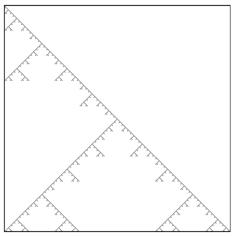
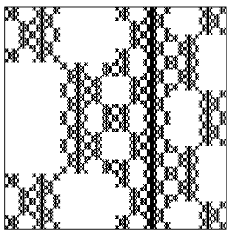
Final Exam Solutions

1. Fractals and their IFS generators.



r	s	θ	φ	e	f
$-.333$	.333	0	0	.333	0
.333	.333	0	0	.667	0
.333	.333	0	0	.333	.333
.333	.333	0	0	0	.667



r	s	θ	φ	e	f
$-.5$	.5	0	0	1	0
$-.5$	.5	0	0	1	.5
.25	.25	0	0	0	0
.25	.25	0	0	0	.75
$-.25$	.25	0	0	.5	.25
$-.25$	.25	0	0	.5	.5

or

r	s	θ	φ	e	f
.5	.5	180	180	1	.5
.5	.5	180	180	1	1
.25	.25	0	0	0	0
.25	.25	0	0	0	.75
.25	.25	180	180	.5	.5
.25	.25	180	180	.5	.75

2. (a) This fractal consists of $N = 4$ pieces, each scaled by a factor of $r = 1/3$, so the dimension is $d = \log(4)/\log(3)$.

(b) This fractal consists of 6 pieces, with scaling factors $r_1 = r_2 = 1/2$, $r_3 = r_4 = r_5 = r_6 = 1/4$. The Moran equation is

$$2 \cdot (1/2)^d + 4 \cdot (1/4)^d = 1$$

Taking $x = (1/2)^d$ this becomes the quadratic equation $2x + 4x^2 = 1$. This gives $x = (-1 + \sqrt{5})/4$ and so $d = \log((-1 + \sqrt{5})/4)/\log(1/2)$.

3. Suppose for each $n > 0$, the minimum number of boxes of side length $\epsilon = 1/4^n$ needed to cover a fractal is

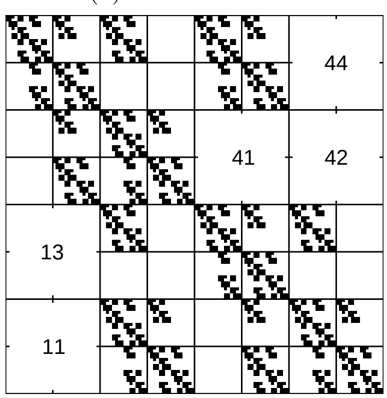
$$N(\epsilon) = 2^n + 3^n + 5$$

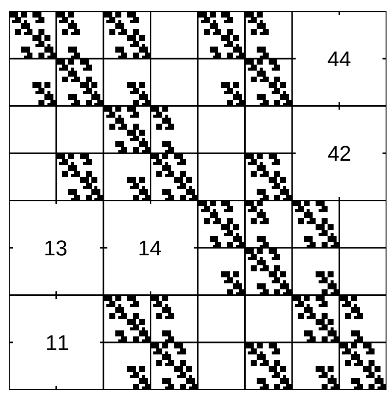
Then the box-counting dimension is

$$
\begin{aligned}
d &= \lim_{\epsilon\to 0} \frac{\log(N(\epsilon))}{\log(1/\epsilon)} \\
&= \lim_{n\to\infty} \frac{\log(2^n+3^n+5)}{\log(4^n)} \\
&= \lim_{n\to\infty} \frac{\log(3^n((2/3)^n+1+5/3^n)}{\log(4^n)} \\
&= \lim_{n\to\infty} \frac{\log(3^n)+\log((2/3)^n+1+5/3^n)}{\log(4^n)} \\
&= \lim_{n\to\infty} \frac{n\log(3)}{n\log(4)} + \lim_{n\to\infty} \frac{\log((2/3)^n+1+5/3^n)}{n\log(4)} \\
&= \frac{\log(3)}{\log(4)}
\end{aligned}
$$

4. In driven IFS (a) we see the empty length-2 addresses are 11, 13, 41, 42, and 44; in driven IFS (b) we see the empty length-2 addresses are 11, 13, 14, 42, and 44.

In the graph we see the forbidden transtions are $1 \to 1$, $1 \to 4$, $2 \to 4$, $3 \to 1$, and $4 \to 4$. The IFS driven by iterating this function will have forbidden addresses 11, 41, 42, 13, and 44. That is, iterating this function produces the driven IFS (a).





5. For this IFS

r	s	θ	φ	e	f	prob
.25	.25	0	0	0	0	0.2
.25	.25	0	0	.25	0	0.2
.25	.25	0	0	.5	0	0.1
.25	.25	0	0	.75	0	0.1
.25	.25	0	0	.25	.25	0.2
.25	.25	0	0	.5	.5	0.1
.25	.25	0	0	0	.75	0.025
.25	.25	0	0	.25	.75	0.025
.25	.25	0	0	.5	.75	0.025
.25	.25	0	0	.75	.75	0.025

because all the swcaling factors are the same, 1/4, the maximum probability corresponds to the minimum α, and the minimum probability corresponds to the maximum α. Then

$$\alpha_{\text{min}} = \frac{\log(\text{max prob})}{\log(1/4)} = \frac{\log(0.2)}{\log(1/4)}$$
$$\alpha_{\text{max}} = \frac{\log(\text{min prob})}{\log(1/4)} = \frac{\log(0.025)}{\log(1/4)}$$

Next, $f(\alpha_{\text{min}})$ is the dimension of the shape determined by the maximum probability. This is the shape determined by $N = 3$ transformations, both with $r = 1/4$, so

$$f(\alpha_{\text{min}}) = \frac{\log(3)}{\log(4)}$$

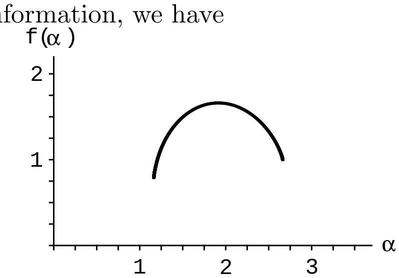
Similarly, $f(\alpha_{\text{max}})$ is the dimension of the shape determined by the minimum probability. This is the shape determined by $N = 4$ transformations, both with $r = 1/4$, so

$$f(\alpha_{\text{max}}) = \frac{\log(4)}{\log(4)} = 1$$

Finally, the maximum value of $f(\alpha)$ is the dimension of the whole IFS. That is,

$$\text{maximum } \{f(\alpha)\} = \frac{\log(10)}{\log(4)}$$

Combining this information, we have

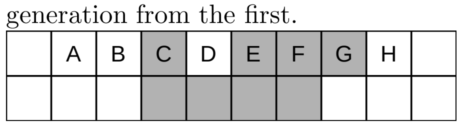


6. (a) To show there is no $N = 3$ CA producing this pattern, in the first generation we nust find two copies of a neighborhood configuration, one making the central cell live, the other making the central cell alive. Otherwise, list all the neighborhood configurations giving a live cell, and say all the other configurations make the central cell dead.

First, note that cells C, D, E, and F give life cells. That is, the CA rule has

$$DLD \to L(C), LDL \to L(D), DLL \to L(E), LLL \to L(F)$$

No configuration occurs in this list, so the CA with these four configurations making the central cell alive and all others making the central cell dead does produce the second generation from the first.



(b) To test if there are other CA producing the second gewneration from the first, observe the configurations making the central cell dead:

$$DDD \to D(A), DDL \to D(B), LLD \to D(G), LDD \to D(H)$$

All other configurations in the first generation are $DDD \to D$. The neighborhoods of A through H constitute all eight $N = 3$ configurations, so this is the only CA producing the second generation from the first.

7. To find the number of 35 cycle discs attached to a 5-cycle disc and the number attached to a 7-cycle disc, we find the number of 5-cycle discs and the number of 7-cycle discs attached to the main cardioid, and apply the multiplier rule.

Because the Mandelbrot set is symmetric across the x-axis, we count the number of these components on the upper part of the Mandelbrot set, then multiply by 2.

The only 5-cycle components attached to the main cardioid are the principal series 5-cycle disc and the Farey sequence 5-cycle disc between the principal 2- and 3-cycle discs. This gives four 5-cycle discs attached to the main cardioid.

The 7-cycle discs attached to the main cardioid are the principal series 7-cycle disc, the Farey 7-cycle disc between the principal 3- and 4-cycle discs, and the 7-cycle disc between the principal series 2-cycle and the Farey 5-cycle discs. This gives six 7-cycle discs attached to the main cardioid.

By the multiplier rule, the number of 35-cycle discs attached to a 5-cycle disc equals the number fo 7-cycle discs attached to the main cardioid, so six. Similarly, the number of 35-cycle discs attached to a 7-cycle disc equals the number fo 5-cycle discs attached to the main cardioid, so four. Consequently, there are more 35-cycle discs attached to a 5-cycle disc.

8. (a) We apply the randomized Moran equation

$$E(r_1^d) + E(r_2^d) = 1$$

From the description of the scaling factors, we see

$$E(r_1^d) = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^d + \frac{1}{2} \cdot \left(\frac{1}{4}\right)^d$$
$$E(r_2^d) = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^d + \frac{1}{2} \cdot \left(\frac{1}{4}\right)^d$$

Then the randomized Moran equation becomes

$$\left(\frac{1}{2}\right)^d + \left(\frac{1}{4}\right)^d = 1$$

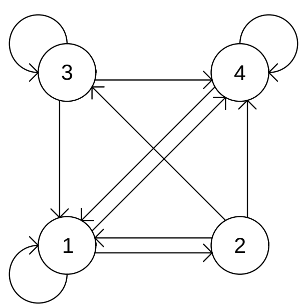
Setting $x = (1/2)^d$, this is the quadratic equation $x + x^2 = 1$. The positive solution is $x = (-1+\sqrt{5})/2$, and so the expected value of the dimension is

$$d = \frac{\log((-1+\sqrt{5})/2)}{\log(1/2)}$$

(b) Call the two copies of the Cantor set A and B. Then by the intersection formula,

$$\begin{aligned} d(A \cap B) &= d(A) + d(B) - 2 \\ &= \frac{\log((-1+\sqrt{5})/2)}{\log(1/2)} + \frac{\log((-1+\sqrt{5})/2)}{\log(1/2)} - 2 \end{aligned}$$

9. From the transition graph, we see that 1 and 4 are romes, so addresses 1 and 4 are copies of the whole fractal scaled by 1/2.
The transition $1 \to 2$ gives a copy scaled by 1/4 in address 21.
The transition $1 \to 2 \to 3$ gives a copy scaled by 1/8 in address 321.
The transition $1 \to 2 \to 3 \to 3$ gives a copy scaled by 1/16 in address 3321.
and so on.



Combining these, the Moran equation becomes

$$2 \cdot (1/2)^d + (1/4)^d + (1/8)^d + (1/16)^d + \cdots = 1$$

As usual, taking $x = (1/2)^d$, we find

$$\begin{aligned}
x + x + x^2 + x^3 + x^4 + \cdots &= 1 \\
x + x\left(1 + x + x^2 + x^3 + \cdots\right) &= 1 \\
x + \frac{x}{1-x} &= 1 && \text{summing the geometric series} \\
x - x^2 + x &= 1 - x && \text{multiplying by } 1 - x
\end{aligned}$$

This is the quadratic equation $x^2 - 3x + 1 = 0$. The solutions are

$$x = \frac{3 \pm \sqrt{5}}{2}$$

Both are positive, but recall that in order to sum the geometric series, we must have $|x| < 1$. Consequently, $x = (3 - \sqrt{5})/2$. This gives the dimension

$$d = \frac{\log((3 - \sqrt{5})/2)}{\log(1/2)}$$