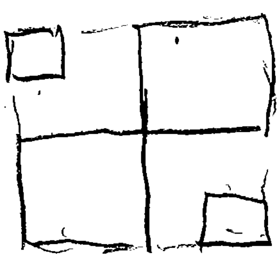
Oct 7 Review

Different scaling factors $r_1, r_2, \ldots, r_N$
Moran equation

$$r_1^d + r_2^d + \cdots + r_N^d = 1$$

This can be extended to an infinite collection of scaling factors - using the geometric series formula.

① All the scalings are powers of one number: $r_1 = r_2 = 1/2$, $r_3 = r_4 = 1/4$



$$\left(\tfrac{1}{2}\right)^d + \left(\tfrac{1}{2}\right)^d + \left(\tfrac{1}{4}\right)^d + \left(\tfrac{1}{4}\right)^d = 1$$

Because $\frac{1}{4} = \left(\frac{1}{2}\right)^2$

$\left(\frac{1}{4}\right)^d = \left(\left(\frac{1}{2}\right)^d\right)^2$. Write $x = \left(\frac{1}{2}\right)^d$. Note x must be positive. Then the Moran equation becomes

$$2x + 2x^2 = 1$$

or $$2x^2 + 2x - 1 = 0$$

$a = 2, \quad b = 2, \quad c = -1$

Then $x = \dfrac{-2 \pm \sqrt{(2)^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2}$

$$x = \frac{-2 \pm \sqrt{4+8}}{4}$$

$$x = \frac{-2 \pm \sqrt{12}}{4}$$

$$x = \frac{-2 \pm 2\sqrt{3}}{4}$$

$$x = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}$$

x must be positive, so $x = -\frac{1}{2} + \frac{1}{2}\sqrt{3}$

$$\left(\tfrac{1}{2}\right)^d = x = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}$$

$$\mathrm{Log}\left(\left(\tfrac{1}{2}\right)^d\right) = \mathrm{Log}\left(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\right)$$

$$d\,\mathrm{Log}\left(\tfrac{1}{2}\right) = \mathrm{Log}\left(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\right)$$

$$d = \frac{\mathrm{Log}\left(-\frac{1}{2} + \frac{1}{2}\sqrt{3}\right)}{\mathrm{Log}\left(1/2\right)}$$

(2) Suppose a fractal consists of 2 copies scaled by 1/2, 2 copies scaled by $\frac{1}{4}$, 2 copies scaled by 1/8, ...

The Moran equation is

$$2\left(\tfrac{1}{2}\right)^d + 2\left(\tfrac{1}{4}\right)^d + 2\left(\tfrac{1}{8}\right)^d + \cdots = 1$$

Take $x = \left(\frac{1}{2}\right)^d$. The Moran equation becomes

$$2x + 2x^2 + 2x^3 + \cdots = 1$$

$$2x\Big(\underbrace{1 + x + x^2 + x^3 + \cdots}_{\text{geometric series}}\Big) = 1$$

If $|x| < 1$, the series sums to

$$\frac{1}{1-x}$$

$$2x\left(\frac{1}{1-x}\right) = 1$$

$$\cancel{(1-x)}\,\frac{2x}{\cancel{1-x}} = 1\cdot(1-x)$$

$$2x = 1 - x$$

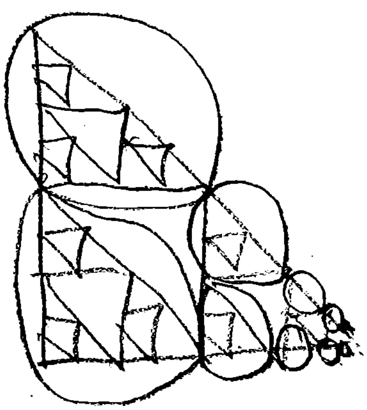
$$3x = 1$$

$$x = 1/3$$

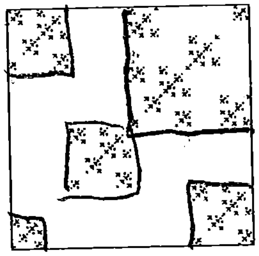
$x = \left(\frac{1}{2}\right)^d$

$\operatorname{Log}(x) = \log\left(\left(\frac{1}{2}\right)^d\right) = d\log(1/2)$

$$d = \frac{\log(x)}{\log(1/2)} = \frac{\log(\frac{1}{3})}{\log(1/2)}$$



$$= \frac{\log(3^{-1})}{\log(2^{-1})} = \frac{\cancel{-1}\cdot\log(3)}{\cancel{-1}\cdot\log(2)} = \frac{\log(3)}{\log(2)}$$



one copy scaled by $1/2$
three by $1/4$, one by $1/8$

Moran: $\left(\frac{1}{2}\right)^d + 3\left(\frac{1}{4}\right)^d + \left(\frac{1}{8}\right)^d = 1$

Write $x = (1/2)^d$. Then $\left(\frac{1}{4}\right)^d = x^2$ and $\left(\frac{1}{8}\right)^d = x^3$

Moran becomes $x + 3x^2 + x^3 = 1$

$$x^3 + 3x^2 + x - 1 = 0$$

If $x-1$ were a factor, $x=1$ is the only solution and $d = \frac{\log(x)}{\log(1/2)} = 0$

$$
\begin{array}{r|l}
 & x^2 + 2x - 1 \\
\hline
x+1 & x^3 + 3x^2 + x - 1 \\
 & -\ (x^3 + x^2) \\
\hline
 & 2x^2 + x \\
 & -\ (2x^2 + 2x) \\
\hline
 & -x - 1 \\
 & -x - 1 \\
\hline
\end{array}
$$

So $x^3 + 3x^2 + x - 1 = (x+1)(x^2 + 2x - 1)$

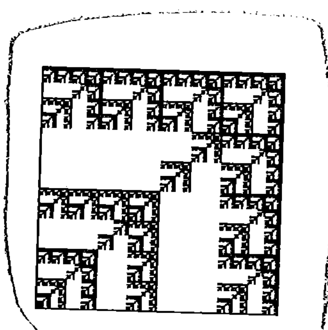
Then $x + 1 = 0$ or $x^2 + 2x - 1 = 0$

$x = -1$ impossible

quadratic formula $x = -1 \pm \sqrt{2}$

Take $x = -1 + \sqrt{2}$

$$d = \frac{\log(-1+\sqrt{2})}{\log(1/2)} \approx 1.272$$



Try this one