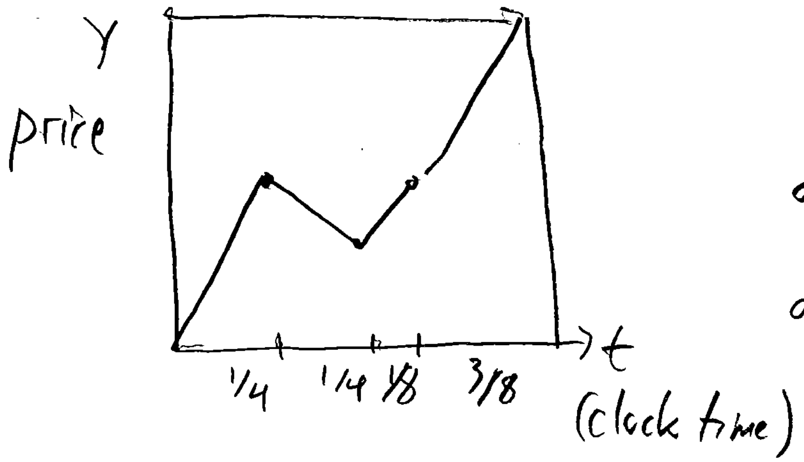
Nov 3 Notes



$dY_1 = 1/2 \quad dt_1 = \frac{1}{4}$

$dY_2 = -1/4 \quad dt_2 = \frac{1}{4}$

$dY_3 = 1/4 \quad dt_3 = 1/8$

$dY_4 = 1/2 \quad dt_4 = 3/8$

Is this unifractal or multifractal?

$|dY_i| = (dt_i)^{H_i}$ scaling hypothesis

$\log |dY_i| = \log((dt_i)^{H_i})$

$\log|dY_i| = H_i \log(dt_i)$

$$H_i = \frac{\log|dY_i|}{\log(dt_i)}$$

$H_1 = \dfrac{\log(1/2)}{\log(1/4)} \quad H_2 = \dfrac{\log(1/4)}{\log(1/4)}$ These two are different (same denominator, different numerator)

So this is a multifractal cartoon.

Convert the time scale to "Trading Time"
Measure time in more detail when large jumps occur, and compress time when the jumps are routine.

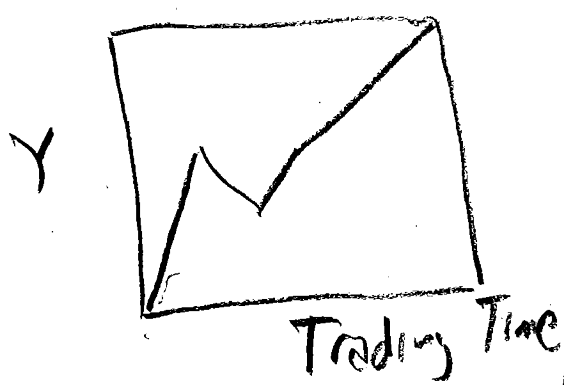
We find the Trading Time generators by solving

$$|dY_1|^D + |dY_2|^D + |dY_3|^D + |dY_4|^D = 1$$

Solve this for D. Then the Trading Time generators are

$dT_1 = |dY_1|^D$ $\quad dT_2 = |dY_2|^D$, $\quad dT_3 = |dY_3|^D$

$dT_4 = |dY_4|^D$



$dY_1 = +1/2$ $\quad dT_1 = |dY_1|^D$

$dY_2 = -1/4$ $\quad dT_2 = |dY_2|^D$

$dY_3 = +1/4$ $\quad dT_3 = |dY_3|^D$

$dY_4 = +1/2$ $\quad dT_4 = |dY_4|^D$

This cartoon is unifractal:

$$\frac{\log|dY_i|}{\log dT_i} = \frac{\log|dY_i|}{\log|dY_i|^D} = \frac{\cancel{\log}|dY_i|}{D\,\cancel{\log}|dY_i|} = \frac{1}{D}$$

Trading Time Theorem: Any multifractal cartoon is fractional Brownian motion when measured in multifractal Trading Time.

The Moran equation is

$$\left(\frac{1}{2}\right)^D + \left(\frac{1}{4}\right)^D + \left(\frac{1}{4}\right)^D + \left(\frac{1}{2}\right)^D = 1$$

Take $x = \left(\frac{1}{2}\right)^D$

$$2x + 2x^2 = 1$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2}$$

$$= \frac{-2 \pm \sqrt{4+8}}{4} = \frac{-2 \pm \sqrt{12}}{4}$$

$$= \frac{-2 \pm \sqrt{4 \cdot 3}}{4} = \frac{-2 \pm 2\sqrt{3}}{4}$$

Take $x = \frac{-1+\sqrt{3}}{2}$

$x = \left(\frac{1}{2}\right)^D \qquad \frac{-1+\sqrt{3}}{2} = \left(\frac{1}{2}\right)^D$

$$\operatorname{Log}\left(\frac{-1+\sqrt{3}}{2}\right) = \operatorname{Log}\left(\left(\frac{1}{2}\right)^D\right)$$

$$D = \frac{\log\left(\frac{-1+\sqrt{3}}{2}\right)}{\operatorname{Log}(1/2)}$$

$dT_1 = dT_4 = \left(\frac{1}{2}\right)^D$

$dT_2 = dT_3 = \left(\frac{1}{4}\right)^D$

Find expressions for dT_1, dT_2, dT_3, and dT_4 that do not contain logs.

$$dT_1 = \left(\frac{1}{2}\right)^{\log((-1+\sqrt{3})/2)/\log(1/2)}$$

$$\log(dT_1) = \log\left(\left(\frac{1}{2}\right)^{\log((-1+\sqrt{3})/2)/\log(1/2)}\right)$$

$$\log(dT_1) = \frac{\log\left(\frac{-1+\sqrt{3}}{2}\right)}{\cancel{\log\left(\frac{1}{2}\right)}} \cdot \cancel{\log(1/2)}$$

$$\log(dT_1) = \log\left(\frac{-1+\sqrt{3}}{2}\right)$$

Then $dT_1 = \frac{-1+\sqrt{3}}{2}$

Similarly, $dT_4 = \frac{-1+\sqrt{3}}{2}$

Exercise: Find $dT_2 = dT_3$.