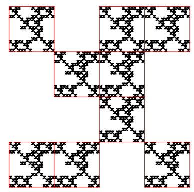
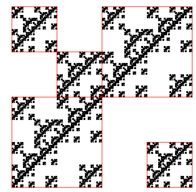
Fourth homework set solutions

a  b 

1. (a) As indicated by the red boxes, we see fractal (a) is composed of $N = 9$ pieces, each scaled by $r = 1/4$, so the similarity dimension of this fractal is

$$d_s = \frac{\log(9)}{\log(4)} = \frac{\log(3^2)}{\log(2^2)} = \frac{2\log(3)}{2\log(2)} = \frac{\log(3)}{\log(2)}.$$

(b) As indicated by the red boxes, this fractal is composed of 2 pieces of size 1/2 and 3 pieces of size 1/4. By the Moran equation, the dimension d satisfies

$$2 \cdot (1/2)^d + 3 \cdot (1/4)^d = 1$$

Writing $x = (1/2)^d$, we see that $(1/4)^d = ((1/2)^2)^d = ((1/2)^d)^2 = x^2$, so the Moran equation can be rewitten

$$2x + 3x^2 = 1$$

Applying the quadratic formula, the positive solution is $x = 1/3$ and so the dimension is

$$d_s = \frac{\log(1/3)}{\log(1/2)} = \frac{\log(3^{-1})}{\log(2^{-1})} = \frac{-\log(3)}{-\log(2)} = \frac{\log(3)}{\log(2)}.$$

2. (a) For this fractal the Moran equation is

$$(1/3)^d + 2(1/9)^d + 2(1/27)^d + 2(1/81)^d + \cdots = 1$$

(b) Writing $x = (1/3)^d$, the Moran equation becomes

$$\begin{aligned} x + 2x^2 + 2x^3 + 2x^4 + \cdots &= 1 \\ x + 2x^2(1 + x + x^2 + x^3 + \cdots) &= 1 \\ x + \frac{2x^2}{1-x} &= 1 \end{aligned}$$

This last line simplifies to

$$x(1-x) + 2x^2 = 1 - x$$

This is the quadratic equation

$$x^2 + 2x - 1 = 0$$

The positive solution is $x = -1 + \sqrt{2}$ and so

$$d = \frac{\log(-1 + \sqrt{2})}{\log(1/3)}$$

3. (a) The Moran equation is

$$2(1/2)^d + 2(1/4)^d + 3(1/8)^d = 1$$

(b) Recall the gasket can be decomposed into three copies, each scaled by 1/2. Suppose we keep two of those, and decompose the third into three copies, each scaled by 1/4. Keep two of these, and decompose the third into three copies, each scaled by 1/8. That is, the gasket consists of 2 copies scaled by 1/2, 2 copies scaled by 1/4, and 3 copies scaled by 1/8. Consequently, this fractal has the same dimension as the gasket, $\log(3)/\log(2)$.

Alternately, writing $x = 1/2$, the Moran equation becomes

$$2x + 2x^2 + 3x^3 = 1$$

The polynomial $3x^3 + 2x^2 + 2x - 1$ factors as

$$3x^3 + 2x^2 + 2x - 1 = (x - 1/3)(3x^2 + 3x + 3)$$

The roots of $3x^2 + 3x + 3$ are $(-1 \pm \sqrt{3}i)/2$, so the (real) positive root is $x = 1/3$. Then the dimension is

$$\frac{\log(1/3)}{\log(1/2)} = \frac{\log(3)}{\log(2)}$$

as we saw in the solution of 1(b).