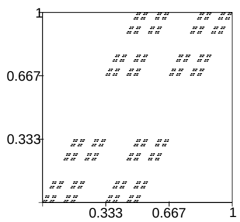
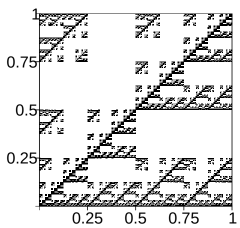
Final Exam Solutions

1. Fractals and their IFS generators.



r	s	θ	φ	e	f
.333	.333	0	0	0	0
.333	.333	0	0	.333	0
.333	.333	0	0	.333	.667
.333	.333	0	0	.667	.667



r	s	θ	φ	e	f
.5	.5	0	0	0	0
.5	.5	0	0	.5	.5
.25	.25	0	0	.5	0
.25	.25	0	0	.75	0
$-.25$	$-.25$	0	0	.25	1

2. (a) This fractal consists of $N = 4$ pieces, each scaled by a factor of $r = 1/3$, so the dimension is $d = \log(4)/\log(3)$.

(b) This fractal consists of 5 pieces, with scaling factors $r_1 = r_2 = 1/2$, $r_3 = r_4 = r_5 = 1/4$. The Moran equation is

$$2 \cdot (1/2)^d + 3 \cdot (1/4)^d = 1$$

Taking $x = (1/2)^d$ this becomes the quadratic equation $2x + 3x^2 = 1$. This gives $x = 1/3$ and so $d = \log(1/3)/\log(1/2) = \log(3)/\log(2)$.

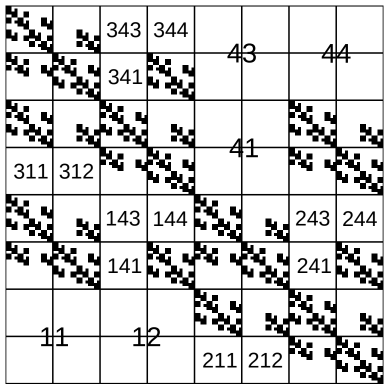
3. Suppose for each $n > 0$, the minimum number of boxes of side length $\epsilon = 1/2^n$ needed to cover a fractal A is

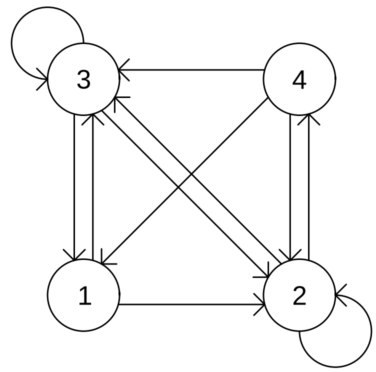
$$N(\epsilon) = 2 \cdot (2^n + 3^n)$$

The box-counting dimension is

$$\begin{aligned}
d &= \lim_{\epsilon \to 0} \frac{\log(N(\epsilon))}{\log(1/\epsilon)} \\
&= \lim_{n\to\infty} \frac{\log(2 \cdot (2^n + 3^n)}{\log(2^n)} \\
&= \lim_{n\to\infty} \frac{\log(2) + \log(3^n((2/3)^n + 1))}{\log(2^n)} \\
&= \lim_{n\to\infty} \frac{\log(2) + \log(3^n) + \log((2/3)^n + 1)}{\log(2^n)} \\
&= \lim_{n\to\infty} \frac{\log(2)}{n\log(2)} + \lim_{n\to\infty} \frac{n\log(3)}{n\log(2)} + \lim_{n\to\infty} \frac{\log((2/3)^n + 1)}{n\log(2)} \\
&= \frac{\log(3)}{\log(2)}
\end{aligned}$$

4. (a) On the left we see the IFS with memory. The forbidden pairs are 11, 12, 41, 43, and 44; the forbidden triples are 141, 143, 144, 211, 212, 241, 243, 244, 311, 312, 341, 343, and 344. Every forbidden pair contains a forbidden triple, so this IFS with memory can be generated by forbidden pairs. The transition graph is shown on the right: each arrow $i \to j$ corresponds to an allowed pair ji.





(b) From the transition graph, we see that 2 and 3 are romes, so 1 and 4 are non-romes.
For each non-rome, there ia a path from some rome to that non-rome: $3 \to 1$ and $2 \to 4$.
The only path through just non-romes is $4 \to 1$, so there is no loop among non-romes.
Consequently, this IFS with memory fractal can be generated with an IFS without memory. The transition graph gives the transformations and their compositions of this IFS:

$$T_2, T_3, T_4 \circ T_2, T_1 \circ T_3, T_1 \circ T_4 \circ T_2$$

and so the IFS table is

r	s	θ	φ	e	f
.5	.5	0	0	.5	0
.5	.5	0	0	0	.5
.25	.25	0	0	0	.25
.25	.25	0	0	.75	.5
.125	.125	0	0	.375	.25

(c) To find the dimension of this fractal, because two copies are scaled by 1/2, two copies are scaled by 1/4, and one by 1/8 the Moran equation becomes

$$2 \cdot (1/2)^d + 2 \cdot (1/4)^d + (1/8)^d = 1$$

Taking $x = (1/2)^d$, the Moran equation becomes $2x + 2x^2 + x^3 = 1$.

5. For this IFS

r	s	θ	φ	e	f	prob
.25	.25	0	0	0	0	0.05
.25	.25	0	0	.25	0	0.05
.25	.25	0	0	.25	.25	0.1
.25	.25	0	0	0	.25	0.1
.25	.25	0	0	.5	.5	0.1
.25	.25	0	0	.75	.75	0.3
.25	.25	0	0	.75	0	0.3

all the scaling factors r_i are the same, .25, so

$$\alpha_{\text{max}} = \frac{\log(\text{min prob})}{\log(.25)} = \frac{\log(.05)}{\log(.25)}$$
$$\alpha_{\text{min}} = \frac{\log(\text{max prob})}{\log(.25)} = \frac{\log(.3)}{\log(.25)}$$

Next, $f(\alpha_{\text{min}})$ is the dimension of the attractor of the IFS consisting of T_6 and T_7, the transformations having the highest probability. Both these transformations have scaling $r = .25$, so

$$f(\alpha_{\text{min}}) = \frac{\log(2)}{\log(1/.25)} = \frac{\log(2)}{\log(4)} = \frac{1}{2}$$

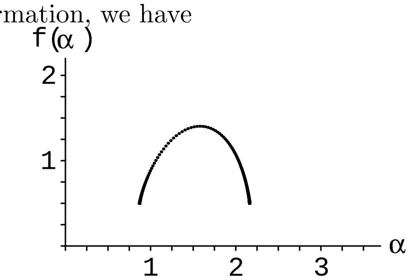
Similarly, $f(\alpha_{\text{max}})$ is the dimension of the attractor of the IFS consisting of T_1 and T_2, the transformations having the lowest probability. Both these transformations have scaling $r = .25$, so

$$f(\alpha_{\text{max}}) = \frac{\log(2)}{\log(1/.25)} = \frac{\log(2)}{\log(4)} = \frac{1}{2}$$

Finally, the maximum value of $f(\alpha)$ is the dimension of the attractor of the fractal generated by the whole IFS. This fractal consists of $N = 7$ pieces, each scaled by $r = .25$. Then the dimension is

$$\frac{\log(7)}{\log(1/.25)} = \frac{\log(7)}{\log(4)}$$

Combining this information, we have

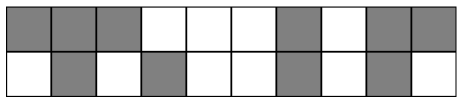


6. (a) Starting with the three consequtive live cells on the left, reading the neighborhood configurations from left to right we find

$$LLL \to L, LLD \to D, LDD \to L, DDD \to D$$

$$DDL \to D, DLD \to L, LDL \to D, DLL \to L$$

This gives a L or D cell outcome to each of the 8 neighborhood configurations. No configuration occurs twice on the list, so there is no possibility of inconsistent assignments.

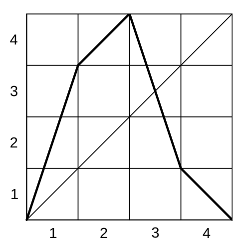


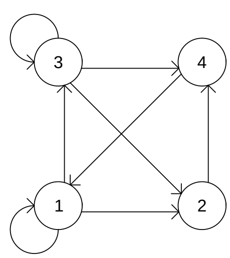
(b) Considering the three left-most cells, we find $LL \to L$. With wrap-around, the two left-most cells and the right-most cell give $LLL \to D$. A CA cannot give both a D cell and a L cell to a single neighborhood configuration, so no $N = 3$ CA can generate the bottom line from the top, if wrap-around is used.

7. Using the multiplier rule, all combinations of cycle numbers for A and B must result from the factorization of 30, namely $30 = 2 \cdot 3 \cdot 5$.

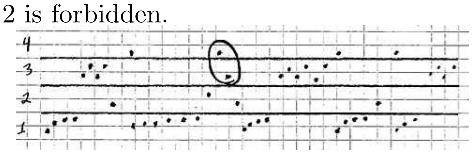
B	A	description
2	$2 \cdot 15$	15 off the 2
3	$3 \cdot 10$	10 off a 3
5	$5 \cdot 6$	6 off a 5
6	$6 \cdot 5$	5 off a 6
10	$10 \cdot 3$	3 off a 10
15	$15 \cdot 2$	2 off a 15

8. (a) Iterating the function graphed on the left gives the graph on the right.



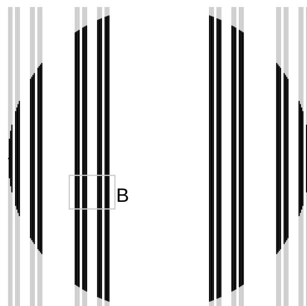


(b) This time series cannot be generated by iterating this function. The circled portion shows a transition $4 \to 3$, forbidden for this function. Also, the $2 \to 1$ after the $4 \to 3 \to 2$ is forbidden.



9. Call the fractal, the dark part of this figure, A. Then certainly A is contained in the product, $C \times I$, of a Cantor middle-thirds set and the unit interval. By the monotonicity rule and the product rule

$$\dim(A) \leq \dim(C \times I) = \dim(C) + \dim(I) = \log(2)/\log(3) + 1$$



Next, the portion of A labled B is a copy of $C \times I$, scaled by 1/9 and so

$$\dim(B) = \dim(C \times I) = \log(2)/\log(3) + 1$$

Applying monotomicity again, we see

$$\dim(B) \leq \dim(A)$$

Combining these results, we find

$$\begin{aligned} \log(2)/\log(3) + 1 &= \dim(B) \\ &\leq \dim(A) \\ &\leq \dim(C \times I) \\ &= \log(2)/\log(3) + 1 \end{aligned}$$

That is,

$$\dim(A) = \log(2)/\log(3) + 1$$