Main Topics:

IFS - find the IFS rules to make a given fractal.

IFS with memory - addresses - how to recognize the number of steps of memory (forbidden pairs, triples, ...)

Romes and associated concepts

Driven IFS and time series

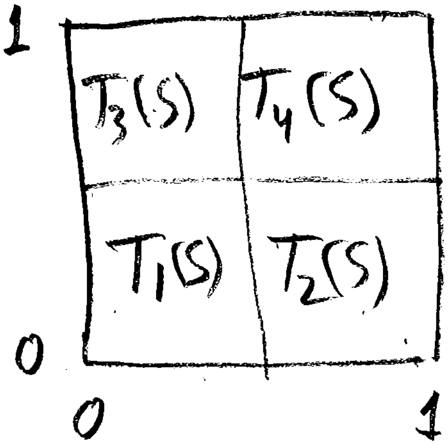
Box-counting dimension

Similarity dimension - basic formula, Moran equation, quadratic, factoring, geometric series

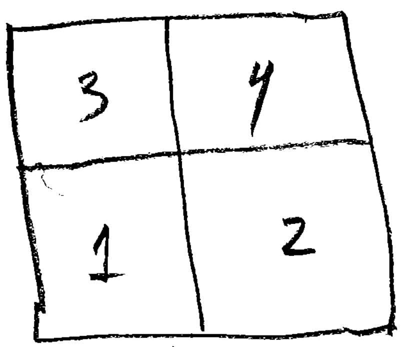
Algebra of dimensions - product, union, intersection

Multifractals max & min α, $f(\max \alpha)$, $f(\min \alpha)$, max of $f(\alpha)$, both from table (r_i, $prob_i$), and by visual inspection of an IFS.

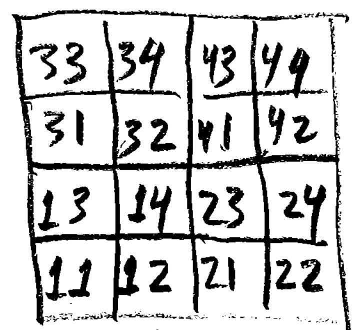
Addresses and their order



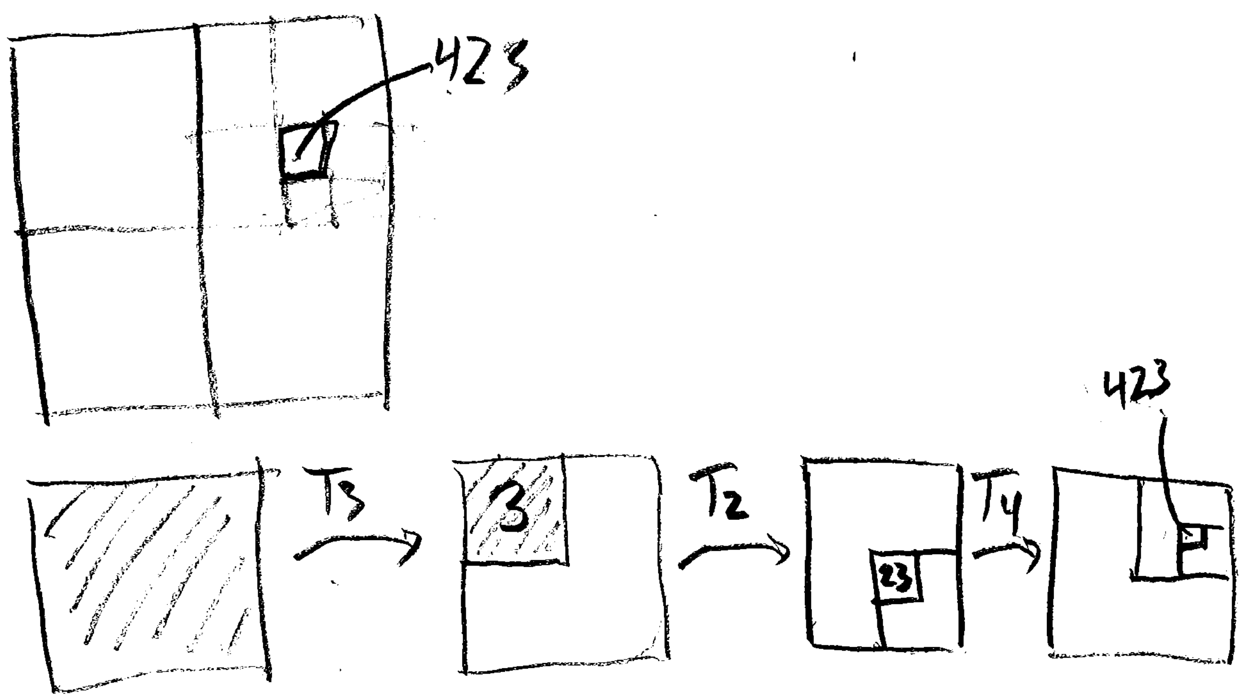
	r	s	θ	φ	e	f
1	.5	.5	0	0	0	0
2	.5	.5	0	0	.5	0
3	.5	.5	0	0	0	.5
4	.5	.5	0	0	.5	.5



length-1 addresses

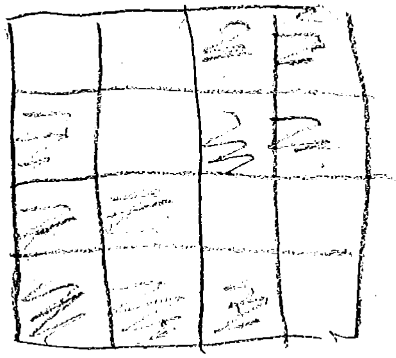


length-2 addresses



Exam 6 3(c)

(i)



empty length-2 addresses

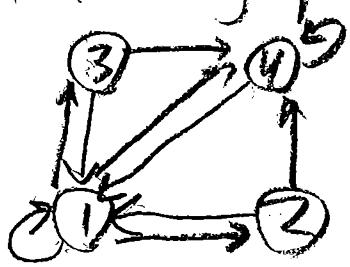
22, 23, 24, 32, 33, 34

empty length-3

122, 123, 124, 132, 133, 134
422, 423, 424, 432, 433, 434

Every empty length-3 address contains an empty length-2 address, so we conclude this IFS is generated by forbidden pairs, or by 1-step memory

Transition graph



Romes? 1 and 4 are romes

Non-romes are 2 and 3

paths from a rome to a non-rome:
$1 \to 2$ and $1 \to 3$

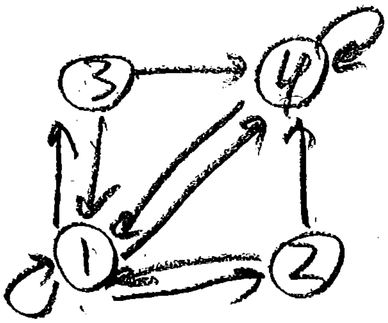
no path involves just non-romes

Then we can generate this IFS without memory

Inspecting the IFS image

r	s	θ	φ	e	f
.5	.5	0	0	0	0
.5	.5	0	0	.5	.5
.25	.25	0	0	0	.5
.25	.25	0	0	.5	0

From the transition graph



1 and 4 are romes

$T_1(x,y) = (x/2,\ y/2)$ .5 .5 0 0 0 0

$T_4(x,y) = (x/2,\ y/2) + (1/2,\ 1/2)$ .5 .5 0 0 .5 .5

The part in address 3 is

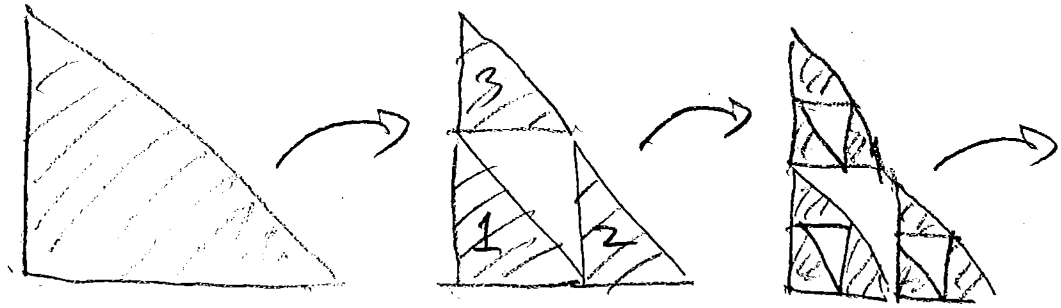
$$T_3(T_1(x,y)) = T_3(x/2,\ y/2)$$

$$= \left(\frac{x/2}{2},\ \frac{y/2}{2}\right) + (0,\ 1/2)$$

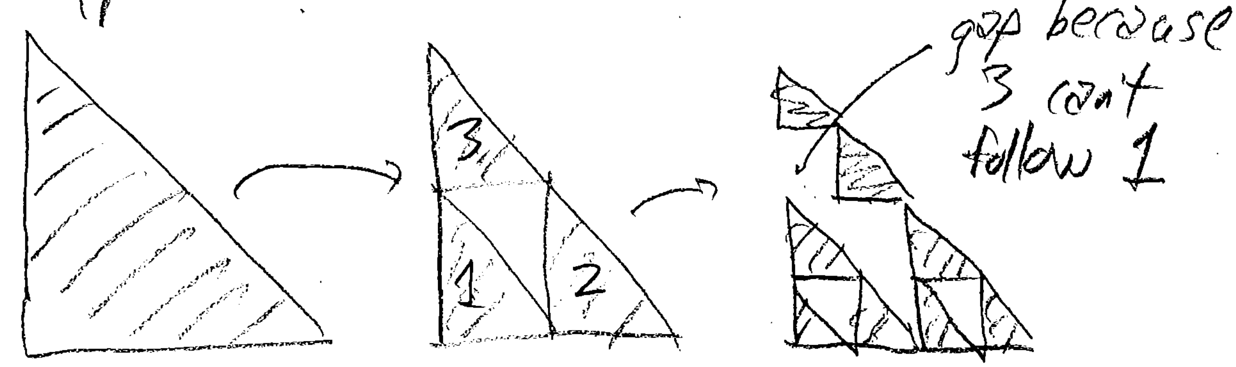
$$= \left(\frac{x}{4},\ \frac{y}{4}\right) + (0,\ 1/2)$$

.25 .25 0 0 0 .5

IFS without memory. All combinations of transformations are allowed.



Suppose 3 never follows 1



Exam 7 # 5

$$d(A) = \frac{\log 3}{\log 2} \qquad d(B) = \frac{\log 2}{\log(1/r)}$$

Intersection formula

dim of the plane

$$d(A \cap B) = d(A) + d(B) - 2$$

$$1 = \frac{\log 3}{\log 2} + \frac{\log 2}{\log(1/r)} - 2$$

Solve for r

dim(point) = 0
dim(line) = 1
dim(plane) = 2
dim(space) = 3

Isolate $\frac{\log 2}{\log(1/r)}$ on one side of the equation

$$3 - \frac{\log 3}{\log 2} = \frac{\log 2}{\log(1/r)}$$

$$\log(1/r)\left(3 - \frac{\log 3}{\log 2}\right) = \frac{\log 2}{\cancel{\log(1/r)}} \cdot \cancel{\log(1/r)} = \log 2$$

$$\log(1/r) = \frac{\log 2}{3 - \frac{\log 3}{\log 2}}$$

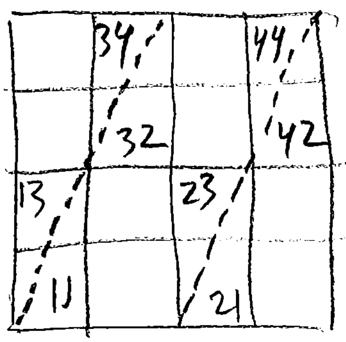
Recall $10^{\log x} = x$ $\quad \log(10^x) = x \cdot \log 10$
$= x$

$$10^{\log(1/r)} = 10^{\log 2/\left(3 - \frac{\log 3}{\log 2}\right)}$$

$$\frac{1}{r} = 10^{\log 2/\left(3 - \frac{\log 3}{\log 2}\right)}$$

$$r = \frac{1}{10^{\log 2/\left(3 - \frac{\log 3}{\log 2}\right)}}$$

Exam 4. 4(c)



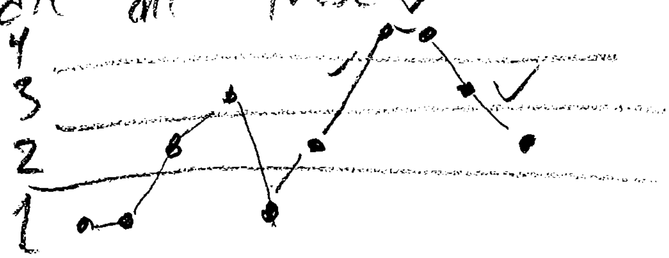
What are the occupied length-2 addresses?

11, 13, 21, 23, 32, 34, 42, 44

Allowed transitions:

$1 \to 1$, $3 \to 1$, $1 \to 2$, $3 \to 2$, $2 \to 3$,
$4 \to 3$, $2 \to 4$, $4 \to 4$

Draw enough of the time series to illustrate all these transitions



In contrast

means $1 \to 1$, $1 \to 2$, $2 \to 2$, $2 \to 1$

Exam 4 5 P is the product of a Cantor MTS and a filled-in square

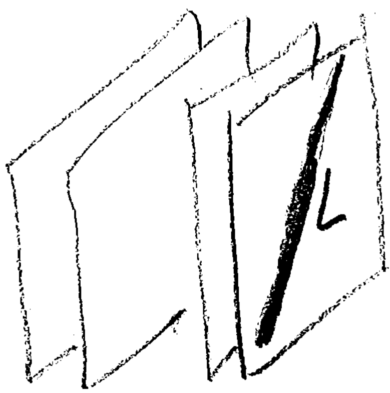
Product rule $d(P) = d(MTS) + d(\text{square})$

$$= \frac{\log 2}{\log 3} + 2$$

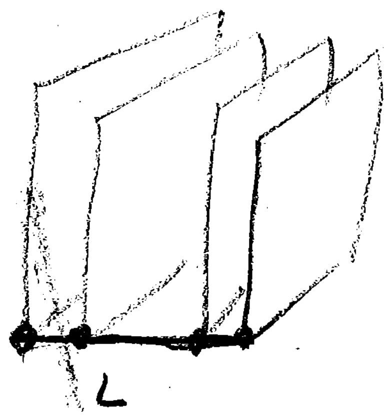
$N = 2$, $r = 1/3$

Typically,

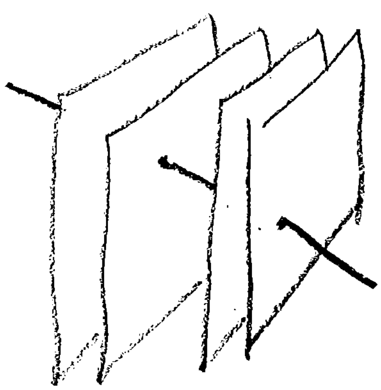
$$d(L \cap P) = d(L) + d(P) - 3$$
$$= 1 + \left(\frac{\log 2}{\log 3} + 2\right) - 3$$
$$= \frac{\log 2}{\log 3}$$



If L lies in one of the squares of P, then $L \cap P = L$, so $d(L \cap P) = d(L) = 1$



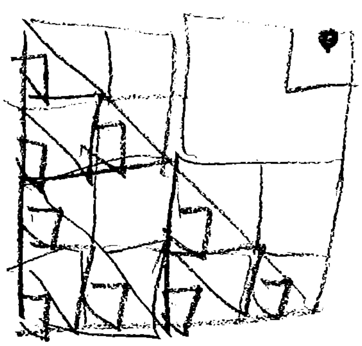
Align L along an edge of P containing the Cantor set, then $L \cap P$ is a Cantor set



Random placement of L will cross each square in one point, so the intersection will be a Cantor set.

To compute box-counting dimension find a formula for $N(r)$ = number of boxes of side length r to cover the shape, and compute

$$\lim_{r\to 0} \frac{\log(N(r))}{\log(1/r)}$$



gasket point

$$N(\tfrac{1}{2}) = 4 = 3 + 1$$

$$N(1/4) = 9 + 1$$

$$N(1/8) = 27 + 1$$

$$N(1/2^n) = 3^n + 1$$

$$d = \lim_{1/2^n \to 0} \frac{\log(3^n+1)}{\log(1/\frac{1}{2^n})} = \lim_{n\to\infty} \frac{\log\left(3^n\left(1+\frac{1}{3^n}\right)\right)}{\log(2^n)}$$

$$= \lim_{n\to\infty} \frac{\log(3^n)}{\log(2^n)} + \frac{\overbrace{\log(1+1/3^n)}^{\to 0}}{\log(2^n)} = \lim_{n\to\infty} \frac{n\log 3}{n\log 2}$$

$$= \frac{\log 3}{\log 2}$$