Sixth homework set solutions

1. (a) The maximum value of the dimension occurs when the scaling factors take on their maximum values: $r_1 = r_2 = r_3 = 1/3$. Because all the r_i have the same value, the dimension is

$$d_{\max} = \frac{\log(3)}{\log(3)} = 1$$

The minimum value of the dimension occurs when the scaling factors take on their minimum values: $r_1 = r_2 = r_3 = 1/9$. Because all the r_i have the same value, the dimension is

$$d_{\min} = \frac{\log(3)}{\log(9)} = \frac{\log(3)}{\log(3^2)} = \frac{1}{2}$$

(b) To find the expected value of the dimension, we use the randomized Moran equation $E(r_1^d) + E(r_2^d) + E(r_3^d) = 1$

$$(1/2)(1/3)^d + (1/2)(1/9)^d + (1/4)(1/3)^d + (3/4)(1/9)^d +$$
$$(1/8)(1/3)^d + (7/8)(1/9)^d = 1$$

Combining like terms gives

$$(7/8)(1/3)^d + (17/8)(1/9)^d = 1$$

Substituting $x = (1/3)^d$, the Moran equation becomes the quadratic equation

$$(7/8)x + (17/8)x^2 = 1$$

The solutions are $x = (-7 \pm \sqrt{593})/34$. Taking the positive solution, we see the expected value of the dimension is

$$d_{\mathrm{E}} = \frac{\log((-7 + \sqrt{593})/34)}{\log(1/3)}$$

(c) The numerical approximation of the dimension of (b) is

$$d_{\mathrm{E}} \approx 0.612$$

The expected value of the dimension is much closer to the minimum than to the maximum. This makes sense, because r_1 takes on its minimum value half the time, r_2 takes on its minimum value three-quarters of the time, and r_3 takes on its minimum value seven-eighths of the time.

2. To find the trading time increments, first we solve the Moran equation using the absolute values of the price increments,

$$(1/4)^d + (1/2)^d + (1/4)^d + (1/2)^d + (1/4)^d + (1/4)^d = 1$$

Substituting in $x = (1/2)^d$, the Moran equation becomes the quadratic equation

$$2x + 4x^2 = 1$$

The positive solution is $x = (-1 + \sqrt{5})/4$, giving

$$d = \frac{\log((-1 + \sqrt{5})/4)}{\log(1/2)}$$

From this we find

$$dT_2 = dT_4 = (1/2)^{\log((-1+\sqrt{5})/4)/\log(1/2)}$$
$$dT_1 = dT_3 = dT_5 = dT_6 = (1/4)^{\log((-1+\sqrt{5})/4)/\log(1/2)}$$

Simplifying, using the exponent rule for logarithms

$$\log(dT_2) = \log(1/2) \cdot \frac{\log((-1 + \sqrt{5})/4)}{\log(1/2)} = \log((-1 + \sqrt{5})/4)$$

and so

$$dT_2 = dT_4 = \frac{-1 + \sqrt{5}}{4}$$

Arguing similarly,

$$\begin{aligned}
\log(dT_1) &= \log(1/4) \cdot \frac{\log((-1 + \sqrt{5})/4)}{\log(1/2)} \\
&= \log((1/2)^2) \cdot \frac{\log((-1 + \sqrt{5})/4)}{\log(1/2)} \\
&= 2 \cdot \log(1/2) \cdot \frac{\log((-1 + \sqrt{5})/4)}{\log(1/2)} \\
&= 2 \cdot \log((-1 + \sqrt{5})/4) \\
&= \log(((-1 + \sqrt{5})/4)^2) \\
&= \log((3 - \sqrt{5})/8)
\end{aligned}$$

and so

$$dT_1 = dT_3 = dT_5 = dT_6 = \frac{3 - \sqrt{5}}{8}$$