$$d = \frac{\text{Log}(N)}{\text{Log}(1/r)}$$

$$d\,\text{Log}\left(\frac{1}{r}\right) = \text{Log}(N)$$

$$\text{Log}\left(\left(\frac{1}{r}\right)^d\right) = \text{Log}(N)$$

$$\left(\frac{1}{r}\right)^d = N$$

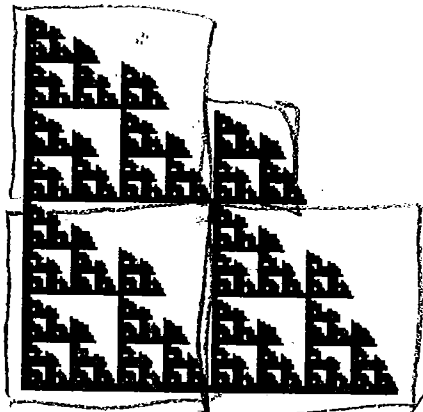
$$\frac{1}{r^d} = N$$

$$1 = N \cdot r^d$$

$$1 = \underbrace{r^d + r^d + \cdots + r^d}_{N \text{ terms}}$$

For a fractal made of N pieces, scaled by $r_1, r_2, \ldots, r_N$, the dimension is the solution of $r_1^d + r_2^d + \cdots + r_N^d = 1$

Moran Equation



$r_1 = 1/2, \quad r_2 = 1/2, \quad r_3 = 1/2, \quad r_4 = 1/4$

$$\left(\frac{1}{2}\right)^d + \left(\frac{1}{2}\right)^d + \left(\frac{1}{2}\right)^d + \left(\frac{1}{4}\right)^d = \underline{1}$$

Let $x = \left(\frac{1}{2}\right)^d$

Then $\left(\frac{1}{4}\right)^d = \left(\left(\frac{1}{2}\right)^2\right)^d = \left(\frac{1}{2}\right)^{2 \cdot d}$

$$= \left(\frac{1}{2}\right)^{d \cdot 2} = \left(\left(\frac{1}{2}\right)^d\right)^2 = x^2$$

$$x + x + x + x^2 = 1$$

$$3x + x^2 = 1$$

$$x^2 + 3x - 1 = 0$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1 \quad b = 3 \quad c = -1$

$$x = \frac{-3 \pm \sqrt{3^2 - 4 \cdot (1) \cdot (-1)}}{2 \cdot 1}$$

$$x = \frac{-3 \pm \sqrt{13}}{2}$$

Recall $x = \left(\frac{1}{2}\right)^d > 0$

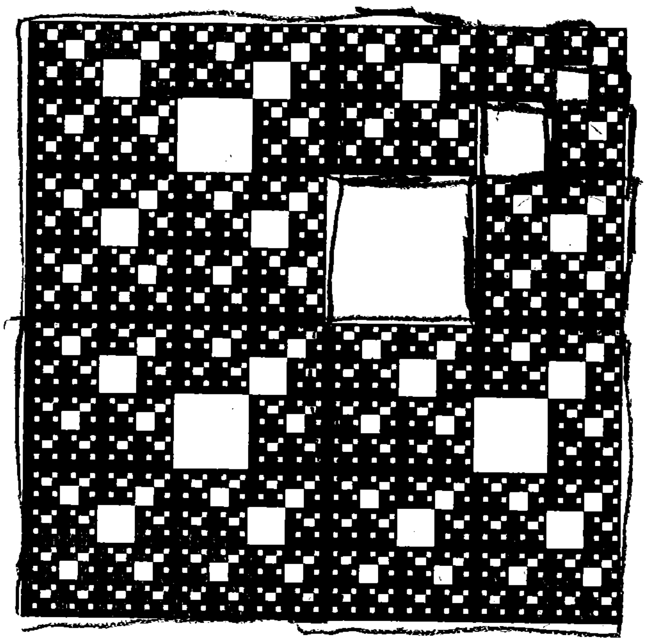
Then $x = \dfrac{-3+\sqrt{13}}{2}$

$$\left(\frac{1}{2}\right)^d = \frac{-3+\sqrt{13}}{2}$$

$$\text{Log}\left(\left(\frac{1}{2}\right)^d\right) = \text{Log}\left(\frac{-3+\sqrt{13}}{2}\right)$$

$$d\,\text{Log}\left(\frac{1}{2}\right) = \text{Log}\left(\frac{-3+\sqrt{13}}{2}\right)$$

$$d = \frac{\text{Log}\left(\frac{-3+\sqrt{13}}{2}\right)}{\text{Log}\,(1/2)} \approx 1.724$$



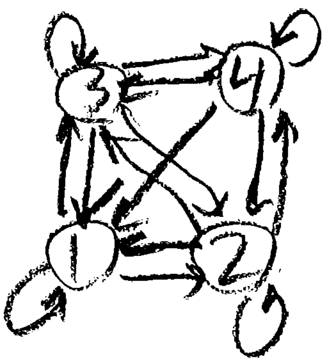
3 copies scaled by $1/2$

2 copies scaled by $1/4$

2 copies scaled by $1/8$

2 copies scaled by $1/16$

$\vdots$



3 romes give 3 copies scaled by $1/2$

$2 \to 4$, $3 \to 4$ give 2 copies scaled by $1/4$

$4 \to 4$ gives 2 copies scaled by $1/8, 1/16, 1/32 \ldots$

Moran equation

$$3\left(\tfrac{1}{2}\right)^d + 2\left(\tfrac{1}{4}\right)^d + 2\left(\tfrac{1}{8}\right)^d + 2\left(\tfrac{1}{16}\right)^d + \cdots = 1$$

Write $x = \left(\tfrac{1}{2}\right)^d$, so $\left(\tfrac{1}{4}\right)^d = x^2$, $\left(\tfrac{1}{8}\right)^d = x^3, \ldots$

$$3x + 2x^2 + 2x^3 + 2x^4 + \cdots = 1$$

If $|x| < 1$, then the geometric series

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$$

$$3x + 2x^2 + 2x^3 + 2x^4 + \cdots = 1$$

$$x(1 + x + x^2 + x^3 + x^4 + \cdots) = x \cdot \frac{1}{1-x}$$

$$x + x^2 + x^3 + x^4 + x^5 + \cdots = \frac{x}{1-x}$$

$$x + 2x + 2x^2 + 2x^3 + 2x^4 + \cdots = 1$$

$$x + 2(x + x^2 + x^3 + x^4 + \cdots) = 1$$

$$x + 2\left(\frac{x}{1-x}\right) = 1$$

$$x + \frac{2x}{1-x} = 1$$

$$x(1-x) + 2x = 1 - x$$

$$x - x^2 + 2x = 1 - x$$

$$0 = x^2 - 4x + 1 \qquad a=1 \quad b=-4 \quad c=1$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$x = \frac{4 \pm \sqrt{12}}{2}$$

$$= \frac{4 \pm \sqrt{4 \cdot 3}}{2}$$

$$= \frac{4 \pm 2\sqrt{3}}{2}$$

$$x = 2 \pm \sqrt{3}$$

Recall that to solve $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$

we must have $|x| < 1$

$x = 2 + \sqrt{3} > 1$

So $x = 2 - \sqrt{3}$

$$\left(\tfrac{1}{2}\right)^d = 2 - \sqrt{3}$$

$$d \operatorname{Log}\left(\tfrac{1}{2}\right) = \operatorname{Log}(2 - \sqrt{3})$$

$$d = \frac{\operatorname{Log}(2 - \sqrt{3})}{\operatorname{Log}(1/2)} \approx 1.900$$