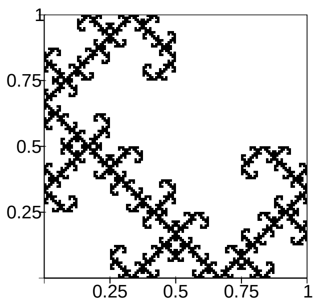
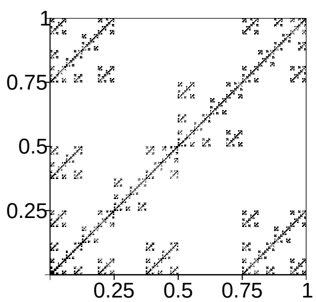
Practice Final Exam 5 Solutions

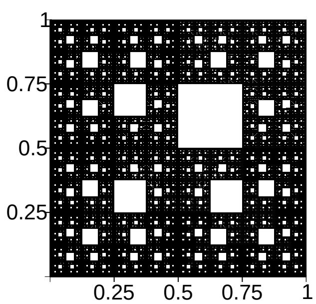
1. Fractals and their IFS generators.



r	s	θ	φ	e	f
$-.5$	.5	90	90	.5	.5
$-.5$	.5	0	0	1	0
.5	$-.5$	0	0	0	1



r	s	θ	φ	e	f
.5	.5	0	0	0	0
.25	.25	0	0	.5	.5
.25	.25	0	0	.75	0
.25	.25	0	0	0	.75
$-.25$	$-.25$	0	0	1	1



r	s	θ	φ	e	f
.5	.5	0	0	0	0
$-.5$	.5	0	0	1	0
.5	$-.5$	0	0	0	1
.25	$-.25$	0	0	.5	1
$-.25$	$-.25$	0	0	1	1
$-.25$	.25	0	0	1	.5

2. (a) This fractal consists of $N = 3$ pieces, each scaled by a factor of $r = 1/2$, so the dimension is $d = \log(3)/\log(2)$.

(b) This fractal consists of 5 pieces, with scaling factors $r_1 = 0.5$, $r_2 = r_3 = r_4 = r_5 = 0.25$. The Moran equation is

$$.5^d + 4 \cdot .25^d = 1$$

Taking $x = .5^d$ this becomes the quadratic equation $x + 4x^2 = 1$. This gives $x = (-1 + \sqrt{17})/8$ and so $d = \log((-1 + \sqrt{17})/8)/\log(1/2)$.

(c) This fractal consists of 6 pieces, with scaling fractors $r_1 = r_2 = r_3 = 0.5$, $r_4 = r_5 = r_6 = 0.25$. The Moran equation is

$$3 \cdot .5^d + 3 \cdot .25^d = 1$$

Taking $x = .5^d$ this becomes the quadratic equation $3x + 3x^2 = 1$. This gives $x = (-3 + \sqrt{21})/6$ and so $d = \log((-3 + \sqrt{21})/6)/\log(1/2)$.

3. Suppose for each $n > 0$, the minimum number of boxes of side length $\epsilon = 1/2^n$ needed to cover a fractal A is

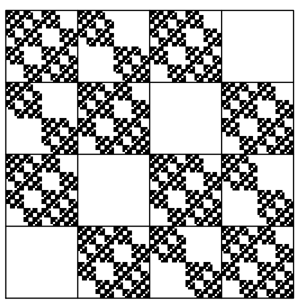
$$N(\epsilon) = n \cdot 3^n$$

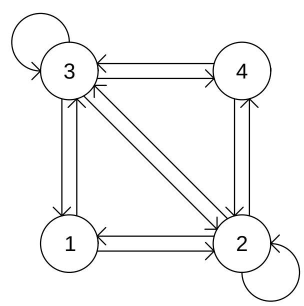
The box-counting dimension is

$$\begin{aligned} d &= \lim_{\epsilon \to 0} \frac{\log(N(\epsilon))}{\log(1/\epsilon)} \\ &= \lim_{n \to \infty} \frac{\log(n) + n\log(3)}{n\log(2)} \\ &= \lim_{n \to \infty} \frac{\log(n)}{n\log(2)} + \lim_{n \to \infty} \frac{\log(3)}{\log(2)} \\ &= \frac{\log(3)}{\log(2)} \end{aligned}$$

because $\lim_{n \to \infty} \log(n)/n = 0$.

4. (a) On the right is the transition graph that generates the IFS with memory on the left.





(b) Corners 2 and 3 are romes, there is a path from each rome to each non-rome, and there are no loops among non-romes, so this fractal can be generated by an IFS without memory. The transition graph gives the transformations and their compositions that constitute this IFS:

$$T_2, T_3, T_1 \circ T_2, T_1 \circ T_3, T_4 \circ T_2, T_4 \circ T_3$$

and so the IFS table is

r	s	θ	φ	e	f
.5	.5	0	0	.5	0
.5	.5	0	0	0	.5
.25	.25	0	0	.25	0
.25	.25	0	0	0	.25
.25	.25	0	0	.5	.75
.25	.25	0	0	.75	.5

(c) To find the dimension of this fractal, because two copies are scaled by .5 and four by .25 we use the Moran equation.

$$2 \cdot .5^d + 4 \cdot .25^d = 1$$

Taking $x = .5^d$, the Moran equation becomes $2x + 4x^2 = 1$, giving $x = (-1 + \sqrt{5})/4$ and so the dimension is $d = \log((-1+\sqrt{5})/4)/\log(1/2)$.

5. For this IFS

r	s	θ	φ	e	f	prob
.5	.5	0	0	0	0	0.3
.5	.5	0	0	.5	0	0.3
.25	.25	0	0	0	.5	0.025
.25	.25	0	0	.25	.5	0.025
.25	.25	0	0	0	.75	0.025
.25	.25	0	0	.25	.75	0.025
.5	.5	0	0	.5	.5	0.3

The minimum and maximum values of α are the minimum and maximum values of $\log(p_i)/\log(r_i)$, these are $\log(.3)/\log(.5) \approx 1.73697$ abd $\log(.025)/\log(.25) \approx 2.66096$. The set on which the minimum α occurs consists of $N = 3$ pieces, each scaled by $r = .5$, so

$$f(\alpha_{\min}) = \log(3)/\log(1/.5) = \log(3)/\log(2).$$

The set on which the maximum α occurs consists of $N = 4$ pieces, each scaled by $r = .25$, so

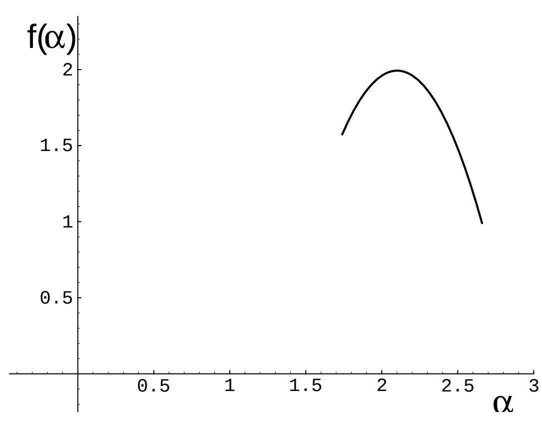
$$f(\alpha_{\max}) = \log(4)/\log(1/.25) = \log(4)/\log(4) = 1$$

The maximum value of the $f(\alpha)$ curve is the dimension of the IFS attractor. This consists of $N = 3$ pieces scaled by $r = .5$, and $N = 4$ pieces scaled by $r = .25$. The Moran equation is

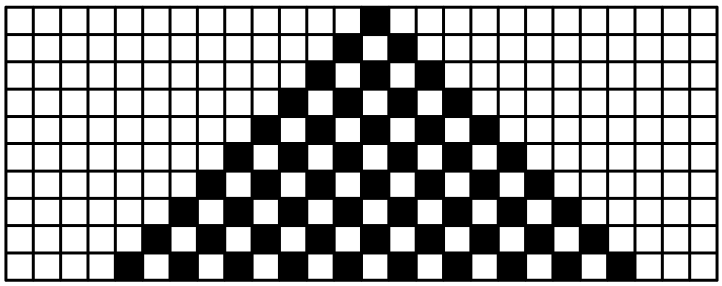
$$3 \cdot .5^d + 4 \cdot .25^d = 1$$

Taking $x = .5^d$, this becomes $3x + 4x^2 = 1$, giving $x = 1/4$, so

$$d = \log(x)/\log(.5) = \log(1/4)/\log(1/2) = 2$$



6. Consider the $N = 3$ CA with this rule, (L, D, D), (D, D, L), and (L, D, L) give L, all other nbhds give D. Here is the CA pattern evolving from a single live cell. In generation 1 the CA has 1 live cell, in generation 2 the CA has 2 live cells, in generation 3 the CA has 3 live cells, and so on. So in generation 10 the CA has 10 live cells. In generation 100 the CA will have 100 live cells.



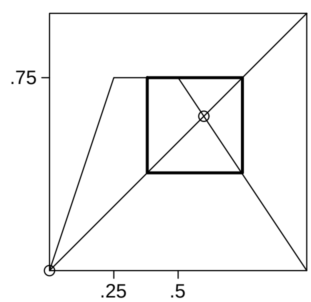
7. For N-cyclic components of the Mandelbrot set,

(a) $N = 17$ must not be the cycle number of a disc attached to a disc attached to the main cardioid, because 17 is prime and the cycle number of a disc attached to a disc must be composite.

(b) $N = 16$ might be be the cycle number of a disc attached to a disc attached to the main cardioid: the period-doubling disc off an 8-cycle disc, and also the 16-cycle disc attached to the cardioid.

(c) $N = 15$ might be be the cycle number of a disc attached to a disc attached to the main cardioid: the period-tripling disc off a 5-cycle disc, and also the 15-cycle disc attached to the cardioid.

8. Pictured here is the graph of a function f inside the unit square, together with the line $y = x$.

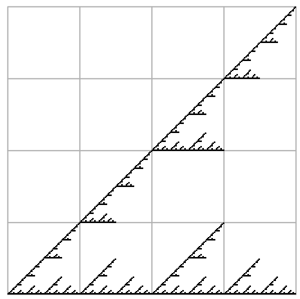


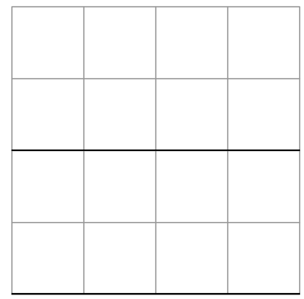
(a) The fixed points are circled. Both are unstable, because the slope of the graph of f is steeper than 1 at the fixed points.

(b) The 2-cycle is plotted as a thick square on the graph.

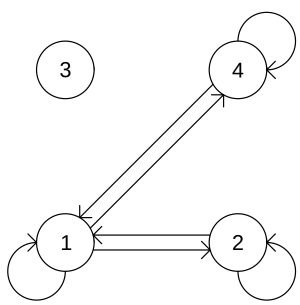
(c) A characteristic of chaos is sensitivity to initial conditions. Iteration of f does not exhibit chaos, because most all iterates converge to the 2-cycle.

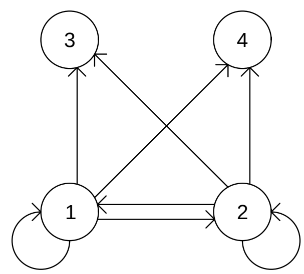
9. (a) To draw the transition graphs of these IFS, note the occupied length 2 addresses. For (i) these are 11, 12, 14, 21, 22, 41, 44. For (ii) these are 11, 12, 21, 22, 31, 32, 41, 42.

(i) 

(ii) 

These are the transitions graphs.

(i) 

(ii) 

9. (b) To gennerate driven IFS (i), the time series exhibits all combinations of points in bins 1 and 2, and all combinations of points in bins 1 and 4. For (ii) no time series will generate the driven IFS: as soon as a data point enters bin 3 or bin 4, it can go nowhere.