frac{2^n}{3^n} + 1 + \frac{n}{3^n} \to 1$

$\log(1) = 0$

$$d = \lim_{n\to\infty} \frac{\log(3^n)}{\log(2^n)} = \lim_{n\to\infty} \frac{\cancel{n}\log 3}{\cancel{n}\log 2} = \frac{\log 3}{\log 2}$$

4 # 8



DDD → D
DDL → L
DLD → L
LDL → D
LDD → D
DLL → D
LLD → L

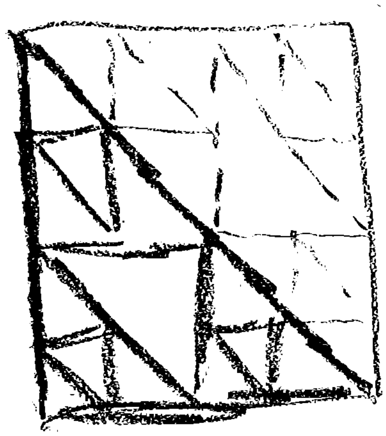
There are 8 n=3 binary CA nbhds, we have not accounted for LLL

[DLL → L, DLD → L, LLD → L, LLL → L, all others give D

and [DLL → L, DLD → L, LLD → L, all others give D

both give the second row from the first.

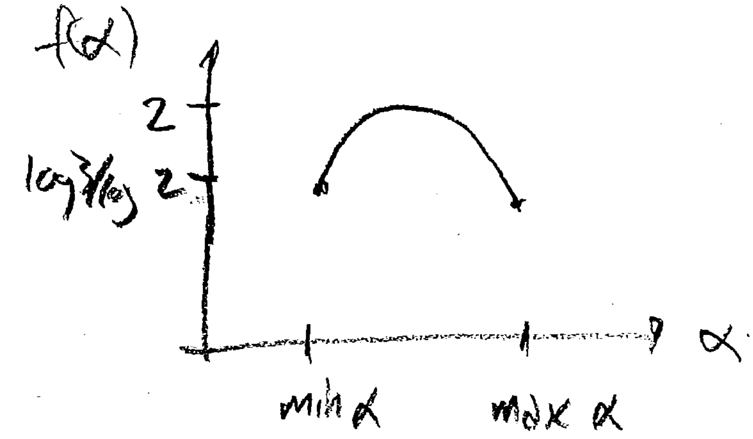
4 # 5



$$f(\min\alpha) = \dim(\text{dark gasket}) = \frac{\log 3}{\log 2}$$

$$f(\max\alpha) = \dim(\text{light gasket}) = \frac{\log 3}{\log 2}$$

$$\max f(\alpha) = \dim \text{IFS} = \dim(\text{filled-in square}) = 2$$



Can this multifractal be generated
by $T_1(x,y)=(x/2, y/2)$, $T_2(x,y)=(x/2, y/2)+(\frac{1}{2},0)$
$T_3(x,y)=(x/2, y/2)+(0,\frac{1}{2})$ $T_4(x,y)=(x/2, y/2)+(\frac{1}{2},\frac{1}{2})$

To get the dark gasket, T_1, T_2, and T_3 must have the same high prob.
To get the light gasket, T_2, T_3, and T_4 must have the same low prob.
This is impossible: T_2 and T_3 cannot have low and high prob simultaneously.

8 # 8 (a) $r_1 = 1/2$, $r_2 = 1/2$ with prob $1/2$
$r_1 = 1/4$, $r_2 = 1/4$ with prob $1/2$
$E(r_1^d) + E(r_2^d) = 1$

$$E(r_1{}^d) = \frac{1}{2}\left(\frac{1}{2}\right)^d + \frac{1}{2}\left(\frac{1}{4}\right)^d$$

$$E(r_2{}^d) = \frac{1}{2}\left(\frac{1}{2}\right)^d + \frac{1}{2}\left(\frac{1}{4}\right)^d$$

$$\underbrace{\left(\frac{1}{2}\left(\frac{1}{2}\right)^d + \frac{1}{2}\left(\frac{1}{4}\right)^d\right)}_{E(r_1{}^d)} + \underbrace{\left(\frac{1}{2}\left(\frac{1}{2}\right)^d + \frac{1}{2}\left(\frac{1}{4}\right)^d\right)}_{E(r_2{}^d)} = 1$$

$$\left(\frac{1}{2}\right)^d + \left(\frac{1}{4}\right)^d = 1 \qquad x = \left(\frac{1}{2}\right)^d,\ x^2 = \left(\frac{1}{4}\right)^d$$

$$x + x^2 = 1$$

$$x^2 + x - 1 = 0 \qquad a = 1 \quad b = 1 \quad c = -1$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1}$$

$$= \frac{-1 \pm \sqrt{5}}{2}$$

$$x = \frac{-1 + \sqrt{5}}{2}$$

$$\left(\frac{1}{2}\right)^d = x = \frac{-1 + \sqrt{5}}{2}$$

$$\log\left(\left(\frac{1}{2}\right)^d\right) = \log\left(\frac{-1 + \sqrt{5}}{2}\right)$$

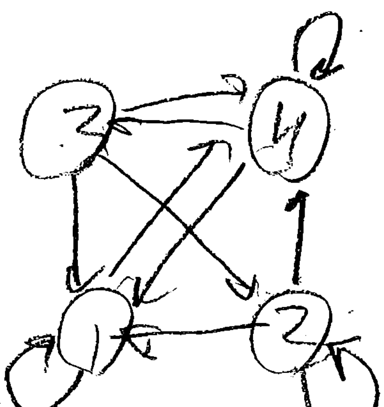
$$d \log\left(\frac{1}{2}\right) = \log\left(\frac{-1 + \sqrt{5}}{2}\right)$$

$$d = \frac{\log\left(\frac{-1 + \sqrt{5}}{2}\right)}{\log\left(1/2\right)}$$

(b) most likely dim of the intersection of two of these in the plane is

$$\frac{\log\left(\frac{-1+\sqrt{5}}{2}\right)}{\log(1/2)} + \frac{\log\left(\frac{-1+\sqrt{5}}{2}\right)}{\log(1/2)} - 2$$

dim of one of these lies between 1 $(r_1 = r_2 = 1/2)$ and $\frac{1}{2}$ $(r_1 = r_2 = \frac{1}{4})$, so $\dfrac{\log\left(\frac{-1+\sqrt{5}}{2}\right)}{\log(1/2)} < 1$



romes: 4 and 1

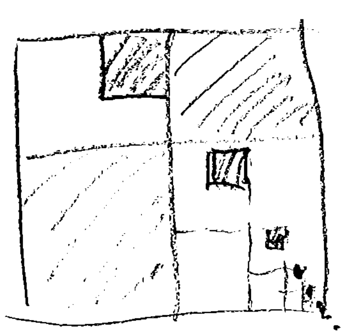
(non-romes: 2 and 3)

path from a rome to each non-rome: $4 \to 3$

$4 \to 3 \to 2$

IFS without memory

Is there a loop among non-romes? 2



2 copies scaled by $1/2$

1 copy scaled by $1/4$

1 copy scaled by $1/8$

1 copy scaled by $1/16$

$\vdots$

$$2\left(\tfrac{1}{2}\right)^d + \left(\tfrac{1}{4}\right)^d + \left(\tfrac{1}{8}\right)^d + \left(\tfrac{1}{16}\right)^d + \left(\tfrac{1}{32}\right)^d + \cdots = 1$$

$$x = \left(\tfrac{1}{2}\right)^d$$

$$2x + x^2 + x^3 + x^4 + x^5 + \cdots = 1$$

Know $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$ if $|x| < 1$

So $x + x^2 + x^3 + x^4 + \cdots = \frac{x}{1-x}$

$$2x + x^2 + x^3 + x^4 + \cdots = 1$$

$$x + \left(x + x^2 + x^3 + x^4 + \cdots\right) = 1$$

$$x + \frac{x}{1-x} = 1$$

$$x(1-x) + x = 1 - x$$

$$x - x^2 + x = 1 - x$$

$$0 = x^2 - 3x + 1$$

$$x = \frac{3 \pm \sqrt{9-4}}{2}$$

$x = \frac{3 \pm \sqrt{5}}{2}$ Recall $|x| < 1$

So $x = \frac{3 - \sqrt{5}}{2}$, $d = \dfrac{\log\left(\frac{3 - \sqrt{5}}{2}\right)}{\log(1/2)}$