Fifth homework set solutions

1. (a) The Cantor set C consists ot $N = 2$ pieces scaled by $r = 1/4$, so by the similarity dimension formula

$$d = \frac{\log(2)}{\log(4)} = \frac{\log(2)}{\log(2^2)} = \frac{\log(2)}{2\log(2)} = \frac{1}{2}$$

(b) By the product formula,

$$d(C \times C \times C) = d(C) + d(C) + d(C) = 3/2$$

(c) By the intersection formula, typically

$$\begin{aligned} d(L \cap (C \times C \times C)) &= d(L) + d(C \times C \times C) - 3 \\ &= 1 + 3/2 - 3 \\ &= -1/2 \end{aligned}$$

(d) Placing L along one of the edges of $C \times C \times C$, we see that $L \cap (C \times C \times C) = C$, and so

$$d(L \cap (C \times C \times C)) = d(C) = 1/2$$

where the last equality follows from part (a).

2. (a) By the intersection formula, typically

$$\begin{aligned} d(S \cap (C \times C \times C)) &= d(S) + d(C \times C \times C) - 3 \\ &= 2 + 3/2 - 3 \\ &= 1/2 \end{aligned}$$

(b) Placing S along a face of $C \times C \times C$, we see that $S \cap (C \times C \times C) = C \times C$, and so

$$d(S \cap (C \times C \times C)) = d(C \times C) = d(C) + d(C) = 1$$

3. To find the maximum and minimum values of α, we must compute

$$\begin{aligned} \log(.025)/\log(.5) &\approx 5.322 \\ \log(.15)/\log(.25) &\approx 1.368 \\ \log(.075)/\log(.125) &\approx 1.246 \end{aligned}$$

(a) The maximum value of α is $\alpha_{\max} = \log(.025)/\log(.5) \approx 5.322$.

(b) The minimum value of α is $\alpha_{\min} = \log(.075)/\log(.125) \approx 1.246$.

(c) The maximum value of α occurs only with transformation 1, so on a set with $N = 1$ pieces scaled by $r = 1/2$. Then

$$f(\alpha_{\max}) = \frac{\log(1)}{\log(2)} = 0$$

(d) The minimum value of α occurs with transformations 7, 8, and 9, so on a set with $N = 3$ pieces scaled by $r = 1/8$. Then

$$f(\alpha_{\min}) = \frac{\log(3)}{\log(8)}$$

(e) The maximum value of $f(\alpha)$ is the dimension of the IFS, which has 1 piece scaled by 1/2, 5 pieces scaled by 1/4, and 3 pieces scaled by 1/8. That is, the dimension is the solution of the Moran equation

$$(1/2)^d + 5 \cdot (1/4)^d + 3 \cdot (1/8)^d = 1$$

Taking $x = (1/2)^d$, so $(1/4)^d = x^2$ and $(1/8)^d = x^3$, the Moran equation becomes

$$3x^3 + 5x^2 + x - 1 = 0$$

Dividing the left-hand side by $x + 1$, we find

$$\begin{aligned} 3x^3 + 5x^2 + x - 1 &= (x + 1)(3x^2 + 2x - 1) \\ &= (x + 1)(3x - 1)(x + 1) \end{aligned}$$

The only positive root is $x = 1/3$. Solving $x = (1/2)^d$ for d,

$$d = \frac{\log(x)}{\log(1/2)} = \frac{\log(1/3)}{\log(1/2)} = \frac{\log(3)}{\log(2)}$$