Geometry of configurations, polylogarithms and motivic cohomology

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Introduction

1. The classical p–logarithm function is defined in the unit disc $|z| \le 1$ by the absolutely convergent series

$$\mathrm{Li}_p(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^p} .$$

It has been investigated widely during the last 200 years – see the book of L. Lewin [L].

The most extensive literature exists for the dilogarithm $\mathrm{Li}_2(z)$, defined by Leibniz (1696) and studied by L. Euler (1776), W. Spence (1809), N.–H. Abel (1828), E. Kummer (1840), ... ([L]). One of the most interesting results was the functional equation for the dilogarithm that generalizes the addition formula $\ln x + \ln y = \ln(xy)$ for the logarithm $(x,y > 0)$.

In the middle seventies the dilogarithm appeared surprisingly in the work of A.M. Gabrielov, I.M. Gelfand and M.V. Losik [GGL] on the combinatorial formula for the first Pontrjagin class, of S. Bloch [Bl 1] in algebraic K–theory and values of zeta–functions at the point 2, and of D. Wigner in continuous cohomology of $GL_2(\mathbb{C})$.

In this paper we propose a geometrical approach to the theory of the classical trilogarithm function $Li_3(z)$ based on the study of configurations of 6 points on the projective plane, and obtain analogues of most of all the above mentioned results related with the classical trilogarithm, including:

a) the generic functional equation for $Li_3(z)$;

b) its connection with algebraic K–theory, weight 3 motivic cohomology, characteristic classes and an explicit formula for a 5–cocycle representing a continuous cohomology class of $GL_3(\mathbb{C})$;

c) the proof of D. Zagier's conjecture [Z3]: the value of the Dedekind zeta–function of an arbitrary number field at the point 3 is expressed by an $(r_1 + r_2)$-determinant whose entries are rational linear combinations of values of the classical trilogarithm at (complex embeddings of) some elements of this field.

2. In § 1 of this paper we construct for an arbitrary field F a complex $\Gamma_F(n)$ that hypothetically after $\otimes \mathbb{Q}$ should give weight n motivic cohomology of Spec F .

Namely, let $\mathbb{Z}[P^1_F]$ be the free abelian group generated by symbols $\{x\}$, where $x \in P^1_F$. We define for every $n \geq 1$ a certain subgroup $\mathcal{R}_n(F) \subset \mathbb{Z}[P^1_F]$ reflecting the functional equations for the classical n–logarithm function (for the precise definition see s. 9 of § 1). For example, $\mathcal{R}_1(F)$ is the subgroup generated by the elements $\{xy\} - \{x\} - \{y\}$ where $x,y \in F^*\backslash 1 \subset P^1_F\backslash\{0,1,\infty\}$, reminiscent of the functional equation for $\log|\cdot|$. We set

$$\mathcal{B}_n(F) := \mathbb{Z}[P^1_F] / \mathcal{R}_n(F) .$$

Note that $\mathcal{B}_1(F) = F^*$. Then we construct the following complex

$$\mathcal{B}_n(F) \xrightarrow{\delta} \mathcal{B}_{n-1}(F) \otimes F^* \xrightarrow{\delta} \mathcal{B}_{n-2}(F) \otimes \Lambda^2 F^* \xrightarrow{\delta} \dots$$
$$\xrightarrow{\delta} \mathcal{B}_2(F) \otimes \Lambda^{n-2} F^* \xrightarrow{\delta} \Lambda^n F^* \qquad (0.1)$$

The differential δ is defined by the following formulae ($\{x\}_m$ is the image of a generator $\{x\}$ in $\mathcal{B}_m(F)$)

$$\delta : \{x\}_m \otimes y_1 \wedge \dots \wedge y_{n-m} \longmapsto \{x\}_{m-1} \otimes x \wedge y_1 \wedge \dots \wedge y_{n-m} \qquad (0.2)$$

if $m \geq 3$ and

$$\delta : \{x\}_2 \otimes y_1 \wedge \dots \wedge y_{n-2} \longmapsto (1-x) \wedge x \wedge y_1 \wedge \dots \wedge y_{n-2} . \qquad (0.3)$$

Then $\delta^2 = 0$ modulo 2–torsion .

Let us denote this complex, where $\mathcal{B}_n(F)$ is placed in degree 1, by $\Gamma_F(n)$.

For an abelian group A we set $A_{\mathbb{Q}} := A \otimes \mathbb{Q}$.

<u>Conjecture A</u> .

$$K_n(F)_{\mathbb{Q}} \cong \bigoplus_{i=0}^{[n/2]} H^{n-2i}(\Gamma_F(n)_{\mathbb{Q}}) .$$

For a more precise conjecture see s. 9 of § 1.

The existence of such complexes was conjectured by A.A. Beilinson [B 1] and S. Lichtenbaum [L 1]. Another construction of complexes that hypothetically should satisfy all Beilinson–Lichtenbaum axioms was proposed by S. Bloch [Bl 2] and S. Landsburg [La].

For number fields $H^i(\Gamma_F(n)_{\mathbb{Q}})$ should be zero for $i \geq 2$ and $K_n(F)_{\mathbb{Q}} = \mathrm{Ker}\,\delta \subset \mathcal{B}_n(F)$. In this case our conjecture coincides with Zagier's conjecture [Z 3].

Weight 2 motivic complexes $\Gamma(X;2)$ for a regular scheme X were constructed by S. Lichtenbaum [L2]. In s. 14 of § 1 we suggest a construction of weight 3 motivic complexes $\Gamma(X;3) \otimes \mathbb{Q}$ for a regular scheme X and – more generally – weight n complexes $\Gamma(X;n) \otimes \mathbb{Q}$ for a smooth curve over an arbitrary field F.

3. A.A. Beilinson conjectured [B 1] that there should exist a mixed Tate category $\mathcal{M}_T(F)$ of mixed Tate motivic sheaves over spec F. So the usual Tannakian arguments tell us that there should exist some graded pro–Lie algebra

$$L(F)_\bullet = \bigoplus_{i=-1}^{-\infty} L(F)_i$$

such that the category of finite dimensional graded representations of $L(F)_\bullet$ is equivalent to the category $\mathcal{M}_T(F)$ (see also s. 10 of § 1). Let $\mathbb{Q}(n)_{\mathcal{M}}$ be the trivial 1–dimensional $L(F)_\bullet$–module placed at degree $-n$ and let γ be the γ–filtration on K–groups [So].

Conjecture (0.1) (A.A. Beilinson [Be 1])

$$\mathrm{Ext}^{\,i}_{\mathcal{M}_T(F)}(\mathbb{Q}(0)_{\mathcal{M}},\mathbb{Q}(n)_{\mathcal{M}}) = \mathrm{gr}^n_\gamma K_{2n-i}(F)_{\mathbb{Q}} .$$

Let us denote by $W^\vee$ the dual space to a vector space W over $\mathbb{Q}$. If W is a profinite vector space, then $W^\vee$ will be an inductive limit of finite dimensional vector spaces and vice versa. Note that $\mathbb{Q}[P^1_F]$ is an inductive limit of finite dimensional vector spaces and, as we will see below, the same is true for $B_n(F)_{\mathbb{Q}}$. So, for example, $F^{*\vee}_{\mathbb{Q}}$ is a profinite vector space.

Conjecture 0.1 is the case $i = 1$; $n = 1$ just means that

$$\mathrm{Ext}^{\,1}_{\mathcal{M}_T(F)}(\mathbb{Q}(0)_{\mathcal{M}},\mathbb{Q}(1)_{\mathcal{M}}) = L(F)^\vee_{-1} \cong F^*_{\mathbb{Q}} .$$

Set $L(F)_{\le -2} := \bigoplus_{i=-2}^{-\infty} L(F)_i$. The space of degree $-n$ generators of the Lie algebra $L(F)_{\le -2}$ is isomorphic to the degree $-n$ subspace of the graded vector space $L(F)_{\le -2}/[L(F)_{\le -2}, L(F)_{\le -2}]$. The Lie algebra $L(F)$ acts on $L(F)_{\le -2}/[L(F)_{\le -2}, L(F)_{\le -2}]$ through its abelian quotient $L(F)/L(F)_{\le -2} \cong F^{*\vee}_{\mathbb{Q}}$. It turns out that in Beilinson's World (*a world where his conjectures are theorems*) Conjecture A is equivalent to the following

Conjecture B.

a) $L(F)_{\le -2}$ is a free graded pro–Lie algebra such that the dual of the space of its degree $-n$ generators is isomorphic to $\mathcal{B}_n(F)_{\mathbb{Q}}$.

b) The dual map to the action of the quotient $L(F)/L(F)_{\leq -2}$ on the space of degree $-(n-1)$ generators of $L(F)_{\leq -2}$ is just the differential

$$\delta : \mathcal{B}_n(F)_{\mathbb{Q}} \longrightarrow \mathcal{B}_{n-1}(F)_{\mathbb{Q}} \otimes F^*_{\mathbb{Q}}$$

in the complex (0.1).

In this paper we will give strong evidence for this conjecture for $n \leq 5$.

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CONTENTS

Let me give some comments that may be helpful to read this paper. Most of all important results and conjectures are formulated in § 1. Moreover, s. 9–15 of § 1 are completely independent from the rest of the paper. To understand the proof of Zagier's conjecture it is sufficient to read s. 0–4, 7 of § 1 and § 3, 4, 6, 10 only. Most important are s. 3 of § 4 and Theorem 4.2. The long calculations in s. 1 of § 5 are given in order to write the explicit formula 1.10 for the functional equation for the trilogarithm; we don't use this explicit formula, only its geometrical interpretation given by Theorem 1.4 (see also s. 6 of § 1); s. 2 of § 5 is a detailed exposition of s. 6 of § 1. The results of § 7 are of independent interest.

§ 1. Main results and conjectures.

0. The single–valued versions of p–logarithms. Note that

$$\mathrm{Li}_1(z) = -\log(1-z)\ ;\ \frac{d}{dz}\,\mathrm{Li}_p(z) = \mathrm{Li}_{p-1}(z)\ d\log z\ . \tag{1.1}$$

So using the inductive formula

$$\mathrm{Li}_p(z) = \int_0^z \mathrm{Li}_{p-1}(t)\,\frac{dt}{t}$$

the p–logarithm can be analytically continued to a multivalued function on $\mathbb{C}P^1\backslash 0,1,\infty$. However, S. Bloch and D. Wigner introduced the function

$$D_2(z) := \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\cdot\log|z|$$

which is single–valued, real–analytic on $\mathbb{C}P^1\backslash 0,1,\infty$ and continuous (but not differentiable) at $0,1,\infty$. It has a singularity of type $x\cdot\ln x$ at these points and

$$D_2(0) = D_2(1) = D_2(\infty) = 0\ . \tag{1.2}$$

It is called the Bloch–Wigner function.

The corresponding function for $\log z$ is just $\log|z|$. Analogous functions $D_p(z)$ for $p \geq 3$ were introduced in [R] and computed explicitly in [Z1].

However, let us consider the slightly modified function

$$\mathcal{L}_3(z) := \mathrm{Re}\Big[\ \mathrm{Li}_3(z) - \log|z| \cdot \mathrm{Li}_2(z) + \tfrac{1}{3}\log^2|z| \cdot \mathrm{Li}_1(z)\Big]\ . \qquad (1.3)$$

Note that $D_3(z) := \mathcal{L}_3(z) + \frac{1}{12}\log^2|z| \cdot \log\left|\frac{z}{(1-z)^2}\right|$. $\mathcal{L}_3(z)$ is single–valued, real–analytic on $\mathbb{C}P^1\backslash 0,1,\infty$, and continuous at $0,1,\infty$. We have

$$\mathcal{L}_3(0) = \mathcal{L}_3(\infty) = 0\ ,\ \ \mathcal{L}_3(1) = \sum_{n=1}^{\infty} \frac{1}{n^3} \equiv \zeta_{\mathbb{Q}}(3)\ . \qquad (1.4)$$

(The advantage of the function $\mathcal{L}_3(z)$ is that, as we will see below, it satisfies functional equations without remainder terms.)

Such modified functions $\mathcal{L}_p(z)$ for all $p \geq 3$ were considered by D. Zagier, A.A. Beilinson and P. Deligne; in [B 1] and [De 2] the Hodge–theoretic interpretation of the functions $D_p(z)$ and $\mathcal{L}_p(z)$ is given. The definition of these functions is as follows ([Z 3]):

$$\mathcal{L}_p(z) = \mathcal{R}_p\Big(\sum_{j=0}^{p} \frac{2^j \cdot B_j}{j\,!} (\log|z|)^j \cdot \mathrm{Li}_{m-j}(z)\Big)$$

where B_j is the j–th Bernoulli number ($B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6, \ldots$) and $\mathcal{R}_m$ is the real part for odd m and the imaginary part for even m , $\mathrm{Li}_0(z) := -1/2$.

1. $\underline{\mathcal{L}_3(z)}$ <u>and</u> $\underline{\zeta_F(3)}$ <u>for a number field F</u> . Let $\mathbb{Z}[P^1_F\backslash 0,1,\infty]$ be the free abelian group, generated by symbols $\{x\}$, where $x \in P^1_F\backslash 0,1,\infty$ (F is a field).

There is a homomorphism

$$\mathcal{L}_3 : \mathbb{Z}[P^1_{\mathbb{C}} \backslash 0,1,\infty] \longrightarrow \mathbb{R}\,, \quad \mathcal{L}_3 : \Sigma\, n_i\{x_i\} \longmapsto \Sigma\, n_i\, \mathcal{L}_3(x_i)\,.$$

A similar homomorphism can be defined for any $\mathbb{R}$–valued function.

Now let F be an arbitrary algebraic number field, d_F the discriminant of F, r_1 resp. r_2 the number of real resp. complex places, so $[F:\mathbb{Q}] = r_1 + 2r_2$, and δ_j the set of all possible embeddings $F \hookrightarrow \mathbb{C}$, $(1 \le j \le r_1 + 2r_2)$ numbered so that $\overline{\sigma_{r_1+k}} = \sigma_{r_1+r_2+k}$.

Let us denote by $R_2(F)$ the subgroup of $\mathbb{Z}[P^1_F \backslash 0,1,\infty]$ generated by the expressions

$$\{x\} - \{y\} + \{y/x\} - \left\{\frac{1-y^{-1}}{1-x^{-1}}\right\} + \left\{\frac{1-y}{1-x}\right\},$$

where $x \neq y$, $x \neq 1$, $y \neq 1$.

Set $B_2(F) := \mathbb{Z}[P^1_F]/R_2(F)$.

Let us consider the following homomorphism

$$\delta : \mathbb{Q}[P^1_F \backslash 0,1,\infty] \longrightarrow B_2(F)_{\mathbb{Q}} \otimes F^*_{\mathbb{Q}}$$

(1.5)

$$\delta : \{x\} \longmapsto \{x\}_2 \otimes x$$

($\{x\}_2$ is the projection of $\{x\}$ onto $B_2(F)$).

Theorem 1.1. Let $\zeta_F(s)$ be the Dedekind zeta–function of F. Then there exist

$$y_1, \dots, y_{r_1+r_2} \in \operatorname{Ker} \delta \subset \mathbb{Q}[P^1_F \backslash 0,1,\infty]$$

such that

$$\zeta_F(3) = \pi^{3r_2} \cdot |d_F|^{-1/2} \cdot \det|\mathcal{L}_3(\sigma_j(y_i))| \quad (1 \le j \le r_1 + r_2).$$

For $s = 2$ a similar result was proved in [Z 4]. It also follows directly from results of A. Borel [Bo 2], S. Bloch [Bl 1] and A.A. Suslin [S 3]. A more elementary proof, which uses only the result of Borel [Bo 2] and the 5–term functional equation for the Bloch–Wigner function is given in § 2.

D. Zagier has conjectured that the analogous fact should be valid for all integers $s \ge 3$ and has given some striking numerical examples [Z 3].

For the proof of Theorem 1.1 we give an explicit formula, expressing the Borel regulator $r_3 : K_5(\mathbb{C}) \longrightarrow \mathbb{R}^{r_1+r_2}$ by $\mathcal{L}_3(z)$, and then use the Borel theorem [Bo 2].

2. The properties of the Bloch–Wigner function. First of all let us recall the remarkable 2–variable functional equation for the dilogarithm, discovered in the 19th century by W. Spence [S], N.H. Abel [Ab] and others [L]. Its version for $D_2(z)$ is as follows. Let $r(x_1, \dots, x_4)$ be the cross–ratio of a 4–tuple of distinct points on P^1. Recall that the cross–ratio is PGL_2–invariant. If $\tilde{x}_i$ are coordinates of the points x_i, then

$$r(x_1, \dots, x_4) := \frac{(\tilde{x}_1 - \tilde{x}_4)(\tilde{x}_2 - \tilde{x}_3)}{(\tilde{x}_1 - \tilde{x}_3)(\tilde{x}_2 - \tilde{x}_4)} \ . \tag{1.6}$$

For every set of 5 distinct points on P^1 set

$$R_2(x_0, \dots, x_4) := \sum_{i=0}^{4} (-1)^i [r(x_0, \dots, \hat{x}_i, \dots, x_4)] \in \mathbb{Z}[P^1 \backslash 0,1,\infty] \ . \tag{1.7}$$

Then for $D_2 : \mathbb{Z}[P^1_{\mathbb{C}} \backslash 0,1,\infty] \longrightarrow \mathbb{R}$ $(D_2[z] := D_2(z))$ we have

$$D_2(R_2(x_0, \dots, x_4)) = 0 \ . \tag{1.8}$$

The Bloch–Wigner function $D_2(z)$ also satisfies the relation

$$D_2(r(x_{\sigma(0)}, \dots, x_{\sigma(3)})) = (-1)^{|\sigma|} D_2(r(x_0, \dots, x_3)) \tag{1.9}$$

where $|\sigma|$ is the sign of the permutation σ. This means that

$$D_2(z) = -D_2(1-z) = -D_2(z^{-1}) \ . \tag{1.9$'$}$$

The relation (1.9) is equivalent to the degenerate case of the functional equation (1.8) when just two points x_i coincide. Indeed, in this case $D_2(r(x_0, \dots, x_3)) = 0$ according to (1.2). So if, for example, $x_2 = x_4 = x$ then (1.8) means that $D_2(r(x_0,x_1,x_3,x)) + D_2(r(x_0,x_1,x,x_3)) = 0$ and so on.

The relation (1.9) can be deduced formally from (1.8). This means that the difference of the left– and right–hand side of (1.9) can be represented as a sum of several expressions (1.8).

Moreover, it seems that any functional equation for the Bloch–Wigner function $D_2(z)$ can be deduced formally from the one (1.8). The reasons lie in algebraic K–theory– see s. 10 below.

It is well–known that $\log|\cdot|$ is (up to a multiple) the unique continuous function satisfying the functional equation $f(xy) = f(x) + f(y)$. Thanks to S. Bloch, we know a similar characterisation of the dilogarithm:

<u>Theorem</u> 1.2 [Bl 1]. Any measurable function on $P^1_{\mathbb{C}}$ satisfying the functional equation (1.8) is proportional to the Bloch–Wigner function $D_2(z)$.

3. <u>The generic functional equation for the trilogarithm</u>. We see that for better understanding of the properties of the dilogarithm we ought to interpret its argument as a cross–ratio of 4 points on a line and then consider 5–tuples of points.

It turns out that the generic functional equation for the trilogarithm also has a geometrical nature: it corresponds to a special configuration of 7 points on the plane. Namely, let x_1,x_2,x_3 be vertices of a triangle in P^2_F (i.e. these points are not on a line); y_1,y_2,y_3 are points on its "sides" $\overline{x_1x_2}$, $\overline{x_2x_3}$ and $\overline{x_3x_1}$ and a point z is in a generic position (see fig. 1.1).

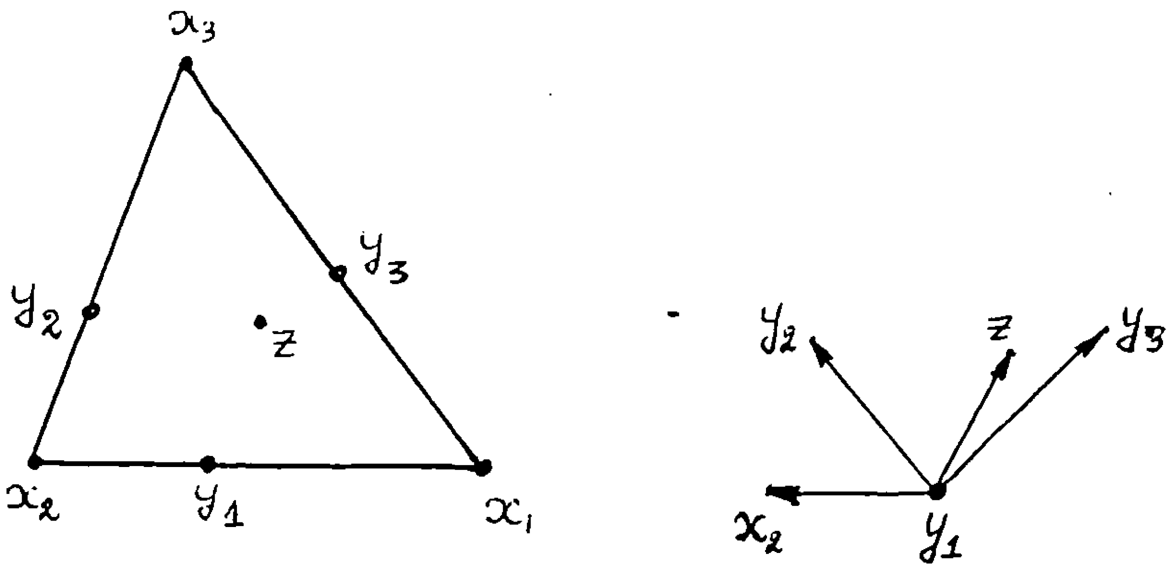


fig. 1.1, 1.2

Further denote by $(y_1 \,|\, y_2,y_3,x_2,z)$ the configuration of 4 points on a line, obtained by projection of the points y_2,y_3,x_2,z with center at the point y_1 (see fig. 1.2). Set

$$R_3(x_i,y_i,z) := (1+\tau+\tau^2) \circ \big[\{r(y_1 \,|\, y_2,y_3,x_2,z)\} - \{r(y_1 \,|\, y_2,y_3,x_3,z)\} +$$

$$+ \{r(z \,|\, x_3,y_3,x_1,y_2)\} + \{r(z \,|\, y_3,y_1,x_1,y_2)\}$$

$$+ \{r(z \,|\, y_1,x_2,x_1,y_2)\} + \{r(z \,|\, x_2,x_3,x_1,y_2)\} - \{r(z \,|\, x_3,y_1,x_1,y_2)\}\big] \tag{1.10}$$

$$+ \{r(y_1 \,|\, y_2,y_3,x_2,x_3)\} - 3\{1\}$$

where $\tau : x_i \longmapsto x_{i+1}$, $y_i \longmapsto y_{i+1}$ (indices modulo 3) (for example $\tau^2 \circ \{r(y_1 | y_2, y_3, x_2, z)\} = \{r(y_3 | y_1, y_2, x_1, z)\}$ and so on) and, by definition, $\{1\} = \{x\} + \{1 - x\} + \{1 - x^{-1}\}$ for some $x \in F^* \backslash 1$. As we will see below, the choice of x is inessential for our purposes.

Theorem 1.3. In the case $F = \mathbb{C}$ the following holds:

a) $\mathcal{L}_3(\{x\} - \{x^{-1}\}) = 0$

b) $\mathcal{L}_3(\{x\} + \{1 - x\} + \{1 - x^{-1}\}) = \mathcal{L}_3(1) \equiv \zeta_{\mathbb{Q}}(3)$.

c) $\mathcal{L}_3(R_3(x_i, y_i, z)) = 0$.

Remark. Let us consider all possible configurations of 4–tuples of points on a line, obtained by projection of some 4 points among x_i, y_i, z with the center at a fifth one. Let us say that two such configurations are equivalent if they differ only by a permutation of points. It is interesting that formula (1.10) contains just one representative for every equivalence class of configurations obtained in this way.

Let us give a more conceptual version of the functional equation. The function $\mathcal{L}_3(r(x_1, \dots, x_4))$ that a priori is defined on configurations of 4 distinct points in $\mathbb{C}P^1$ can be prolonged continuously to the set of 4–tuples such that just 2 of them coincide by the following rule:

$$\mathcal{L}_3(r(x_1, \dots, x_4)) := \begin{cases} \mathcal{L}_3(1) & \text{if } x_1 = x_2 \text{ or } x_3 = x_4 \\ 0 & \text{in other cases} \end{cases}$$

Let $(\ell_0, \dots, \ell_5)$ be a 6–tuple of distinct points in $P^2_{\mathbb{C}}$ such that ℓ_0, ℓ_1, ℓ_2 lie on the same line, but there are no 4 points among the ℓ_i with this property. Put $\mathcal{L}_3'(r(x_1, \dots, x_4)) := -\mathcal{L}_3(r(x_1,x_2,x_3,x_4)) - 2\mathcal{L}_3(r(x_1,x_3,x_2,x_4)) + \mathcal{L}_3(1)$ and set

$$\mathcal{M}_3(\ell_0, \dots, \ell_5) := \frac{1}{3} \sum_{0 \le i, j \le 2} (-1)^{i+j} \mathcal{L}_3'(r(\ell_{2+j} | \ell_0, \dots, \hat{\ell}_i, \dots, \hat{\ell}_{2+j}, \dots, \ell_5)) \quad (1.11)$$

$$\mathcal{M}_3(\ell_{\sigma(0)}, \dots, \ell_{\sigma(5)}) := (-1)^{|\sigma|} \mathcal{M}_3(\ell_0, \dots, \ell_5) .$$

It can be proved using the identity $\mathcal{L}_3(x) + \mathcal{L}_3(1-x) + \mathcal{L}_3(1-x^{-1}) = \mathcal{L}_3(1)$ that these definitions are correct.

Now let $(\ell_0, \dots, \ell_6)$ be a configuration as presented in fig. 1.1′. Then for every i there are 3 points among $(\ell_0, \dots, \hat{\ell}_i, \dots, \ell_6)$ that lie on the same line.

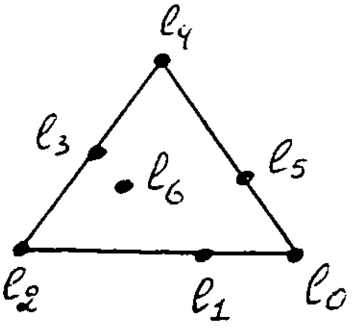


fig. 1.1′

Theorem 1.4. For a configuration $(\ell_0, \dots, \ell_6)$ as in fig. 1.1′

$$\sum_{i=0}^{6} (-1)^i \mathcal{M}_3(\ell_0, \dots, \hat{\ell}_i, \dots, \ell_6) = 0 .$$

We will prove in § 5 that the functional equation (1.10) can be deduced from this one using only the relation from Theorem 1.3 b). (This is not quite obvious: for example, all coefficients in 1.11 are $\pm 1/3$).

Now let $(\ell_0, \dots, \ell_5)$ be a configuration of 6 points in generic position in $P^2_{\mathbb{C}}$. Put $\ell_6 := \overline{\ell_0\ell_1} \cap \overline{\ell_2\ell_3}$ (see fig. 1.11 in s. 6 below) and

$$\mathcal{M}_3(\ell_0, \dots, \ell_5) := \sum_{i=0}^{5} (-1)^{i-1} \mathcal{M}_3(\ell_0, \dots, \hat{\ell}_i, \dots, \ell_6) .$$

(The right–hand side was already defined in (1.11)). We will prove in § 5 that this function $\mathcal{M}_3(\ell_0, \dots, \ell_5)$ is skew–symmetric with respect to permutations of points ℓ_i and satisfies the 7–term relation

$$\sum_{i=0}^{6} (-1)^i \mathcal{M}_3(\ell_0, \dots, \hat{\ell}_i, \dots, \ell_6) = 0 . \tag{1.12}$$

4. <u>Explicit formula for a 5–cocycle representing a class of continuous cohomology of $GL_3(\mathbb{C})$</u>

Choose a point $x \in \mathbb{C}P^2$. Then there is a measurable cocycle

$$f^{(x)} : \underbrace{GL_3(\mathbb{C}) \times \dots \times GL_3(\mathbb{C})}_{6 \text{ times}} \longrightarrow \mathbb{R}$$

$$f^{(x)}(g_0, \dots, g_5) := \mathcal{M}_3(g_0 x, \dots, g_5 x) . \tag{1.13}$$

It is certainly invariant under the left action of $GL_3(\mathbb{C})$. So the 7–term relation (1.12) just means that $f^{(x)}$ is a measurable cocycle of $GL_3(\mathbb{C})$. Different points x give cohomologous cocycles.

The function $\mathcal{L}_3(z)$ is continuous on $\mathbb{C}P^1$ and hence bounded. Therefore the function $f^{(x)}$ is also bounded. Applying Proposition 1.4 from ch. III in [Gu] we see that the cohomology class of the cocycle (1.13) lies in

$$\mathrm{Im}(H^5_{cts}(GL_3(\mathbb{C}),R) \longrightarrow H^5(GL_3(\mathbb{C}),R)) \tag{1.14}$$

where $H^*_{cts}(G,R)$ denotes the continuous cohomology of a Lie group G . Recall that (see [Bo1])

$$H^*_{cts}(GL_n(\mathbb{C}),R) = \Lambda^*_R[u_1,u_3, \dots ,u_{2n-1}] \tag{1.15}$$

where $u_i \in H^{2i-1}_{cts}(GL_n(\mathbb{C}),R)$. The subspace generated by the element u_i is called the indecomposable part of $H^{2i-1}_{cts}(GL_n(\mathbb{C}),R)$. In particular, $\dim H^5_{cts}(GL_3(\mathbb{C}),R) = 1$. The constructed cocycle represents a non–trivial cohomology class.

5. <u>Functional equations for the trilogarithm in coordinates.</u> Let us now write $R_3(x_i,y_i,z)$ in coordinates. Choose homogeneous coordinates for the points x_i,z,y_i as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & c \\ 0 & 1 & 0 & 1 & a & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & b & 1 \\ - & - & - & - & - & - & - \end{bmatrix} .$$
$$\begin{matrix} x_1 & x_2 & x_3 & z & y_1 & y_2 & y_3 \end{matrix}$$

Then $R_3(x_i,y_i,z)$ coincides with $R_3(a,b,c)$, where

$$R_3(a,b,c) := \underset{\text{cycle}}{\oplus}\left(\{ca-a+1\} + \left\{\frac{ca-a+1}{ca}\right\} + \{c\} + \left\{\frac{(bc-c+1)}{(ca-a+1)b}\right\} - \right. \tag{1.16}$$

$$\left. - \left\{\frac{ca-a+1}{c}\right\} + \left\{\frac{(bc-c+1)a}{(ca-a+1)}\right\} - \left\{\frac{(bc-c+1)}{(ca-a+1)bc}\right\} - \{1\}\right) + \{-abc\}\,.$$

Here $\underset{\text{cycle}}{\oplus} f(a,b,c) := f(a,b,c) + f(c,a,b) + f(b,c,a)$, and according to Theorem 1.3 a) we do not distinguish between $\{x\}$ and $\{x^{-1}\}$.

It is interesting that all coefficients in this formula are equal to one.

Let us consider a specialisation of this formula setting $a = 1$. Then we get

$$R_3(1,b,c) = -\left\{\frac{(bc-c+1)}{bc^2}\right\} - \left\{\frac{(bc-c+1)}{b}\right\} - \{(bc-c+1)b\} \tag{1.17}$$

$$+ 2\left(\left\{\frac{(bc-c+1)}{bc}\right\} + \left\{-\frac{(bc-c+1)}{c}\right\} + \{bc-c+1\} + \{-bc\} + \{b\} + \{c\} - \{1\}\right).$$

From the geometrical point of view $R_3(1,b,c)$ corresponds to a configuration of 7 points as in fig. 1.3 (z lies on the line $\overline{x_3y_1}$).

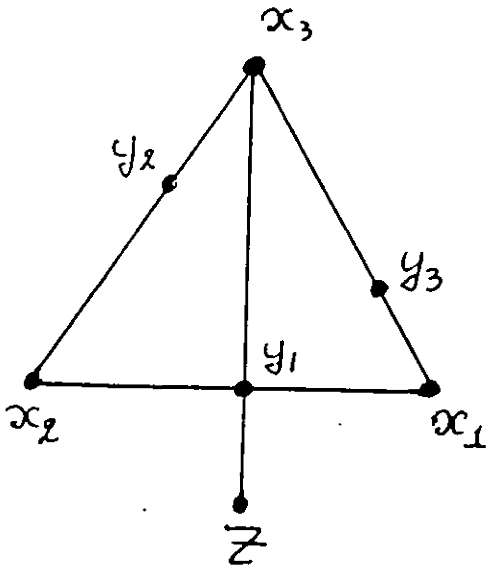


fig. 1.3

The corresponding functional equation for the trilogarithm coincides with the classical Spence–Kummer one, discovered by W. Spence in 1809 [S] and, independently, by E. Kummer in 1840 [K] – see Ch. VI in Lewin's book [L]. To see this let us set $x = \frac{bc-c+1}{bc}$, $y = \frac{b-1}{b}$. Then we get

$$-\left\{\frac{(x-y)x}{1-y}\right\} - \left\{\frac{x(1-y)}{x-y}\right\} - \left\{\frac{x}{(x-y)(1-y)}\right\} +$$

$$+ 2\left(\left\{\frac{1-y}{x-y}\right\} + \left\{\frac{x}{x-y}\right\} + \left\{\frac{x}{y-1}\right\} + \{y-x\} + \{1-y\} + \{x\} - \{1\}\right).$$

Substituting $v = \frac{y-1}{y-x-1}$, $u = \frac{x}{x+1-y}$ we obtain the last formula in section 7.2 of ch. VI in [L].

The Spence–Kummer equation (1.17) can be deduced formally from Theorem 1.3. More precisely, it can be represented as a sum of 3 generic equations (1.16) – see the proof of Proposition 5.6. The validity of the converse statement is an interesting problem.

Let us emphasize that the functional equation for the function $\mathcal{L}_3(z)$ has no remainder terms (such as products of logarithms and dilogarithms). For the Bloch–Wigner–Ramakrishnan function $D_3(z)$ or the ordinary trilogarithm this is no longer true. The functional equations for $\mathcal{L}_p(z)$ have no remainder terms for any p.

Subsequent specialisation of (1.17) gives

$$R_3(1,1,c) = -\{c^2\} + 4\{c\} + 4\{-c\};$$
$$R_3(1,1,1) = 3\{1\} + 4\{-1\}.$$

So we have (compare with the formulae (6.4) and (6.5) in [L]):

$$\mathcal{L}_3(c^2) = 4(\mathcal{L}_3(c) + \mathcal{L}_3(-c)), \quad \mathcal{L}_3(-1) = -3/4\ \mathcal{L}_3(1).$$

The corresponding configurations of 7 points in P^2 can be seen in fig. 1.4 and 1.5.

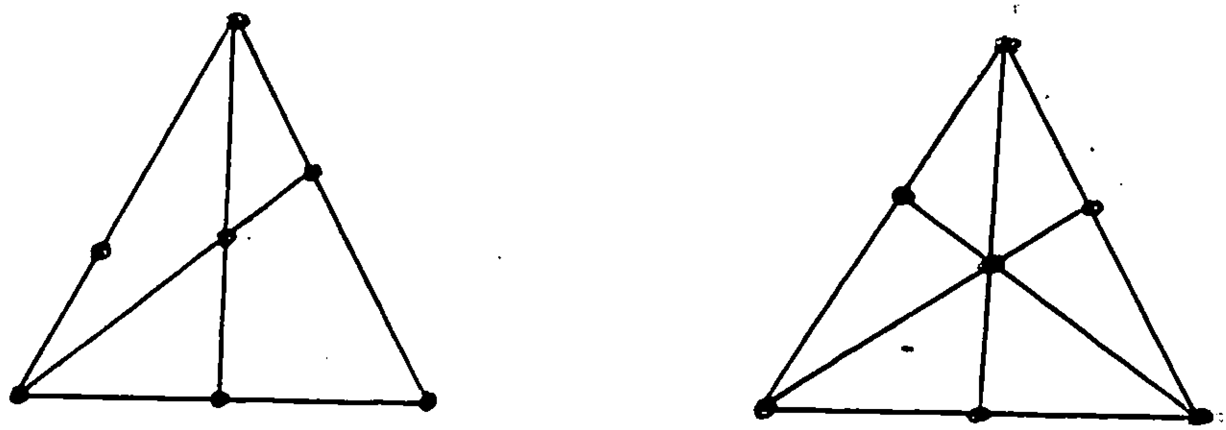
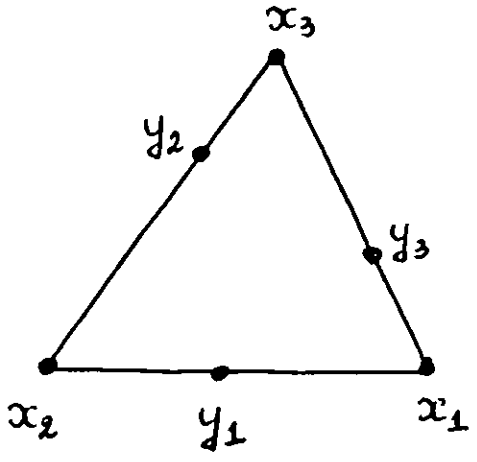


fig. 1.4, 1.5

6. The group of "abstract trilogarithms". For a G–space X points of $G\backslash X \times \ldots \times X$ are called configurations. Let $C_6(P^2_F)$ be the free abelian group generated by all possible configurations $(\ell_0, \ldots, \ell_5)$ of 6 points in P^2_F.

Let $(x_1,x_2,x_3,y_1,y_2,y_3)$ be a configuration of 6 points in P^2_F as in fig. 1.6 (i.e. y_i lies on the line $\overline{x_ix_{i+1}}$, indices modulo 3) such that

$$r(y_3 \,|\, x_1,x_2,y_1,y_2) = x .$$



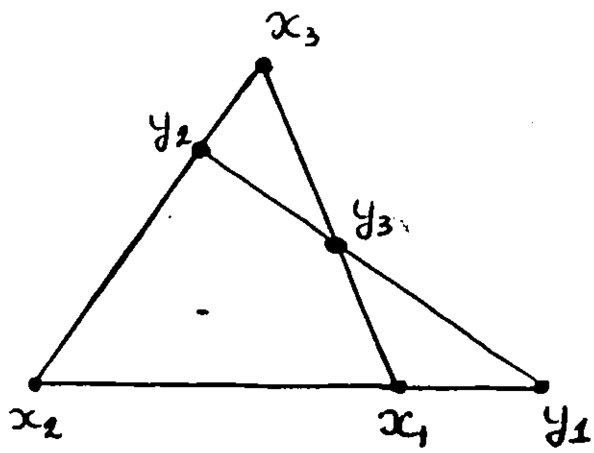


fig. 1.6, 1.7

Let us define a homomorphism

$$\tilde{L}_3 : \mathbb{Z}[P_F^1 \backslash 0,1,\infty] \longrightarrow C_6(P_F^2),$$

setting

$$\tilde{L}_3 : \{x\} \longmapsto (x_1,x_2,x_3,y_1,y_2,y_3).$$

The configuration where y_1,y_2,y_3 are on a line will be denoted by η_3 (see fig. 1.7).

Definition 1.5. $\mathcal{B}_3(F)$ is the quotient of the group $C_6(P_F^2)$ by the following relations

R1) $(\ell_0, \ldots ,\ell_5) = 0$ if 2 of the points ℓ_i coincide or 4 lie on a line.

R2) (The 7–term relation). For any 7 points $\ell_0, \dots, \ell_6$ in P^2_F

$$\sum_{i=0}^{6} (-1)^i(\ell_0, \dots, \hat{\ell}_i, \dots, \ell_6) = 0 .$$

R3) Let $(m_0, \dots, m_5)$ be a configuration of 6 points in P^2_F such that $m_2 = \overline{m_0m_1} \cap \overline{m_3m_4}$ and m_5 is in generic position (see fig. 1.8). Set

$$L_3'\{x\} := -\tilde{\mathcal{L}}_3\{x\} - 2\tilde{\mathcal{L}}_3\{1-x\} + \eta_3 .$$

Then

$$3 \cdot (m_0, \dots, m_5) = \sum_{i=0}^{4} (-1)^i L_3'\{r(m_5 \,|\, m_0, \dots, \hat{m}_i, \dots, m_4)\} .$$

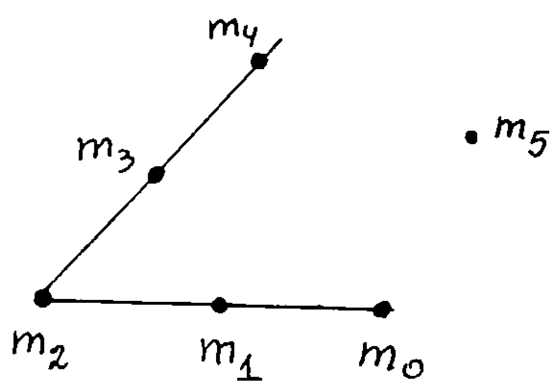


fig. 1.8

Remark 1.6. Let us consider the action of $GL_3(F)$ on 6–tuples of points in P^2_F. Then a configuration $(\ell_0, \dots, \ell_5)$ is stable (respectively semistable) in the sense of Mumford if and only if among the points ℓ_i there are no 2 coinciding or 4 lying on a line (respectively 3 coinciding or 5 lying on a line) – see [Mu]. So relation R1) means that $(\ell_0, \dots, \ell_5) = 0$ if the configuration $(\ell_0, \dots, \ell_5)$ is semistable or unstable.

Lemma 1.7. In the group $\mathcal{B}_3(F)$

$$(\ell_0, \dots, \ell_5) = (-1)^{|\sigma|}(\ell_{\sigma(0)}, \dots, \ell_{\sigma(5)})$$

where $|\sigma|$ is the sign of the permutation σ.

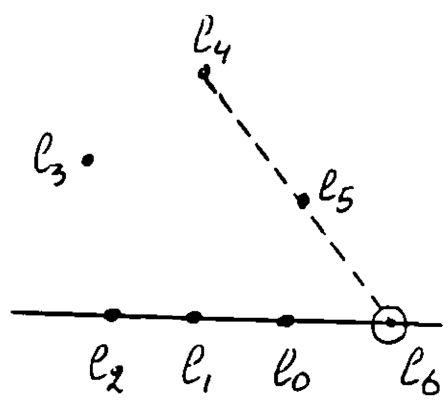
Proof. Consider the relation R2) for a configuration $(\ell_0, \dots, \ell_6)$ where just 2 of the points coincide and apply R1). ■

The homomorphism $\tilde{\mathcal{L}}_3$ induces the homomorphism

$$L_3 : \mathbb{Q}[P^1_F \backslash 0,1,\infty] \longrightarrow \mathcal{B}_3(F)_{\mathbb{Q}}\,.$$

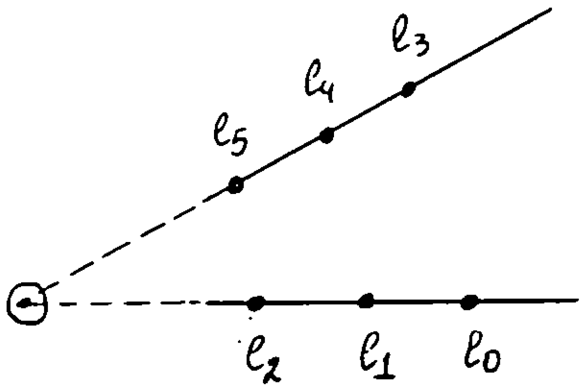
It is not hard to prove that this map is an epimorphism. Indeed, the relation R3) implies that a configuration as in fig. 1.8 lies in the image of L_3. It remains to apply the 7–term relation to configurations as in fig. 1.9 – 1.11.

Let us prove that relation R3 does not follow from the relations R1 and R2. Denote by $C^s_5(P^2)$ the free abelian group generated by configurations of 5 distinct points in P^2 such that there are no 4 points on a line among these points. Then there is a homomorphism $\partial : C_6(P^2) \longrightarrow C^s_5(P^2)$ defined as follows: degenerated configurations satisfying condition R1 map to 0 and $\partial(\ell_0, \dots, \ell_5) := \mathrm{Alt}(\sum_{i=0}^{5} (-1)^i(\ell_0, \dots, \hat{\ell}_i, \dots, \ell_5))$, where Alt is the skew–symmetrization. Note that if the points m_0 , m_1 , m_2 lie on a line ℓ and $m_3, m_4 \notin \ell$, then there exists an element $g \in PGL_3(F)$ such that the 5–tuples of points (m_0,m_1,m_2,m_3,m_4) and (m_0,m_1,m_2,m_4,m_3) are equivalent under the action of $GL_3(F)$. Therefore $\mathrm{Alt}(m_0, \dots, m_4) = 0$. So $\partial(\tilde{L}_3\{x\}) = 0$, but for a configuration $(m_0, \dots, m_5)$ as in fig. 1.8 we have $\partial(m_0, \dots, m_5) = (m_0, \dots, m_4)$.



A configuration $(\ell_0, \dots, \ell_5) \in \mathrm{Im}\, L_3$

fig. 1.9



$$(\ell_0, \ldots, \ell_5) = 0 \quad \text{in} \quad \mathcal{G}_3(F)$$

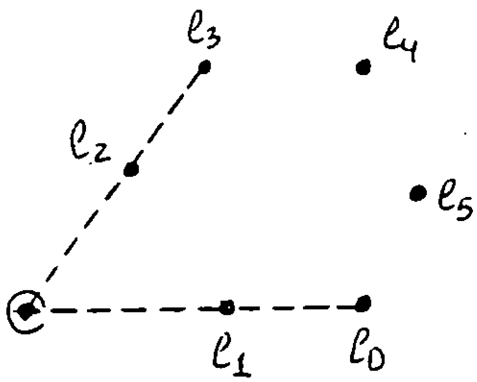


fig.1.10-1.11

Let us denote by $R_3(F)$ the subgroup in $\mathbb{Z}[P^1_F \backslash 0,1,\infty]$ generated by the following elements (compare with Theorem 1.3)

$\{x\} - \{x^{-1}\}$,

$(\{x\} + \{1-x\} + \{1-x^{-1}\}) - (\{y\} + \{1-y\} + \{1-y^{-1}\})$,

$R_3(x_i, y_i, z)$,

where $x,y \in F^*\backslash 1$, $x_i, y_i, z \in P^2_F$ and the configurations of the points $(x_1,x_2,x_3,y_1,y_2,y_3,z)$ are as in fig. 1.1, where there are just 3 lines, containing exactly 3 points of a configuration. Set

$$B_3(F) := \mathbb{Z}[P^1_F\backslash 0,1,\infty]/R_3(F).$$

Theorem 1.8. The homomorphism L_3 induces the isomorphism

$$L_3 : B_3(F)_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{B}_3(F)_{\mathbb{Q}}.$$

The inverse homomorphism

$$M_3 := L_3^{-1} : \mathcal{B}_3(F)_{\mathbb{Q}} \longrightarrow B_3(F)_{\mathbb{Q}}$$

can be defined explicitly on the generators of the group $\mathcal{B}_3(F)$ as follows. Set $\ell_3'\{x\} = -\{x\} - 2\{1-x\} + \{1\}$. Then for a configuration $(\ell_0, \dots, \ell_5)$ as in fig. 1.9

$$M_3(\ell_0, \dots, \ell_5) := \frac{1}{3} \sum_{0\le i, j\le 2} (-1)^{i+j} \ell_3'\{r(\ell_{2+j} | \ell_0, \dots, \hat{\ell}_i, \dots, \hat{\ell}_{2+j}, \dots, \ell_5)\}$$

compare with (1.11) and for a generic configuration $(\ell_0, \dots, \ell_5)$

$$L_3^{-1}(\ell_0, \dots, \ell_5) := \sum_{i=0}^{5} (-1)^{i-1} L_3^{-1}\{\ell_0, \dots, \hat{\ell}_i, \dots, \hat{\ell}_6)$$

(ℓ_6 is defined in fig. 1.1), where the right–hand side was already defined above. The proof of the correctness of this definition uses the basic relation $R_3(x_i, y_i, z)$ in the group $B_3(F)$ – see s.2 of § 5.

7. <u>The trilogarithm is determined by its functional equation</u>. Let Meas $C_m(\mathbb{C}P^n)$ be the space of all measurable functions on configurations of m points in $\mathbb{C}P^n$. Define a map

$$\text{Meas } C_m(\mathbb{C}P^n) \xrightarrow{d_m^*} \text{Meas } C_{m+1}(\mathbb{C}P^n)$$

by the formula $(d_m^* f)(x_0, \dots, x_m) = \sum_{i=0}^{m} (-1)^i f(x_0, \dots, \hat{x}_i, \dots, x_m)$.

Recall ([Mu]) that a configuration $(\ell_0, \dots, \ell_{m-1})$ of m points in $\mathbb{C}P^n$ is stable if and only if for any subspace $L \subset \mathbb{C}P^n$

$$\frac{\text{The number of points } \ell_i \text{ in } L}{\dim L+1} < \frac{m}{n+1}.$$

Let Cont $C_m^s(\mathbb{C}P^n)$ be the space of all continuous functions on stable configurations of m points in $\mathbb{C}P^n$. We have the following complexes Meas $C_\bullet(\mathbb{C}P^2)$ and Cont $C_\bullet^s(\mathbb{C}P^2)$:

$$\longrightarrow \text{Meas } C_5(\mathbb{C}P^2) \xrightarrow{d_5^*} \text{Meas } C_6(\mathbb{C}P^2) \xrightarrow{d_6^*} \text{Meas } C_7(\mathbb{C}P^2) \qquad (1.18\text{ a})$$

$$\longrightarrow \text{Cont } C_5^s(\mathbb{C}P^2) \xrightarrow{d_5^*} \text{Cont } C_6^s(\mathbb{C}P^2) \xrightarrow{d_6^*} \text{Cont } C_7^s(\mathbb{C}P^2) \qquad (1.18\text{ b}).$$

Note that the complex conjugation $z \longrightarrow \bar{z}$ acts on $\mathbb{C}P^2$ and hence on these complexes. Denote by $H^6(\text{Cont}_\bullet^s(\mathbb{C}P^2))^+$ the subspace of invariants of this action.

<u>Theorem</u> 1.9.

a) $\dim H^6(\text{Cont } C_\bullet^s(\mathbb{C}P^2)) = \dim H^6(\text{Meas } C_\bullet(\mathbb{C}P^2)) = 2$

b) $H^6(\text{Cont } C_\bullet(\mathbb{C}P^2))^+$ is canonically isomorphic to $H^5_{cts}(GL_3(\mathbb{C}),R)$

c) The function $\mathcal{M}_3(\ell_0, \dots ,\ell_5)$ represents a nonzero element in $H^6(\text{Cont } C_\bullet^s(\mathbb{C}P^2))^+$.

Let us consider a degenerate configuration $(\ell_0, \dots ,\ell_5)$ presented in fig. 1.12. It depends on one parameter $z := r(\ell_5 \,|\, \ell_0,\ell_2,\ell_1,\ell_3) \in \mathbb{C}^*$. In this case $\mathcal{M}_3(\ell_0, \dots ,\ell_5) = \mathcal{L}_3(z)$.

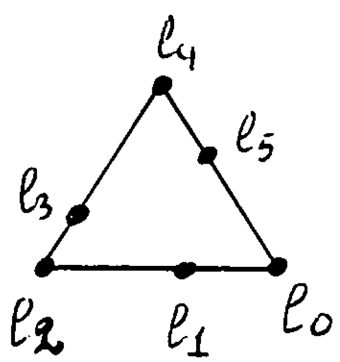


fig. 1.12

Theorem 1.10.

a) The space of continuous functions $f(z)$ on $\mathbb{C}P^1\backslash 0,1,\infty$ that satisfy the functional equation $f(R_3(a,b,c)) = 0$ (see 1.11) is generated by the functions $\mathcal{L}_3(z)$ and $D_2(z) \cdot \ln|z|$.

b) Let $F(\ell_0, \dots, \ell_5)$ be a continuous function on the set of all stable configurations of 6 points in $\mathbb{C}P^2$ that is skew–symmetric with respect to permutations of ℓ_i and satisfies the 7–term relation $\sum_{i=0}^{6} (-1)^i F(\ell_0, \dots, \hat{\ell}_i, \dots, \ell_6) = 0$ (for any stable configuration $(\ell_0, \dots, \ell_6)$). Then the restriction of F to the degenerate configuration $(\ell_0, \dots, \ell_5)$ presented in fig. 1.12 is a linear combination of $\mathcal{L}_3(z)$ and $D_2(z) \cdot \ln|z|$.

Remark. Let $f \in \text{Cont}\ C_5^s(\mathbb{C}P^5)$. Then, of course, $d_6^*(d_5^* f) = 0$, but the value of $d_5^* f$ at a configuration $(\ell_0, \dots, \ell_5)$ as shown in fig. 1.12 can be an arbitrary continuous function on $\mathbb{C}P^1\backslash 0,\infty$. So the skew–symmetry relation does not follow from the 7–term one for stable configurations (compare with Lemma 1.7).

The following proposition proves part c) of Theorem 1.9.

Proposition 1.11. $(\mathcal{L}_3 \circ M_3)(x_0, \dots, x_5) \notin d_5^*\ \text{Cont}\ C_5^s\ (\mathbb{C}P^2)$.

Proof. Suppose that $(\mathcal{L}_3 \circ M_3)(x_0, \dots, x_5) = d_5^* f$. The left–hand side is skew–symmetric with respect to permutations of points x_i. So we have

$$(\mathscr{L}_3 \circ M_3)(x_0, \dots ,x_5) = \text{Alt}\ d_5^* f = d_5^* \text{Alt}\ f$$

where $\text{Alt}\ g(x_0, \dots ,x_{n-1}) = \sum_{\sigma \in S_n} (-1)^{|\sigma|} g(x_{\sigma(0)}, \dots ,x_{\sigma(n-1)})$.

Note that if for a configuration $(v_0, \dots ,v_4)$ of 5 points on the plane v_0, v_1, v_2 lie on a line and v_3, v_4 are not on this line, then the configurations (v_0,v_1,v_2,v_3,v_4) and (v_0,v_1,v_2,v_4,v_3) are equal. But for a configuration $(\ell_0, \dots ,\ell_5)$ as in fig. 1.12 all configurations $(\ell_0, \dots ,\hat{\ell}_i, \dots ,\ell_5)$ are of this type, so $d_5^*\ \text{Alt}\ f(\ell_0, \dots ,\ell_5) = 0$. ∎

Now let us construct a representative of $H^6(\text{Cont}\ C_\bullet^s(\mathbb{C}P^2))$. Let V_3 be a 3–dimensional vector space with a volume form $\omega_3 \in \det V_3^*$, $\Delta(\ell_0\ell_1\ell_2) := \langle w_3, \ell_0 \wedge \ell_1 \wedge \ell_2 \rangle$. Set for a generic configuration of 5 vectors $(\ell_0, \dots ,\ell_4)$ in V_3

$$f_1^{(3)}(\ell_0, \dots ,\ell_4) = \sum_{i=0}^{4} (-1)^i D_2(\ell_i \mid \ell_0, \dots ,\hat{\ell}_i, \dots ,\ell_4) \cdot \prod_{\substack{0 \le j_1 < j_2 < j_3 \le 4 \\ i \notin \{j_1, j_2, j_3\}}} \ln|\Delta(\ell_{j_1}\ell_{j_2}\ell_{j_3})| \ .$$

It does not depend on ω (Proposition 3.7). Further, $(d^* f_1^{(3)})(\ell_0, \dots ,\ell_5) := \sum_{j=0}^{5} (-1)^j f_1^{(3)} (\ell_0, \dots ,\hat{\ell}_j, \dots ,\ell_5)$ does not depend on the length of the vectors ℓ_i (Proposition 3.9) and so defines a function on configurations of 6 points in V_3 . It can be prolonged to all stable configurations (see § 4). The restriction of the function so obtained to a degenerate configuration presented in fig. 1.12 is just $D_2(z) \cdot \ln|z|$ (see Lemma 4.7). The constructed function is skew–symmetric, so the proof of Proposition 1.11 shows that it does not lie in $d_5^*\ \text{Cont}\ C_5^s(\mathbb{C}P^2)$.

We will see in § 9 that this function can be obtained by transgression of a non–zero element in $H^4_{cts}(GL_2(\mathbb{C}),R)$ in some spectral sequence.

Finally, $H^4(\text{Cont } C^s_\bullet(\mathbb{C}P^1)) = \ker d_4^*$, because all 3–tuples of distinct points on $\mathbb{C}P^1$ are $PGL_2(\mathbb{C})$–equivalent and so for a generic configuration of 4 points on $\mathbb{C}P^1$ we have $\sum_{i\geq 0}^{3} (-1)^i(\ell_0, \dots, \hat{\ell}_i, \dots, \ell_3) = 0$.

<u>Theorem</u> 1.12. There exists a C^∞ – function $\mathcal{M}_n(\ell_0, \dots, \ell_{2n-1})$ on the manifold of generic configurations of $2n$ vectors in an n–dimensional $\mathbb{C}$–vector space such that

a) $\sum_{i=0}^{2n} (-1)^i \mathcal{M}_n(\ell_0, \dots, \hat{\ell}_i, \dots, \ell_{2n}) = 0$ for a generic configuration $(\ell_0, \dots, \ell_{2n})$.

b) $\mathcal{M}_n(g_0\ell_0, g_1\ell_1, \dots, g_{2n-1}\ell_{2n-1})$, $g_i \in GL_n(\mathbb{C})$, is a measurable $2n-1$ – cocycle of $GL_n(\mathbb{C})$ representing the indecomposable class in $H^{2n-1}_{cts}(GL_n(\mathbb{C}),R)$.

The proof of this theorem uses a variant of Suslin's spectral sequence [S 1].

The existence of such a function was conjectured in [HM], see also [GGL] and [GM].

8. <u>The classical trilogarithm and weight 3 motivic cohomology</u>. Now let F be an arbitrary field. The groups $K_n(F)$ were defined by Quillen [Q 1] as homotopy groups

$$K_n(F) := \pi_n(BGL(F)^+)$$

where $BGL(F)^+$ is the H–space having the same homology as $BGL(F)$, i.e. the same as the homology of the discrete group $GL_\infty(F) \equiv GL(F)$.

By the Milnor–Moore theorem [MM]

$$K_n(F) \otimes \mathbb{Q} = \text{Prim } H_n(GL(F),\mathbb{Q}) . \qquad (1.19)$$

Recall that $K_0(F) = \mathbb{Z}$, $K_1(F) = F^*$.

On the other hand, we have the Milnor ring $K_*^M(F)$ which is defined as a quotient ring of the tensor algebra

$$T(F^*) := \bigoplus_n F^* \underset{\mathbb{Z}}{\otimes} \dots \underset{\mathbb{Z}}{\otimes} F^* \qquad \text{(n times)}$$

by the homogeneous ideal generated by all tensors $(1-x) \otimes x \in T_2(F^*) = F^* \otimes F^*$. It is not hard to prove ([M 1]) that

$$K_n^M(F) = \Lambda^n F^* / \{(1-x) \wedge x \wedge y_1 \wedge \dots \wedge y_{n-2}\} . \qquad (1.20)$$

where $\Lambda^n F^* := \otimes^n F^* / \{\dots x_i \otimes x_{i+1} \dots + \dots x_{i+1} \otimes x_i \dots\}$.

There is the canonical ring homomorphism $m : K_*^M(F) \longrightarrow K_*(F)$. Thanks to Matsumoto we know that it is an isomorphism for $n = 2$ (see [M 2]). It is injective modulo torsion ([S 1]). But the Coker(m) can be rather big.

Set

$$B_p(F) := \mathbb{Z}[P^1_F \backslash 0,1,\infty] / R_p(F) \quad (p \le 3)$$

where the subgroups R_p $(p = 1,2)$ are generated by the following elements

$$R_1(F) = \{ [xy] - [x] - [y] , x,y \in F^* \backslash 1 \} ,$$
$$R_2(F) = \{ \sum_{i=0}^{4} (-1)^i [r(x_0, \dots, \hat{x}_i, \dots, x_4)] , \; x_i \in P^1_F , \; x_i \neq x_j \} ,$$

and the subgroup $R_3(F)$ is defined in s. 5 in a similar way. The definition of these groups is reminiscent of the functional equations for the classical p–logarithms, $p \le 3$.

Note that the map $[x] \longrightarrow x$ defines an isomorphism $B_1(F) \xrightarrow{\sim} F^*$.

Let us consider the Bloch–Suslin complex $B_F(2)$:

$$B_2(F) \xrightarrow{\delta_2} \Lambda^2 F^*$$

$$\delta_2 : [x] \longmapsto (1-x) \wedge x$$

with the group $B_2(F)$ placed in degree 1. Note that $\delta_2(R_2(F)) = 0$, so the definition is correct.

Let $K_3^{ind}(F) := \mathrm{Coker}(K_3^M(F) \longrightarrow K_3(F))$. Using some ideas of S. Bloch, A.A. Suslin proved the following remarkable theorem (see also closely related results of [DS] and [Sa]).

Theorem 1.13 [S 2]. There is an exact sequence

$$0 \longrightarrow \mathrm{Tor}(F^*,F^*)^{\sim} \longrightarrow K_3^{ind}(F) \longrightarrow \mathrm{Ker}\,\delta_2 \longrightarrow 0$$

where $\mathrm{Tor}(F^*,F^*)^{\sim}$ is the unique nontrivial extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathrm{Tor}(F^*,F^*)$.

Historical remark. The kernel of the homomorphism $\delta : \mathbb{Z}[P^1_F] \longrightarrow \Lambda^2 F^*$ is called the Bloch group. It was introduced by Spencer Bloch in his pioneering work [Bl 3]. The relation of this group to $K_3^{ind}(F)$, and also the interpretation of elements of $R_2(\mathbb{C})$ as functional equations for the Bloch–Wigner function together with a geometrical spectral sequence for the computation of $H^3_{cts}(CL_2(\mathbb{C}),R)$ appeared in [Bl1]. Influenced by these ideas, J. Dupont and C.H. Sah and independently A.A. Suslin divided $\mathbb{Z}[P^1_F]$ by $R_2(F)$ and clarified the relation of $H^1(B_F(2))$ with $H_3(GL_2(F))$ and $K_3^{ind}(F)$. I recommend also to read the excellently written first part of [D]. The group $B_2(\mathbb{C})$ has a beautiful interpretation as a scissors congruence group of tetrahedra in the Lobachevsky space.

Theorems of Matsumoto and Suslin claim that

$$H^1(B_F(2) \otimes \mathbb{Q}) \cong K_3^{ind}(F)_{\mathbb{Q}}$$

$$H^2(B_F(2)) \cong K_2(F)\,.$$

Let us define the complex $B_F(3) \otimes \mathbb{Q}$ as follows:

$$B_3(F)_{\mathbb{Q}} \xrightarrow{\ \delta\ } (B_2(F) \otimes F^*)_{\mathbb{Q}} \xrightarrow{\ \delta\ } (\Lambda^3 F^*)_{\mathbb{Q}} \qquad (1.21)$$

where the left group is placed in degree 1 and

$$\delta\{x\} = [x] \otimes x\ ;\ \delta([x] \otimes y) = (1-x) \wedge x \wedge y\ .$$

(Here $\{x\}$ is a generator in the group $B_3(F)$ and $[x]$ is a generator in the group $B_2(F)$).

The correctness of the definition is provided by the following theorem.

Theorem 1.3′. $\delta_3(R_3(F)) = 0$ in $B_2(F) \otimes F^*$ modulo 6–torsion.

Now let us introduce the rank filtration on $K_n(F)$. According to the stabilisation theorem of A.A. Suslin [S 1]

$$H_n(GL_n(F),\mathbb{Z}) = H_n(GL(F),\mathbb{Z}) \tag{1.22}$$

so

$$K_n(F)_{\mathbb{Q}} = \mathrm{Prim}\ H_n(GL_n(F),\mathbb{Q})\ .$$

Therefore

$$\mathrm{Im}(H_n(GL_{n-i}(F),\mathbb{Q}) \longrightarrow H_n(GL_n(F),\mathbb{Q}))$$

gives the canonical filtration on $H_n(GL_n(F),\mathbb{Q})$ and hence defines a filtration

$$K_n(F)_{\mathbb{Q}} \supset K_n^{(1)}(F)_{\mathbb{Q}} \supset K_n^{(2)}(F)_{\mathbb{Q}} \supset \dots\ . \tag{1.23}$$

Set

$$K_n^{[i]}(F)_{\mathbb{Q}} := K_n^{(i)}(F)_{\mathbb{Q}}/K_n^{(i+1)}(F)_{\mathbb{Q}} . \tag{1.24}$$

Theorem 1.14. There are canonical maps

$$c_1 : K_5^{[2]}(F)_{\mathbb{Q}} \longrightarrow H^1(B_F(3) \otimes \mathbb{Q})$$

$$c_2 : K_4^{[1]}(F)_{\mathbb{Q}} \longrightarrow H^2(B_F(3) \otimes \mathbb{Q}) .$$

Remark. A. Suslin proved in [S 1] that $K_n^{[0]}(F)_{\mathbb{Q}} = K_n^M(F)_{\mathbb{Q}}$. More precisely he proved that the homological multiplication

$$H_1(F^*) \times \dots \times H_1(F^*) \longrightarrow H_n(F^* \times \dots \times F^*) \longrightarrow H_n(GL_n(F))$$

defines an isomorphism modulo $(n-1)!$ – torsion

$$K_n^M(F) \longrightarrow H_n(GL_n(F))/H_n(GL_{n-1}(F)) .$$

In particular we have

$$K_3^{[0]}(F)_{\mathbb{Q}} \cong H^3(B_F(3) \otimes \mathbb{Q}) .$$

Conjecture 1.15. c_1 and c_2 in Theorem 1.14 are isomorphisms.

9. Polylogarithms and the weight p motivic complexes $\Gamma_F(p)$. In this section we give an inductive definition of subgroups $\mathcal{R}_p(F) \subset \mathbb{Z}[P^1_F]$ and hence define for all p groups

$$\mathcal{B}_p(F) := \mathbb{Z}[P^1_F] / \mathcal{R}_p(F) .$$

Set $\mathcal{B}_1(F) = F^*$. Let us consider the homomorphisms

$$\mathbb{Z}[P^1_F] \xrightarrow{\delta} F^* \wedge F^*$$

$$\delta : \{x\} \longmapsto (1-x) \wedge x \,, \quad x \in P^1_F \backslash 0,1,\infty \,; \quad \delta : \{0\},\{1\},\{\infty\} \longmapsto 0$$

and

$$\mathbb{Z}[P^1_F] \xrightarrow{\delta} \mathcal{B}_{p-1}(F) \otimes F \,, \quad p \geq 3 \,;$$

$$\delta : \{x\} \longmapsto \{x\}_{p-1} \otimes x \,, \quad x \in P^1_F \backslash 0,1,\infty \,; \quad \delta : \{0\},\{1\},\{\infty\} \longmapsto 0$$

where $\{x\}_{p-1}$ is the image of a generator $\{x\}$ in $\mathbb{Z}[P^1_F] / \mathcal{R}_{p-1}(F)$. These formulae reflect the differential equation (1.1) for $Li_p(z)$. Then the subgroup $\mathcal{R}_p(F)$ is defined as follows:

Let X be a curve over F and F(X) be the field of rational functions on X. Consider an element

$$\alpha = \sum_{i=1}^{N} n_i \{f_i\} \in \mathbb{Z}[P^1_{F(X)}] .$$

A rational function f_i defines a map $f_i : X \longrightarrow P^1_F$. So for any point $u \in X$ there is a specialisation

$$\alpha(u) = \sum_{i=1}^{N} n_i\{f_i(u)\} \in \mathbb{Z}[P^1_F] .$$

Set $\mathcal{A}_p(F) := \mathrm{Ker}(\mathbb{Z}[P^1_F] \xrightarrow{\delta} \mathcal{B}_{p-1}(F) \otimes F^*)$, $p \geq 3$, and

$$\mathcal{A}_2(F) := \mathrm{Ker}(\mathbb{Z}[P^1_F] \xrightarrow{\delta} F^* \wedge F^*) .$$

Let us denote by $\mathcal{R}_p(F)$, where $p \geq 2$, the subgroup of $\mathbb{Z}[P^1_F]$, generated by $\{0\}$, $\{\infty\}$ and $\alpha(u) - \alpha(u')$ where X runs through all connected smooth curves over F , u , u' run through all points of X and $\alpha \in \mathcal{A}_p(F(X))$.

It is easy to prove by induction that $\{x\}_p + (-1)^p\{x^{-1}\} \in \mathcal{A}_p(F(P^1))$. So $(\{x\}_p + (-1)^p\{x^{-1}\}_p) - (\{0\}_p + (-1)^p\{\infty\}_p) \in \mathcal{R}_p(F)$ and hence $\{x\}_p + (-1)^p\{x^{-1}\}_p \in \mathcal{R}_p(F)$. Therefore $2 \cdot \{1\}_p \in \mathcal{R}_p(F)$ for p even. We will see below that $\{1\}_p \notin \mathcal{R}_p(\mathbb{C})$ for p odd.

<u>Lemma</u> 1.16. $\delta(\mathcal{R}_p(F)) = 0$ in $\mathcal{B}_{p-1}(F) \otimes F^*$.

<u>Proof</u>. First of all let us prove by induction that for a variety X/F and an irreducible codimension 1 subvariety $X_0 \hookrightarrow X$ over F there is a specialisation map s_0 : $\mathcal{B}_p(F(X)) \longrightarrow \mathcal{B}_p(F(X_0))$ that is defined on generators by $s_0 : \{f\} \longrightarrow \{f|_{X_0}\}$ if $X_0 \not\subset \mathrm{Supp\,div}\, f$ and $s_0\{f\} = 0$ in the opposite case. We need to check that $s_0(\mathcal{R}_p(F(X))) \subset \mathcal{R}_p(F(X_0))$. Suppose that we have already proved this for $\mathcal{B}_{p-1}$.

Then there is a homomorphism (we suppose that $p \geq 3$; the case $p = 2$ can be considered analogously)

$$s_0 : \mathcal{B}_{p-1}(F(X)) \otimes F(X)^* \longrightarrow \mathcal{B}_{p-1}(F(X_0)) \otimes F(X_0)^*$$

$$s_0 : \{f\} \otimes g \longmapsto \begin{cases} \left\{ f\big|_{X_0} \right\} \otimes g\big|_{X_0} & \text{if } X_0 \subset \text{Supp div } g \cup \text{Supp div } f \\ 0 & \text{otherwise} \end{cases}$$

and $s_0 \circ \delta = \delta \circ s_0$. So if $Y \longrightarrow X$ is a curve over $F(X)$; $i_0, i_1 : X \longrightarrow Y$; $j : X_0 \hookrightarrow X$; $Y(X_0)$ is a fiber of Y over X_0 :

$$\begin{array}{ccc} Y(X_0) & \hookrightarrow & Y \\ i_0 \uparrow \; i_1 \uparrow \; \downarrow & & i_0 \uparrow \; i_1 \uparrow \; \downarrow \\ X_0 & \stackrel{j}{\hookrightarrow} & X \end{array}$$

$g = \sum n_i \{g_i\}_p \in \mathcal{A}_p(F(X)(Y))$, $\tilde{g} = \Sigma\, n_i \left\{ g\big|_{Y(X_0)} \right\}_p$.

Then $\tilde{g} \in \mathcal{A}_p(F(Y(X_0)))$ and so $s_0(i_0^* g - i_1^* g) = i_0^* \tilde{g} - i_1^* \tilde{g} \in \mathcal{R}_p(F(X_0))$.

Lemma 1.16 follows immediately from $s_0 \circ \delta = \delta \circ s_0$. ∎

Set

$$\mathcal{B}_p(F)_{\mathbb{Q}} = \mathbb{Z}[P^1_F] \,/\, \mathcal{R}_p(F) . \qquad (1.24)$$

Let us define a complex $\Gamma_F(p)$ as follows

$$\mathcal{B}_p(F) \xrightarrow{\delta} \mathcal{B}_{p-1}(F) \otimes F^* \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathcal{B}_2(F) \otimes \Lambda^{p-2}F^* \longrightarrow \Lambda^p F^* \qquad (1.25)$$

where $\mathcal{B}_p(F)$ is placed in degree 1 and $\Lambda^p F^*$ in degree p ,

$$\delta(\{x\}_k \otimes y_1 \wedge \dots \wedge y_{p-k}) = \{x\}_{k-1} \otimes x \wedge y_1 \wedge \dots \wedge y_{p-k} \qquad (1.26)$$

for $k > 2$ and

$$\delta(\{x\}_2 \otimes y_1 \wedge \dots \wedge y_{p-2}) = (1-x) \wedge x \wedge y_1 \wedge \dots \wedge y_{p-2} . \qquad (1.27)$$

Then $\delta^2 = 0$ modulo 2–torsion.

<u>Conjecture A</u>: There is a canonical isomorphism

$$H^i\left[\,\Gamma_F(n) \otimes \mathbb{Q}\,\right] = K_{2n-i}^{[n-i]}(F)_{\mathbb{Q}} .$$

Let us denote by $K_n^{\{m\}}(F)_{\mathbb{Q}}$ the subspace of $K_n(F)_{\mathbb{Q}}$ where the Adams operations ψ_ℓ act by multiplication on ℓ^m (see [So]).

<u>Conjecture</u> 1.17. The rank filtration (1.22) is opposite to the Adams γ–filtration after $\otimes \mathbb{Q}$:

$$K_n^{(p)}(F)_{\mathbb{Q}} = \bigoplus_{i \le n-p} K_n^{\{i\}}(F)_{\mathbb{Q}} .$$

It seems that this conjecture was first stated by A.A. Suslin (unpublished).

In the case of number fields $K_n^{\{m\}}(F)_{\mathbb{Q}} \neq 0$ only if $n=2m-1$. Recently J. Yang [Y1] showed that an improvement of arguments of A. Borel [Bo 2] permits to prove for number fields F different from $\mathbb{Q}$ that $K_n^{(p)}(F)_{\mathbb{Q}} \neq 0$ only if $n=2p+1$. So Conjecture 1.17 is valid for number fields $F \neq \mathbb{Q}$.

Now let us give a motivation for the definition of the groups $\mathcal{B}_n(F)$ and the complexes (1.25) and prove that Zagier's conjecture on $\zeta_F(n)$ for number fields is an immediate consequence of (the refined) Conjecture A and Borel's theorem.

<u>Proposition</u> 1.18. Set $\beta_k = \frac{2^k \cdot B_k}{k!}$, where B_k are the Bernoulli numbers, then

$$d\mathcal{L}_{2n}(z) = \mathcal{L}_{2n-1}(z)\, d\arg z - \Big(\sum_{k=2}^{2n-2} \beta_k \cdot \log^{k-1}|z| \cdot \mathcal{L}_{2n-k}(z)\Big) \cdot d\log|z| \quad (1.28a)$$

$$d\mathcal{L}_{2n+1}(z) = -\mathcal{L}_{2n}(z)\, d\arg z - \Big(\sum_{k=2}^{2n-1} \beta_k \cdot \log^{k-1}|z| \cdot \mathcal{L}_{2n+1-k}(z)\Big) \cdot d\log|z|$$

$$-\beta_{2n} \cdot \log^{2n-1}|z| \cdot (\log|z| \cdot d\log|1-z| - \log|1-z| \cdot d\log|z|) \quad (1.28b)$$

<u>Proof</u>. Straightforward calculation using the identities $-\sum_{k=1}^{r} \beta_k \cdot \beta_{r-k} = r \cdot \beta_r + 2\beta_{r-1}$, $-\sum_{k=1}^{r} (-1)^k \beta_k \cdot \beta_{r-k} = r \cdot \beta_r$ that follow easily from the generating function for the β_k, $\Sigma\, \beta_r x^r = \frac{2x}{e^{2x}-1}$.

There is a little bit more natural formula for the function

$$\mathscr{P}_n(z) := \begin{cases} \mathscr{L}_n(z), & n \text{ odd} \\ i \cdot \mathscr{L}_n(z), & n \text{ even} \end{cases}$$

$$d\,\mathscr{P}_n(z) = \mathscr{P}_{n-1}(z)\, d(i \arg z) - \sum_{k=2}^{n-2} \beta_k \log^{k-1}|z|\; \mathscr{P}_{n-k}(z)\, d\log|z| \tag{1.28c}$$
$$-\beta_n \log^{n-1}|z|\,(\log|z|\, d\log|1-z| - \log|1-z|\; d\log|z|)\,.$$

For another formula for $d\mathcal{L}_n(z)$ (without Bernoulli numbers on the right–hand side) see [Z3].

Corollary 1.19. Let us define a homomorphism $\mathcal{L}_n : \mathbb{Q}[P^1_{\mathbb{C}}] \longrightarrow \mathbb{R}$ setting $\mathcal{L}_n(\{z\}) := \mathcal{L}_n(z)$. Then the restriction of $\mathcal{L}_n$ to the subgroup $\mathcal{R}_n(\mathbb{C})_{\mathbb{Q}} \subset \mathbb{Q}[P^1_{\mathbb{C}}]$ is identically zero, so we have a correctly defined homomorphism

$$\mathcal{L}_n : \mathcal{B}_n(\mathbb{C}) \longrightarrow \mathbb{R}.$$

The proof follows by induction from the formulae (1.28). More precisely, there are homomorphisms Δ_k :

$$\Delta_k : \mathbb{Q}[P^1_F] \longrightarrow (\mathcal{B}_k(F)_{\mathbb{Q}} \otimes S^{n-k-1}F^*_{\mathbb{Q}}) \otimes F^*_{\mathbb{Q}}, \quad k \geq 3$$

$$\Delta_2 : \mathbb{Q}[P^1_F] \longrightarrow S^{n-2}F^*_{\mathbb{Q}} \otimes \Lambda^2 F^*_{\mathbb{Q}}$$

that are defined as the following compositions

$$\Delta_k : \mathbb{Q}[P^1_F] \xrightarrow{\delta_n} \mathcal{B}_{n-1}(F)_{\mathbb{Q}} \otimes F^*_{\mathbb{Q}} \xrightarrow{\delta_{n-1}\otimes \mathrm{id}} \dots \longrightarrow$$

$$(\mathcal{B}_k(F)_{\mathbb{Q}} \otimes F^{*\otimes n-k-1}_{\mathbb{Q}}) \otimes F^*_{\mathbb{Q}} \longrightarrow (\mathcal{B}_k(F)_{\mathbb{Q}} \otimes S^{n-k-1}F^*_{\mathbb{Q}}) \otimes F^*_{\mathbb{Q}}$$

$$\Delta_2 : \mathbb{Q}[P^1_F] \xrightarrow{\delta_n} \mathcal{B}_{n-1}(F)_{\mathbb{Q}} \otimes F^*_{\mathbb{Q}} \xrightarrow{\delta_{n-1}\otimes \mathrm{id}} \dots \longrightarrow$$

$$\mathcal{B}_2(F)_{\mathbb{Q}} \otimes S^{n-2}F^*_{\mathbb{Q}} \xrightarrow{\delta_2\otimes \mathrm{id}} \Lambda^2 F^*_{\mathbb{Q}} \otimes S^{n-2}F^*_{\mathbb{Q}}.$$

Let X be a curve over $\mathbb{C}$ and $\alpha(z) = \Sigma\, n_i\{f_i(z)\} \in \mathcal{A}_n(\mathbb{C}(X))$. Consider $\mathcal{L}_n(\alpha(z))$ as a function on X . Then d $\mathcal{L}_n(\alpha(z)) = 0$. Indeed, in this case every term of the right–hand side of (1.28) is zero because of $\Delta_k(\alpha(z)) = 0$ and the induction assumption if $k > 2$. So $\mathcal{L}_n(\alpha(z)) = \text{const}$ and hence $\mathcal{L}_n(\alpha(z_0) - \alpha(z_1)) = 0$. ■

When F is a number field Conjecture A should be refined by the assumption that the composition

$$K_{2n-1}(F) \longrightarrow \mathcal{B}_m(F) \longrightarrow \bigoplus_{\text{Hom}(F,\mathbb{C})} \mathcal{B}_m(\mathbb{C}) \longrightarrow \bigoplus_{\text{Hom}(F,\mathbb{C})} R(m-1)$$

is just the Borel regulator.

Then it implies, of course, Zagier's conjecture on values of Dedekind zeta–functions at integer points.

Corollary 1.19 means that $\mathcal{R}_n(\mathbb{C})$ is just the subgroup of "functional equations" for the n–logarithm $\mathcal{L}_n(z)$. In the definition of $\mathcal{R}_n(F)$ we have an infinite number of 1–variable functional equations. However I believe that there exists an universal many–variable functional equation such that $\mathcal{R}_n(F)$ is generated by its specialisations. Let me state the precise conjecture in the cases n = 2, 3 . Recall that the subgroups $R_2(F)$ and $R_3(F)$ are generated respectively by 5–term relations and relations $R_3(a,b,c)$ see s. 8 and 5. I claim $R_n(F) \subset \mathcal{R}_n(F)$ for n = 2,3 . The proof in the case n = 3 is as follows. We will see in § 4 – 5 that $\delta\, R_3(a,b,c) = 0$ in $B_2(F(a,b,c)) \otimes F(a,b,c)^*$. In fact, this is not so hard to prove directly. Consider $R_3(a,b,c)$ as a function in the variable a , i.e. $R_3(a,b,c) \subset \mathbb{Z}[P^1_{F(a)}]$, b , c fixed. Then by our definition $R_3(a,b,c) - R_3(1,b,c) \in \mathcal{R}_3(F)$. Further, considering $R_3(1,b,c)$

as a function in the variable b we get $R_3(1,b,c) - R_3(1,1,c) \in \mathcal{R}_3(F)$. Finally, $R_3(1,1,c) - R_3(1,1,\infty) \in R_3(F)$, but $R_3(1,1,\infty) = 7\{\infty\} \subset \mathcal{R}_3(F)$. The case $n = 2$ is similar and even simpler.

Conjecture 1.20. $R_n(F) = \mathcal{R}_n(F)$ for $n = 2, 3$.

Let F_0 be the subfield of all constants of F (i.e. of the elements that are algebraic over the prime subfield of F) . There is the following rigidity conjecture of Merkurjev–Suslin for $K_3^{ind}(F)$ (see conjecture 4.10 in [MS]).

Conjecture 1.21. $K_3^{ind}(F_0) = K_3^{ind}(F)$.

Proposition 1.22. If F_0 is algebraically closed then Conjecture 1.20 for $n = 2$ is equivalent to Conjecture 1.21.

Proof.

a) (Conjecture 1.21) ⇒ (Conjecture 1.20). Let X be a curve over F . By Suslin's theorem [S3] $\mathcal{A}_2(F(X))/R_2(F(X)) \otimes \mathbb{Q} \cong K_3^{ind}(F(X))_{\mathbb{Q}}$. So Conjecture 1.21 implies that for any $\alpha(z) = \Sigma\, n_i\{f_i(z)\} \in \mathcal{A}_2(F(X))$ there is a $\beta \in \mathcal{A}_2(F)$ such that $\alpha(z) - \beta \in R_2(F(X))$. Hence specialising we have $\alpha(z_0) - \alpha(z_1) \in R_2(F)$.

b) The claim (Conjecture 1.20) ⇒ (Conjecture 1.21) is a special case of the following

Proposition 1.23. Let F_0 be algebraically closed. Then for $n > 1$

$$H^1(\Gamma_{F_0}(n)_{\mathbb{Q}}) \cong H^1(\Gamma_F(n)_{\mathbb{Q}}) .$$

Proof. Let $\alpha = \sum_{i=1}^{m} n_i\{x_i\} \in \mathscr{A}_n(F)$. Set $I_\alpha := \{f \in F_0[t_1, \dots, t_m] \mid f(x_1, \dots, x_m) = 0\}$. If $\alpha \notin \mathscr{A}_n(F_0)$ then $\dim \operatorname{Spec} F_0[t_1, \dots, t_n]/I_\alpha \geq 1$. Let $(x_1^0, \dots, x_n^0)$ be a point of this variety defined over F_0 . Set $\alpha_0 = \sum_{i=1}^{m} n_i\{x_i^0\}$. Choose a curve Y (over F) containing $(x_1^0, \dots, x_n^0)$ and $(x_1, \dots, x_n)$. Then $\sum_{i=1}^{m} n_i\{t_i\}$ gives an element of $\mathscr{A}_n(F(Y))$ (t_i considered as functions on Y) . So we have $\alpha - \alpha_0 \in \mathscr{R}_n(F)$. ■

Note that Proposition 1.23 and Conjecture A imply the following conjecture.

Conjecture 1.24. Let F_0 be algebraically closed, $F_0 = \overline{\mathbf{F}}$, then for $n \geq 1$ $K_{2n+1}^{[n]}(F_0) = K_{2n+1}^{[n]}(F)$.

This conjecture was stated by A.A. Beilinson (for the Adams filtration), who also has shown that it follows from (his) standard conjectures about categories of motivic sheaves and independently by D. Ramakrishnan in the case when F is algebraically closed (see [R2]).

Remark. Let $\tilde{\mathscr{R}}_p(F) \subset \mathscr{R}_p(F)$ be the subgroup generated by $\{0\}$, $\{\infty\}$ and $\alpha(1) - \alpha(0)$, where $\alpha \in \mathscr{A}_p(F(t))$. (the difference in the definition of $\tilde{\mathscr{R}}_p(F)$ and

$\mathcal{R}_p(F)$ is that we use only P^1_F instead of all curves X/F). Undoubtedly $\tilde{\mathcal{R}}_p(F) = \mathcal{R}_p(F)$. However this is not known even for $p = 2$, where it is equivalent to the rigidity conjecture for $K_3^{ind}(F)$. In any case we can set $\tilde{\mathcal{B}}_p(F) := \mathbb{Z}[P^1_F]/\tilde{\mathcal{R}}_p(F)$ and define motivic complexes $\tilde{\Gamma}_F(n)$. It seems that these complexes are much more convenient for the construction of a natural homomorphism from motivic cohomology to algebraic K–theory ($\otimes \mathbb{Q}$) .

10. <u>The mixed Tate Lie algebra L(Spec F)</u> . For the convenience of the reader we repeat in this section some basic definitions given in [Be 2], [B–D], see also [D1–2], [BMS].

A mixed Tate category is a Tannakian category $\mathcal{M}$ together with a fixed invertible object $\mathbb{Q}(1)_{\mathcal{M}}$ such that any simple object in $\mathcal{M}$ is isomorphic to $\mathbb{Q}(m)_{\mathcal{M}} := \mathbb{Q}(1)_{\mathcal{M}}^{\otimes m}$ for some $m \in \mathbb{Z}$ and

$$\dim \operatorname{Hom}(\mathbb{Q}(0)_{\mathcal{M}}, \mathbb{Q}(m)_{\mathcal{M}}) = \delta_{0,m} \tag{1.29}$$

$$\operatorname{Ext}^1_{\mathcal{M}}(\mathbb{Q}(0)_{\mathcal{M}}, \mathbb{Q}(m)_{\mathcal{M}}) = 0 \ \text{ for } \ m \leq 0 . \tag{1.30}$$

An object $\mathcal{F}$ of $\mathcal{M}$ carries a canonical finite increasing filtration $\ldots \subset \mathcal{F}_{\leq i} \subset \mathcal{F}_{\leq i+1} 1 \subset \ldots$ such that $\mathcal{F}_i := \mathcal{F}_{\leq i} / \mathcal{F}_{\leq i-1}$ is isomorphic to a direct sum of $\mathbb{Q}(-i)_{\mathcal{M}}$'s . Let $\operatorname{Vect}^{\bullet}(\mathbb{Q})$ be the category of finite dimensional vector spaces over $\mathbb{Q}$. Then there is a canonical fiber functor $\omega_{\mathcal{M}}: \mathcal{M} \longrightarrow \operatorname{Vect}^{\bullet}(\mathbb{Q})$, $\omega_{\mathcal{M}}: \mathcal{F} \longmapsto \oplus \operatorname{Hom}(\mathbb{Q}(-i)_{\mathcal{M}}, \mathcal{F}_i)$. Let us denote by $L(\mathcal{M})$ the Lie algebra of derivations of ω : an element $\varphi \in L(\mathcal{M})$ is a natural endomorphism of ω such that $\varphi_{\mathcal{F}_1 \otimes \mathcal{F}_2} =$

$\varphi_{\mathcal{F}_1} \otimes \mathrm{id}_{\omega(\mathcal{F}_2)} + \mathrm{id}_{\omega(\mathcal{F}_1)} \otimes \varphi_{\mathcal{F}_2}$. Then $L(\mathcal{M})$ is a graded pronilpotent Lie algebra such that $L(\mathcal{M})_i = 0$ for $i \geq 0$. Such a Lie algebra is called a mixed Tate Lie algebra. For any mixed Tate Lie algebra L the category L–mod of finite dimensional graded L–modules is a mixed Tate category with $\mathbb{Q}(1) :=$ a trivial one dimensional L–module placed in degree -1 . The fiber functor is just the forgetting of the L–module structure functor. So for any mixed Tate category $\mathcal{M}$ the fiber functor $\omega_{\mathcal{M}}$ defines the equivalence of mixed Tate categories $\omega_{\mathcal{M}}: \mathcal{M} \longrightarrow L(\mathcal{M})$. Any Tate functor $F: \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ (that is, by definition, an exact $\otimes$ functor such that $F(\mathbb{Q}(1)_{\mathcal{M}_1}) = \mathbb{Q}(1)_{\mathcal{M}_2}$) defines a morphism $F_\bullet : L(\mathcal{M}_1) \longrightarrow L(\mathcal{M}_2)$. For $\mathcal{F} \in \mathrm{Ob}\,\mathcal{M}$ set $H^\bullet_{\mathcal{M}}(\mathcal{F}) := \mathrm{Ext}^\bullet_{\mathcal{M}}(\mathbb{Q}(0)_{\mathcal{M}}, \mathcal{F}) = H^\bullet(L(\mathcal{M}), \omega_{\mathcal{M}}(\mathcal{F}))$.

A.A. Beilinson conjectured that for any scheme S there exists a certain mixed Tate category $\mathcal{M}_T(S)$ of (mixed) motivic Tate (perverse) $\mathbb{Q}$–sheaves over S – see [B1]. He also conjectured that in the case $S = \mathrm{Spec}\, F$, where F is a field, $\mathcal{M}_T(F) := \mathcal{M}_T(\mathrm{Spec}\, F)$, the following holds:

$$\mathrm{Ext}^i_{\mathcal{M}_T(F)}(\mathbb{Q}(0)_{\mathcal{M}_T(F)}, \mathbb{Q}(n)_{\mathcal{M}_T(F)}) = K^{\{n\}}_{2n-i}(F)_{\mathbb{Q}} . \tag{1.31}$$

11. Conjecture A $\Leftrightarrow$ Conjecture B in the Beilinson World.

Conjecture A′. The complex $\Gamma_F(n)_{\mathbb{Q}}$ represents $R\,\mathrm{Hom}(\mathbb{Q}(0)_{\mathcal{M}_T(F)} , \mathbb{Q}(n)_{\mathcal{M}_T(F)})$ in the derived category of $\mathcal{M}_T(F)$.

Let L(F) be the mixed Tate Lie algebra corresponding to the category $\mathcal{M}_T(F)$. Then (1.31) can be rewritten as

$$H^i(L(F),\mathbb{Q}(n)) = K^{\{n\}}_{2n-i}(F)_{\mathbb{Q}} . \tag{1.32}$$

In particular we have an isomorphism of $\mathbb{Q}$–vector spaces

$$L(F)^{\vee}_{-1} = H^1(L(F),\mathbb{Q}(1)) = F^*_{\mathbb{Q}} . \tag{1.33}$$

Set $L(F)_{\leq -2} := \bigoplus_{i=-1}^{-\infty} L(F)_i$. Sometimes we will write L , $L_{\leq -2}$ and so on, omitting F . It is well–known that for a nilpotent Lie algebra $\mathfrak{g}$ the space $H_1(\mathfrak{g}) = \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ can be interpreted as the space of generators of $\mathfrak{g}$. So the space of degree $-n$ generators of $L_{\leq -2}$ is isomorphic to $L_{-n}/[L_{\leq -2},L_{\leq -2}]_{-n}$, where $[L_{\leq -2},L_{\leq -2}]_{-n} := \sum_{k=2}^{n-2} [L_{-k},L_{-(n-k)}]$. The Lie algebra L acts on $L_{\leq -2}/[L_{\leq -2},L_{\leq -2}]$ through its abelian quotient $L/L_{\leq -2}$. The action is described by a map

$$f_n : L/L_{\leq -2} \otimes H_1(L_{\leq -2})_{-(n-1)} \longrightarrow H_1(L_{\leq -2})_{(-n)} .$$

Let

$$f^*_n : H^1(L_{\leq -2})_{(n)} \longrightarrow H^1(L_{\leq -2})_{(n-1)} \otimes (L/L_{\leq -2})^{\vee} \tag{1.34}$$

be the dual map.

Conjecture B.

a) For an arbitrary field F $L(F)_{\leq -2}$ is a free graded pro–Lie algebra.

b) $H_1(L(F)_{\leq -2})^{\vee}_{-n} \cong \mathcal{B}_n(F)_{\mathbb{Q}}$, $n \geq 1$.

c) The map f_n^* (see 1.34) coincides with the differential $\delta : \mathcal{B}_n(F)_{\mathbb{Q}} \longrightarrow \mathcal{B}_{n-i}(F)_{\mathbb{Q}} \otimes F_{\mathbb{Q}}^*$ in the complex $\Gamma(n)_{\mathbb{Q}}$, $(n \geq 2)$.

Proposition 1.25. Conjecture B implies Conjecture A.

Proof. Let us compute $H^*(L(F),\mathbb{Q}(n))$ using the graded version of the Hochschild–Serre spectral sequence related to the ideal $L(F)_{\leq -2} \subset L(F)$. Then

$$E_1^{p,q} = \Lambda^p F_{\mathbb{Q}}^* \otimes H^q(L(F)_{\leq -2},\mathbb{Q}(n-p)) .$$

We have: $H^q(L(F)_{\leq -2},\mathbb{Q}(n)) = 0$ for $q \geq 2$ because the Lie algebra $L(F)_{\leq -2}$ is free, $H^1(L(F)_{\leq -2},\mathbb{Q}(m)) = \mathcal{B}_m(F)_{\mathbb{Q}}$ $(m \geq 2)$, $H^0(L(F)_{\leq -2},\mathbb{Q}(m)) = \mathbb{Q}$ if $m = 0$ and 0 in other cases. So

$$E_1^{p,q} = \begin{cases} 0 & \text{if } q = 0 \\ \Lambda^p F_{\mathbb{Q}}^* \otimes \mathcal{B}_{n-p}(F)_{\mathbb{Q}} & \text{if } q = 1 , \ 0 \leq p \leq n-2 \\ \Lambda^n F_{\mathbb{Q}}^* & \text{if } q = 0 , \ p = n \\ 0 & \text{if } q = 0 , \ p \neq n \end{cases}$$

$d_1^{p,1} : E_1^{p,1} \longrightarrow E_1^{p+1,1}$ coincides with the differential in (1.26)

$$\delta : \Lambda^{p}F^{*}_{\mathbb{Q}} \otimes \mathcal{B}_{n-p}(F)_{\mathbb{Q}} \longrightarrow \Lambda^{p+1}F^{*}_{\mathbb{Q}} \otimes \mathcal{B}_{n-p-1}(F)_{\mathbb{Q}} \quad (p+1 \le n-2)$$

and $d_2^{n-2,0} : E_2^{n-2,1} \longrightarrow E_2^{n,0}$ with the one in (1.27)

$$\delta : \Lambda^{n-2}F^{*}_{\mathbb{Q}} \otimes \mathcal{B}_2(F)_{\mathbb{Q}}/\delta(\Lambda^{n-3}F^{*}_{\mathbb{Q}} \otimes \mathcal{B}_3(F)_{\mathbb{Q}}) \longrightarrow \Lambda^{n}F^{*}_{\mathbb{Q}}.$$

Other differentials in the spectral sequence are zero, so we get just the complex $\Gamma_F(n)_{\mathbb{Q}}$.

Now let us prove that under some natural assumptions Conjecture A implies Conjecture B.

Let $(\Lambda^{\bullet}(L^{\vee}_{\bullet}),\partial)$ be the cochain complex of the Lie algebra $L_{\bullet}$. It has a natural grading: $\Lambda^{\bullet}(L_{\bullet}) = \bigoplus_{n=1}^{\infty} \Lambda^{\bullet}(L^{\vee}_{\bullet})_n$. Suppose that there are homomorphisms $\varphi_k : \mathcal{B}_k(F)_{\mathbb{Q}} \longrightarrow L(F)^{\vee}_{-k}$, $k \ge 1$, such that the following diagrams are commutative

$$\begin{array}{ccc} \mathcal{B}_k(F)_{\mathbb{Q}} & \xrightarrow{\ \delta\ } & \mathcal{B}_{k-1}(F)_{\mathbb{Q}} \otimes F^{*}_{\mathbb{Q}} \\ \varphi_k \big\downarrow & & \big\downarrow \varphi_{k-1} \wedge \varphi_1 \\ L(F)^{\vee}_{-k} & \xrightarrow{\ \partial\ } & (\Lambda^2 L(F)^{\vee})_{-k} \end{array} \quad , \quad k \ge 3 \qquad (1.35a)$$

$$\begin{array}{ccc} \mathcal{B}_2(F) & \xrightarrow{\ \delta\ } & \Lambda^2 F^{*} \\ \varphi_2 \big\downarrow & & \big\downarrow \varphi_1 \wedge \varphi_1 \\ L(F)^{\vee}_{-2} & \xrightarrow{\ \partial\ } & \Lambda^2 L(F)^{\vee}_{-1} \end{array} \qquad (1.35b)$$

Then we have a homomorphism of complexes

$$\psi_k : \Gamma_F(k)_{\mathbb{Q}} \longrightarrow (\Lambda^{\bullet}(L(F)^{\vee}),\partial)_k \tag{1.36}$$

$$\psi_k : \{x\}_m \otimes y_1 \wedge \ldots \wedge y_{k-m} \longmapsto \varphi_m(\{x\}_m) \otimes \varphi_1(y_1) \wedge \ldots \wedge \varphi_1(y_{k-m}) .$$

Now let $(\Lambda^{\bullet}(L^{\vee}_{\leq -2}),\partial')$ be the cochain complex of the Lie algebra $L_{\leq -2}$. Then it follows from (1.35) that $\partial' \circ \varphi_k(\mathcal{B}_k(F)_{\mathbb{Q}}) = 0$. So φ_k gives the map

$$\overline{\varphi}_k : \mathcal{B}_k(F)_{\mathbb{Q}} \longrightarrow H^1(L(F)_{\leq -2},\mathbb{Q}(k)) \tag{1.37}$$

where $H^1(L(F)_{\leq -2},\mathbb{Q}(k))$ is the subspace of $L(F)^{\vee}_{-k}$ consisting of functionals with zero restriction to $[L(F)_{\leq -2},L(F)_{\leq -2}] \cap L(F)_{-k}$. It is isomorphic to the dual to the space of degree $-k$ generators of the Lie algebra $L(F)_{\leq -2}$.

<u>Proposition</u> 1.26. Suppose that ψ_k (see 1.36) is a quasiisomorphism of complexes. Then $L(F)_{\leq -2}$ is a free graded Lie algebra and $\overline{\varphi}_k$ is an isomorphism.

<u>Proof</u>. $\psi_1 : \mathcal{B}_1(F)_{\mathbb{Q}} = F^*_{\mathbb{Q}} \xrightarrow{\sim} L(F)^{\vee}_{-1}$ and an easy induction shows that $\ker \varphi_k = 0$ for all k.

Let us prove using induction on the degree that $H^i(L_{\leq -2}) = 0$ for $i \geq 2$. There is a filtration $\mathcal{F}^*$ on the complex $\Lambda^{\bullet}(L^{\vee}_{\bullet})$:

$$\mathcal{F}^k \Lambda^{\bullet}(L^{\vee}_{\bullet}) = \Lambda^k L^{\vee}_{-1} \otimes \Lambda^{\bullet}(L^{\vee}_{\bullet})$$

$$\mathrm{gr}^k_{\mathcal{F}} \Lambda^{\bullet}(L^{\vee}_{\bullet}) = \Lambda^k L^{\vee}_{-1} \otimes (\Lambda^{\bullet}(L_{\leq -2}),\partial') .$$

Let $a \in \Lambda^i(L^\vee_{\leq -2})_{-k}$, $i \geq 2$ and $\partial' a = 0$. Then $\partial a \in \mathcal{F}^1\Lambda^\bullet(L^\vee_\bullet)$; let b_1 be the image of ∂a in $gr^1\Lambda^\bullet(L^\vee_\bullet)$. Denote by $\partial_{(k)}$ the coboundary in the complex $gr^k_{\mathcal{F}}\Lambda^\bullet(L^\vee_\bullet)$. Then $\partial_{(1)}\partial a = 0$. By the induction assumption there is an $a_1 \in L^\vee_{-1} \otimes \Lambda^{i-1}(L^\vee_{\leq -2})$ such that $\partial_{(1)}a_1 = b_1$. Therefore $\partial(a - a_1) \in \mathcal{F}^2\,\Lambda^\bullet(L^\vee_\bullet)$. Let b_2 be the image of $\partial(a - a_1)$ in $gr^2\Lambda^\bullet(L^\vee_\bullet)$. Then $\partial_{(2)}b_2 = 0$ and we can find an $a_2 \in \Lambda^2(L^\vee_{-1}) \otimes \Lambda^{i-2}(L^\vee_{\leq -2})$ such that $\partial_{(2)}a_2 = b_2$; consider $a - a_1 - a_2$ and so on. Finally we get an element $b_i \in \Lambda^i(L^\vee_{-1}) \otimes L^\vee_{-(k-i)}$ such that $\partial b_i = 0$. The quotient $\Lambda^\bullet(L^\vee_\bullet)_k/\psi_k(\Gamma_F(k)_{\mathbb{Q}})$ is an acyclic complex because $\ker \psi_k = 0$ and ψ_k is a quasiisomorphism. So there is an $a_i \in \Lambda^\bullet(L^\vee_\bullet)_k$ such that $\partial a_i = b_i \mod \Lambda^i(L^\vee_{-1}) \otimes \varphi_{k-i}(\mathcal{B}_{-(k-i)})$. Let $a - a_1 - \ldots - a_i = x + x'$ where $x' \in gr^1\Lambda^\bullet(L^\vee_\bullet)$ and $x \in \Lambda^\bullet(L_{\leq -2})$. Then $\partial' x = a$. So we have proved that $H^i(L_{\leq -2}) = 0$ for $i \geq 2$.

Now let us prove by induction that the homomorphism $\overline{\varphi}_k$ (see 1.37) is an isomorphism. We have by definition that

$$H^1(L_\bullet,\mathbb{Q}(k)) = \ker\left(H^1(L_{\leq -2},\mathbb{Q}(k)) \xrightarrow{\ \partial\ } L^\vee_{-(k-1)} \otimes F^*_{\mathbb{Q}}\right)$$

and it is easy to see that the image of ∂ lies in $H^1(L_{\leq -2},\mathbb{Q}(k-1)) \otimes F^*$ which is just $\mathcal{B}_{k-1}(F)_{\mathbb{Q}} \otimes F^*_{\mathbb{Q}}$ by the induction assumption $(k \geq 3)$. Therefore there is a quasiisomorphism of complexes

$$\begin{array}{ccccc} \mathcal{B}_k(F)_{\mathbb{Q}} & \xrightarrow{\quad\delta\quad} & \mathcal{B}_{k-1}(F)_{\mathbb{Q}} \otimes F^*_{\mathbb{Q}} & \xrightarrow{\ \delta\ } & \ldots \\ {\scriptstyle\overline{\varphi}_k}\downarrow & & \Vert & & \\ H^1(L_{\leq -2},\mathbb{Q}(k)) & \xrightarrow{\quad\partial\quad} & \mathcal{B}_{k-1}(F)_{\mathbb{Q}} \otimes F^*_{\mathbb{Q}} & \xrightarrow{\ \delta\ } & \ldots \end{array}$$

So $\overline{\varphi}_k$ is an isomorphism. Q.E.D.

We see that if we assume homomorphisms $\varphi_k : \mathcal{B}_k(F)_{\mathbb{Q}} \longrightarrow (L(F)_{-k})$ providing quasiisomorphisms $\psi_k : \Gamma_F(k)_{\mathbb{Q}} \longrightarrow (\Lambda^\bullet(L(F)_\bullet^\vee),\partial)$, and assume also Beilinson's conjecture (1.31) (together with the rank conjecture (1.17)), then Conjecture A $\Rightarrow$ Conjecture B .

12. <u>Some evidence for Conjectures A and B</u> .

The motivic category $\mathcal{M}_T(\mathrm{Spec}\, F)$ and the Lie algebra $L(F)_\bullet$ are rather mysterious objects, whose existence is not proved yet. However Beilinson's conjecture relating Ext's in the category $\mathcal{M}_T(\mathrm{Spec}\, F)$ with algebraic K–theory together with a symbolic description of the first pieces of K–groups gives a key for an understanding of the structure of $L(F)_\bullet$. For example, we have already seen before that $L(F)_{-1}^\vee$ should be isomorphic as a profinite $\mathbb{Q}$–vector space to $F_{\mathbb{Q}}^{*\vee}$. Further, assume that there is a homomorphism $\varphi_2 : B_2(F) \longrightarrow L(F)_{-2}^\vee$ making the following diagram commutative

$$\begin{array}{ccc} B_2(F)_{\mathbb{Q}} & \xrightarrow{\ \delta\ } & \Lambda^2 F_{\mathbb{Q}}^* \\ {\scriptstyle\varphi_2}\big\downarrow & & \big\downarrow {\scriptstyle \mathrm{id}}\ \int \\ L(F)_{-2}^\vee & \xrightarrow{\ \delta\ } & \Lambda^2 F_{\mathbb{Q}}^* \end{array} \tag{1.39}$$

(On the right–hand side of the bottom line stands $\Lambda^2 L(F)_{-1}^\vee$ but it should be isomorphic to $\Lambda^2 F_{\mathbb{Q}}^*$) The cohomology of the Bloch–Suslin complex (upper line in (1.39)) is isomorphic to $K_3^{ind}(F)_{\mathbb{Q}}$ and $K_2(F)_{\mathbb{Q}}$. Beilinson's conjecture (see (1.31)) predicts that the bottom line should have the same cohomology. So it is natural to assume that the vertical arrows induce an isomorphism on cohomologies. Then $\varphi_2 : B_2(F)_{\mathbb{Q}} \longrightarrow L(F)^{\vee}_{-2}$ must be an isomorphism!

In fact the last assumption (that the morphism of complexes (1.39) is a quasiisomorphism) can be deduced from the Borel theorem and standard conjectures: rigidity and existence of the Hodge realisation. Indeed, if F is a number field, then the following diagram should be commutative

$$\begin{array}{ccc} H_1(B_F(2) \otimes \mathbb{Q}) & \overset{r_B}{\searrow} & \\ \tilde{\varphi}_2 \downarrow & & H^1_{\mathcal{D}}(\text{Spec } F, \mathbb{R}(2)) = \mathbb{R}^{r_2} \\ H^1(L(F)_\bullet, \mathbb{Q}(2)) & \underset{r_{\mathcal{H}}}{\nearrow} & \end{array}$$

where r_B is the Borel regulator and $r_{\mathcal{H}}$ is the regulator from the motivic cohomology of Spec F to the Deligne cohomology provided by the expected functor of the Hodge realisation of our motivic category. By the Borel theorem r_B is injective, so $\tilde{\varphi}_2$ is also incjective. By Suslin's theorem both $\mathbb{Q}$–vector spaces have the same dimension: $\dim K_3^{ind}(F)_{\mathbb{Q}} = r_2$. So $\tilde{\varphi}_2$ is an isomorphism. Now the rigidity conjecture tells us that $\tilde{\varphi}_2$ should be an isomorphism for an arbitrary field F . Therefore φ_2 is injective. Further, we have

$$\Lambda^2\varphi_1 : \Lambda^2 F^* / \delta(B_2(F)_{\mathbb{Q}}) \xrightarrow{\sim} \Lambda^2 L(F)^{\vee}_{-1} / \partial \circ \varphi_2(B_2(F)_{\mathbb{Q}}) \, .$$

The left hand side is isomorphic to $K_2(F)_{\mathbb{Q}}$. So $\partial(L(F)^{\vee}_{-2}) = \partial \circ \varphi_2(B_2(F)_{\mathbb{Q}})$ and therefore φ_2 is an isomorphism.

Now let us consider weight 3 motivic complexes. Assume a homomorphism $\varphi_3 : \mathcal{B}_3(F) \longrightarrow L(F)^{\vee}_{-3}$ making diagram (1.35a) commutative. There is a homomorphism $B_3(F) \longrightarrow \mathcal{B}_3(F)$ (see s. 9), so we get a morphism of complexes (assuming $L(F)^{\vee}_{-2} \cong B_2(F)_{\mathbb{Q}}$, $L(F)^{\vee}_{-1} \cong F^*_{\mathbb{Q}}$)

$$\begin{array}{ccccc} B_3(F) & \xrightarrow{\delta} & B_2(F) \otimes F^* & \xrightarrow{\delta} & \Lambda^3 F^* \\ \varphi_3 \big\downarrow & & \int \big\downarrow \varphi_2 \otimes \varphi_1 & & \int \big\downarrow \Lambda^3 \varphi_1 \\ L(F)^{\vee}_{-3} & \xrightarrow{\partial} & L(F)^{\vee}_{-2} \otimes L(F)^{\vee}_{-1} & \xrightarrow{\partial} & \Lambda^3 L(F)^{\vee}_{-1} \end{array}$$

The bottom complex is just $(\Lambda^{\bullet}(L(F)_{\bullet})_3, \partial)$ – the part of grading 3 of the cochain complex of the Lie algebra $L(F)_{\bullet}$.

The main results of this paper give considerable evidence for the expected isomorphisms

$$H^i(B_F(3) \otimes \mathbb{Q}) \xrightarrow{\sim} H^i(\Lambda^{\bullet}(L(F)_{\bullet})_3) . \tag{1.40}$$

More precisely, the same arguments as above show that in order to be convinced of (1.40) it suffices to prove Conjecture 1.15. It is easy to see that (1.40) implies that $\varphi_3 : B_3(F)_{\mathbb{Q}} \longrightarrow L(F)^{\vee}_{-3}$ is an isomorphism.

In any case the complexes $(\Lambda^{\bullet}(L(F)_{\bullet})_n, 0)$ for $n = 2,3$ look like the complexes $\Gamma_F(n)$. But already the weight 4 part of the cochain complex of $L(F)_{\bullet}$, that is

$$L(F)^{\vee}_{-4} \xrightarrow{\partial} \begin{array}{c} L(F)^{\vee}_{-3} \otimes L(F)^{\vee}_{-1} \\ \oplus \\ \Lambda^2 L(F)^{\vee}_{-2} \end{array} \xrightarrow{\partial} \tag{1.41}$$

$$\xrightarrow{\partial} L(F)^{\vee}_{-2} \otimes \Lambda^2 L(F)^{\vee}_{-1} \xrightarrow{\partial} \Lambda^4 L(F)^{\vee}_{-1} , \tag{1.41}$$

looks quite different from $\Gamma_F(4)$, because we have an extra term $\Lambda^2 L(F)^{\vee}_{-2}$ $(4 = 2 + 2)$ that has no analog in $\Gamma_F(4)$. So assuming a homomorphism $\varphi_4 : \mathcal{B}_4(F)_{\mathbb{Q}} \longrightarrow L(F)^{\vee}_{-4}$ making (1.35a) commutative we get a homomorphism $\tilde{\varphi}_4$ of $\Gamma_F(4)$ to the complex (1.41), but it can't be an isomorphism. However, the following important lemma shows that $\tilde{\varphi}_4 : H^3(\Gamma_F(4) \otimes \mathbb{Q}) \longrightarrow H^3(\Lambda^\bullet(L(F)_\bullet)_4)$ is an isomorphism (assuming $L(F)^{\vee}_{-n} \cong \mathcal{B}_n(F)_{\mathbb{Q}} \cong B_n(F)_{\mathbb{Q}}$ for $n = 1,2,3)$.

Lemma 1.27.

$$\delta\Big[(-\{1-x\}_3 - \{1-y\}_3 + \left\{\frac{1-x}{1-y}\right\}_3 - \left\{\frac{1-x^{-1}}{1-y^{-1}}\right\}_3 - \{\tfrac{x}{y}\}_3) \otimes \frac{x}{y} +$$

$$+ \{x\}_3 \otimes (1-y) - \{y\}_3 \otimes (1-x) + \{\tfrac{x}{y}\}_3 \otimes \frac{1-x}{1-y}\Big] = \delta[\{x\}_2 \wedge \{y\}_2] =$$

$$= \{y\}_2 \otimes (1-x) \wedge x - \{x\}_2 \otimes (1-y) \wedge y .$$

Proof. Direct calculation. ∎

More precisely, this lemma proves that $\partial(\Lambda^2 L(F)^{\vee}_{-2}) \subset \partial(L(F)^{\vee}_{-3} \otimes L(F)^{\vee}_{-1})$ if we assume only $L(F)^{\vee}_{-2} \cong B_2(F)_{\mathbb{Q}}$ and $\varphi_3 : B_3(F) \longrightarrow L(F)^{\vee}_{-3}$ making (1.35a) commutative (we need not assume that φ_3 is an isomorphism).

Corollary 1.28. Assume that for $n = 1,2,3$ there exist isomorphisms $\varphi_n : B_n(F)_{\mathbb{Q}} \xrightarrow{\sim} L(F)^{\vee}_{-3}$ making diagram (1.35) commutative. Then

$$H^{n-1}(L(F)_\bullet,\mathbb{Q}(n)) \cong \ker(B_2(F)_\mathbb{Q} \otimes \Lambda^{n-2}F^*_\mathbb{Q} \longrightarrow \Lambda^n F^*_\mathbb{Q})/\{x\}_2 \otimes x \wedge \Lambda^{n-3}F^*_\mathbb{Q} . \quad (1.42)$$

Proof. The left hand side is just the cohomology of the following complex

$$\begin{matrix} L^\vee_{-3} \otimes \Lambda^{n-3}L^\vee_{-1} \\ \oplus \\ \Lambda^2 L^\vee_{-2} \otimes \Lambda^{n-4}L^\vee_{-1} \end{matrix} \xrightarrow{\ \partial\ } L^\vee_{-2} \otimes \Lambda^{n-2}L^\vee_{-1} \xrightarrow{\ \partial\ } \Lambda^n L^\vee_{-1} .$$

It remains to apply Lemma 1.27. ■

Note that the right–hand side of (1.42) maps to $H^{n-1}(\Gamma_F(n) \otimes \mathbb{Q})$ and this map should be an isomorphism because of the rigidity conjecture: $B_2(F)_\mathbb{Q} = \mathcal{B}_2(F)_\mathbb{Q}$.

Consider the following element $(\varphi_k\{a\} := \varphi_k\{a\}_k)$:

$$\varphi_3\left[-\{1-x\} - \{1-y\} + \left\{\frac{1-x}{1-y}\right\} - \left\{\frac{1-x^{-1}}{1-y^{-1}}\right\} - \left\{\frac{x}{y}\right\}\right] \otimes \frac{x}{y} +$$

$$+ \varphi_3\{x\} \otimes (1-y) - \varphi_3\{y\} \otimes (1-x) + \varphi_3\left\{\frac{x}{y}\right\} \otimes \frac{1-x}{1-y} \qquad (1.43)$$

$$- \varphi_2\{x\} \wedge \varphi_2\{y\}$$

that lies in $L^\vee_{-3} \otimes L^\vee_{-1} \oplus \Lambda^2 L^\vee_{-2}$. By Lemma 1.27 its coboundary is zero, so there should be an element $\phi_4(x,y) \subset L^\vee_{-4}$ whose coboundary is (1.43). Let us assume that such an element $\phi_4(x,y)$ exists.

Now look at the weight 5 part of the cochain complex of $L(F)_\bullet$:

$$L^\vee_{-5} \xrightarrow{\partial} \begin{matrix} L^\vee_{-4} \otimes L^\vee_{-1} \\ \oplus \\ L^\vee_{-3} \otimes L^\vee_{-2} \end{matrix} \xrightarrow{\partial} \begin{matrix} L^\vee_{-3} \otimes \Lambda^2 L^\vee_{-1} \\ \oplus \\ \Lambda^2 L^\vee_{-2} \otimes L^\vee_{-1} \end{matrix} \xrightarrow{\partial} L^\vee_{-2} \otimes \Lambda^3 L^\vee_{-1} \xrightarrow{\partial} \Lambda^5 L^\vee_{-1} .$$

We would like to prove that the component $\partial_{3,2} : L^\vee_{-5} \longrightarrow L^\vee_{-3} \otimes L^\vee_{-2}$ of the coboundary ∂ is an epimorphism. Unfortunately it is not quite clear how to construct an element in $L^\vee_{-5}$ because $L^\vee_{-5}$ itself is a quite mysterious object. However, assuming the existence of $\phi_4(x,y)$ we can find an element in $L^\vee_{-4} \otimes L^\vee_{-1} \oplus L^\vee_{-3} \otimes L^\vee_{-2}$ with zero coboundary, whose component in $L^\vee_{-3} \otimes L^\vee_{-2}$ is $\varphi_3\{x\} \otimes \varphi_2\{y\}$. We expect that such a cycle should be in $\partial(L^\vee_{-5})$.

Let's do this. We assume a $\varphi_4 : \mathcal{B}_4(F) \longrightarrow L(F)^\vee_{-4}$ making (1.35a) commutative. Consider the following element

$$\phi_5(x,y) := \phi_4(x,y) \otimes \frac{x}{y} + \varphi_4 \left\{\frac{x}{y}\right\} \otimes \frac{1-x}{1-y} + \varphi_4\{x\} \otimes (1-y) +$$

$$+ \varphi_4(y) \otimes (1-x) - \varphi_3\{x\} \otimes \varphi_2\{y\} - \varphi_3\{y\} \otimes \varphi_2\{x\} .$$

Lemma 1.29. $\partial\phi_5(x,y) = 0$.

Proof. Direct calculation using formula (1.43) for $\partial\phi_4(x,y)$. ∎

The $L^\vee_{-3} \otimes L^\vee_{-2}$ component of $-1/2\,(\phi_5(x,y) + \phi_5(x,y^{-1}))$ is equal to $\varphi_3\{x\} \otimes \varphi_2\{y\}$ because $\{y\}_2 + \{y^{-1}\}_2 = 0$ in $B_2(F)_{\mathbb{Q}}$ and $\{y\}_3 = \{y^{-1}\}_3$ in $B_3(F)_{\mathbb{Q}}$.

We can pursue this idea further and "construct" by induction elements $\phi_n(x,y) \in L(F)^{\vee}_{-n}$ (using the same assumptions as above) such that $(\varphi_1\{a\} := 1 - a \in F^*)$

$$\partial\phi_n(x,y) = \phi_{n-1}(x,y) \otimes \frac{x}{y} + \varphi_{n-1}\left\{\frac{x}{y}\right\} \otimes \frac{1-x}{1-y} + \tag{1.44$_{(n)}$}$$

$$+ \sum_{k=1}^{\left[\frac{n}{2}\right]} (-1)^{k-1}(\varphi_{n-k}\{x\} \otimes \varphi_k\{y\} + (-1)^{n-k}\varphi_{n-k}\{y\} \otimes \varphi_k\{x\})$$

for n odd; for n even we have the same formula, but the last term will be $(-1)^{n/2-1} \cdot \varphi_{n/2}\{x\} \wedge \varphi_{n/2}\{y\}$.

Proposition 1.30. Suppose that $\partial\phi_{n-1}(x,y)$ is given by formula $(1.44)_{n-1}$. Then the coboundary of the right hand side of $(1.44)_n$ is equal to 0.

Proof. Direct calculation using the formula

$$\partial(\phi_{n-1}(x,y) \otimes \frac{x}{y} + \varphi_{n-1}\left\{\frac{x}{y}\right\} \otimes \frac{1-x}{1-y}) = +$$

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^{k-1}(\varphi_{n-1-k}\{x\} \otimes \varphi_k\{x\} + (-1)^{n-1-k}\varphi_{n-1-k}\{y\} \otimes \varphi_k\{x\})$$

(for n odd the last term in this sum should be $(-1)^{\frac{n-1}{2}-1}\varphi_{\frac{n-1}{2}}\{x\} \wedge \varphi_{\frac{n-1}{2}}\{y\}$.

13. A topological consequence of Conjecture B. This section grew up in discussions with A.A. Beilinson in May 1990. In the Beilinson World any morphism of schemes $f: X \longrightarrow Y$ defines an exact "inverse image" function $f^*: M_T(Y) \longrightarrow M_T(X)$ and so a morphism of corresponding Lie algebras $L(Y)_\bullet \longrightarrow L(X)_\bullet$. In particular, if X is a variety over a field F we have a canonical epimorphism $L(X)_\bullet \longrightarrow L(\mathrm{Spec}\, F)_\bullet$. Its kernel is called the geometrical fundamental Lie algebra of X: $L^{geom}(X)$.

For a generic point of X we have

$$0 \longrightarrow L^{geom}(F(X))_\bullet \longrightarrow L(F(X))_\bullet \longrightarrow L(F)_\bullet \longrightarrow 0 .$$

The commutant of $L^{geom}(F(X))_\bullet$ lies in $L(F(X))_{\leq -2}$ and so it should be a free graded pro–Lie algebra. Now suppose that $F = \mathbb{C}$. Let $\mathcal{H}_X$ be the category of "good" unipotent variations of mixed Hodge–Tate structures over a complex manifold X (see [H–Z]). There is a canonical fiber functor $\omega_H: H_X \longrightarrow \mathrm{Vect}_{\mathbb{Q}}^\bullet$, $\omega_{\mathcal{H}}: H \longmapsto \bigoplus_i \mathrm{Hom}(\mathbb{Q}(-i)_{\mathcal{H}}, \mathrm{gr}_i^W H)$, where $\mathbb{Q}(-i)_{\mathcal{H}} \in \mathcal{H}_X$ is a constant variation of the Tate structure of weight i. There should be a canonical Hodge realization functor $r_{\mathcal{H}}: \mathcal{M}_T(X) \longrightarrow \mathcal{H}_X$ commuting with these fiber functors:

$$\begin{array}{ccc} \mathcal{M}_T(X) & \xrightarrow{\ \omega_{\mathcal{M}}\ } & \\ {\scriptstyle r_{\mathcal{H}}}\downarrow & & \mathrm{Vect}^{\bullet}_{\mathbb{Q}} \\ \mathcal{H}_X & \xrightarrow{\ \omega_{\mathcal{H}}\ } & \end{array}$$

So we get a morphism of the corresponding mixed Tate Lie algebras $L^{geom}_{\mathcal{H}}(X)_\bullet \longrightarrow L^{geom}(X)_\bullet$ that should be an isomorphism in the Beilinson World. More precisely, A.A. Beilinson proved that this follows from standard conjectures including the Hodge conjecture – see a future version of [B–D].

On the other hand, $L^{geom}_{\mathcal{H}}(X)_\bullet$ is isomorphic to the Lie algebra of the maximal Tate quotient of the pronilpotent completion of the classical fundamental group $\pi_1(X,x)$, $x \in X$.

More precisely, Hain and Zucker introduced the notion of "good" variation of a mixed Hodge structure on an open manifold X (that is some special conditions at infinity – see conditions 1.5 i), 1.5 ii) in [H–Z]) and proved the following

<u>Theorem</u> 1.31 ([H–Z]). Fix any $x \in X$. Then the monodromy representation functor defines an equivalence of categories

$$\left\{\begin{array}{l}\text{"good" unipotent variations}\\ \text{of mixed Hodge structures}\\ \text{defined over } \mathbb{Q}\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{mixed Hodge theoretic representations}\\ \text{of } \varprojlim_{r} \mathbb{Q}\,\pi_1(X,x)/\mathcal{J}^r \text{ defined over } \mathbb{Q}\end{array}\right\}$$

($\mathcal{J}$ is the kernel of the usual augmentation of the group ring)

On the other hand, let $L_{\mathcal{H}}(*)_\bullet$:= mixed Tate Lie algebra corresponding to the category of mixed Tate $\mathbb{Q}$–Hodge structures (over a point $*$). Then we have

$$0 \longrightarrow L_{\mathscr{H}}^{geom}(X)_\bullet \longrightarrow L_{\mathscr{H}}(X)_\bullet \longrightarrow L_{\mathscr{H}}(*)_\bullet \longrightarrow 0$$

A point $x \in X$ defines a splitting $L_{\mathscr{H}}(*)_\bullet \longrightarrow L_{\mathscr{H}}(X)_\bullet$ of this exact sequence. So a representation of $L_{\mathscr{H}}(X)$ is just a mixed Hodge theoretic representation of $L_{\mathscr{H}}^{geom}(X)$. Therefore the Hain–Zucker theorem implies that the universal enveloping algebra of $L_{\mathscr{H}}^{geom}(X)_\bullet$ is isomorphic to the maximal Tate quotient of $\lim\limits_{r} \mathbb{Q}[\pi_1(X,x)] / \mathscr{I}^r$.

Summarizing we see that $L^{geom}(\mathbb{C}(X))_\bullet$ should be isomorphic to the maximal Tate quotient of the pronilpotent completion of the classical fundamental group of the generic point of $X(\mathbb{C})$. (In fact to give a precise definition of the last object we should work with the mixed Tate category of good unipotent variations of mixed Tate Hodge structures over the generic point of $X(\mathbb{C})$ and use Beilinson's arithmetical fiber functor, because we cannot choose a point x of the generic point of $X(\mathbb{C})$). Combining all this we see that in the Beilinson World Conjecture B implies

Conjecture 1.32. The commutant of the maximal Tate quotient of the pronilpotent completion of the classical fundamental group of the generic point of an arbitrary algebraic variety over $\mathbb{C}$ is free.

This conjecture can be considered as a geometric analog of the following Bogomolov conjecture

Conjecture 1.33 (Bogomolov, 1986). The commutant of the maximal pro–p–quotient of the Galois group of the geometric type field K containing a closed subfield is free as a pro–p–group.

The condition on the field K means that it has a realisation as a field of functions on an algebraic variety over an algebraically closed field k, or is obtained as a completion of such a field with respect to some valuation.

Conjecture 1.32 suggests that Conjecture B can be considered as a motivic version of the following Schafarevich conjecture.

Conjecture 1.34 (Schafarevich) The commutant of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is a free profinite group.

14. The residue homomorphism for the complexes $\Gamma_F(\underline{n})$. Suppose that the field F has a discrete valuation v with residue class field $\bar{F}_v (= \bar{F})$. The group of units will be denoted by U, and the natural homomorphism $U \longrightarrow \bar{F}^*$ by $u \longmapsto \bar{u}$. An element π of F^* is prime if $\mathrm{ord}_v \pi = 1$. Milnor constructed a canonical homomorphism (see [M2], § 2)

$$\partial_v : K_n^M(F) \longrightarrow K_{n-1}^M(\bar{F}_v) . \tag{1.45}$$

It is defined uniquely by the following properties

i) $\partial_v\{\pi, u_2, \dots, u_n\} = \{\bar{u}_2, \dots, \bar{u}_n\}$

ii) $\partial_v\{u_1, \dots, u_n\} = 0$

where $u_1, \dots, u_n$ are (arbitrary) units and π is a prime element.

In this section we will construct a canonical homomorphism of complexes

$$\partial_v : \Gamma_F(n) \longrightarrow \Gamma_{\bar{F}_v}(n-1)[-1] \tag{1.46}$$

such that the induced homomorphism

$$\partial_v : H^n(\Gamma_F(n)) \longrightarrow H^{n-1}(\Gamma_{\bar{F}}(n-1))$$

coincides with (1.45).

Let us adjoin to the ring $\Lambda^{\bullet}\bar{F}^{*}$ a new symbol ξ of degree 1 which anticommutes with the elements of $\bar{F}^{*}$ and satisfies the identity $\xi \wedge \xi = \xi \wedge (-1)$. We denote the enlarged ring by $\Lambda^{*}(\bar{F})[\xi]$. It is a free $\Lambda^{\bullet}(\bar{F}^{*})$ – module with basis $\{1,\xi\}$.

The correspondence $\pi^{i}u \longrightarrow i\xi + \bar{u}$ extends uniquely to a ring homomorphism $\theta_{\pi} : \Lambda^{\bullet}(F^{*}) \longrightarrow \Lambda^{\bullet}(\bar{F}^{*})[\xi]$. Setting $\theta_{\pi}(\alpha) = \psi(\alpha) + \xi \cdot \partial_{v}(\alpha)$ with $\psi(\alpha)$, $\partial_{v}(\alpha) \in \Lambda^{\bullet}(F^{*})$, we get the required homomorphism ∂_{v}. It obviously satisfies conditions i), ii), and so does not depend on the choice of the prime element π.

Now let us define a homomorphism $s_{v} : \mathbb{Z}[P^{1}_{F}] \longrightarrow \mathbb{Z}\left[P^{1}_{\bar{F}_{v}}\right]$ as follows:

$$s_{v}\{x\} = \begin{cases} \{\bar{x}\} & \text{if } x \text{ is a unit} \\ \{0\} & \text{in other cases} \end{cases}.$$

Then it induces a homomorphism

$$s_{v} : \mathcal{B}_{k}(F) \longrightarrow \mathcal{B}_{k}(\bar{F}_{v})$$

(see s. 9).

Consider the homomorphism

$$\partial_{v} : \mathcal{B}_{k}(F) \otimes \Lambda^{n-k}F^{*} \longrightarrow \mathcal{B}_{k}(\bar{F}_{v}) \otimes \Lambda^{n-k-1}(\bar{F}_{v}^{*})$$

(1.47)

$$\partial_v : \{x\}_k \otimes y_1 \wedge \ldots \wedge y_{n-k} \longmapsto s_v\{x\}_k \otimes \partial_v(y_1 \wedge \ldots \wedge y_{n-k}) .$$

Lemma–definition. The homomorphisms ∂_v commute with the coboundary ∂ and hence define a homomorphism of the complexes (1.46).

Proof. Let $x = \pi^i \cdot u$, where u is a u–unit. We have the following special cases:

1) $k = 2$, $i = 0$, $\bar{u} = 1$. Then $u = 1 + \pi v$ and
$\partial_v \delta(\{u\}_2 \otimes y_1 \wedge \ldots \wedge y_{n-2}) = \partial_v((-\pi v) \wedge (1+\pi v) \wedge y_1 \wedge \ldots \wedge y_{n-2}) = 0$.
On the other hand, $\delta \partial_v(\{u\}_2 \otimes \ldots \}) = \delta(\{1\}_2 \otimes \ldots) = 0$.

2) $k = 2$, $i = 0$, $\bar{u} \neq 1$ or $k > 2$, $i = 0$: is an immediate consequence of

$$\bar{u} \wedge \partial_v(y_1 \wedge \ldots \wedge y_{n-2}) = \partial_v(u \wedge y_1 \wedge \ldots \wedge y_{n-2}) .$$

3) $k = 2$, $i > 0$. Then $\partial_v(\{x\}_2 \otimes \ldots) = 0$ and
$\partial_v \delta(\{x\}_2 \otimes \ldots) = \partial_v(\pi^i u \wedge (1 - \pi^i u) \wedge \ldots) = 0$.

4) $k > 2$, $i \neq 0$. Then $\partial_v(\{x\}_k \otimes \ldots) = 0$ and
$\partial_v \delta(\{x\}_k \otimes \ldots) = \partial_v(\{x\}_{k-1} \otimes x \wedge \ldots) = 0$.

5) $i < 0$. In this case we may use the relation $2(\{x\}_k + (-1)^k\{x^{-1}\}_k) = 0$ in $\mathcal{B}_k(F)$. If we don't want to neglect 2–torsion, it is sufficient to check that $\partial_v((1 - \pi^{-a}u) \wedge (\pi^{-a}u)) = 0$, $a > 0$. We have $1 - \pi^{-a}u = (-1) \cdot \frac{1-\pi^a u^{-1}}{\pi^a u^{-1}}$, so

$$\partial_v((1-\pi^{-a}u) \wedge (\pi^{-a}u)) = -\,\partial_v(((-1)\cdot\pi^{a}u^{-1}) \wedge (\pi^{-a}u)) = 0$$

because

$$\theta_\pi(((-1)\cdot\pi^{a}\cdot u^{-1})\cdot(\pi^{-a}\cdot u)) = (a\cdot\xi + (-\overline{u}^{\,-1})) \wedge (-a\,\xi + \overline{u}) =$$

$$= a^2\cdot\xi \wedge (-1) + a\cdot\xi \wedge \overline{u} + a\cdot\xi \wedge (-\overline{u}^{\,-1}) + (-1) = (-1)\,.$$

15. The motivic complexes $\Gamma(X;n)$. a) Set $\Gamma(\mathrm{Spec}\, F,n) := \Gamma_F(n)$.

b) Now let X be a smooth curve over F and X^1 the set of all points of X. A point $x \in X^1$ defines a discrete valuation v_x of the field $F(X)$. Denote by $F(x)$ the residue class field of v_x. Let us define the motivic complex $\Gamma(X,n)$ as the simple complex associated with the following bicomplex

$$\begin{array}{ccccccc} & & \coprod_{x\in X^1} \mathcal{B}_{n-1}(F(x)) & \longrightarrow & \dots & \longrightarrow & \coprod_{x\in X^1} \Lambda^{n-1}F(x)^* \\ & & \uparrow \oplus\partial_{v_x} & & & & \uparrow \oplus\partial_{v_x} \\ \mathcal{B}_n(F(X)) & \xrightarrow{\delta} & \mathcal{B}_{n-1}(F(X))\otimes F(X)^* & \xrightarrow{\delta} & \dots & \xrightarrow{\delta} & \Lambda^n F(X)^* . \end{array}$$

Conjecture 1.36. $H^i(\Gamma(X,n)\otimes \mathbb{Q}) = \mathrm{gr}^n_\gamma K_{2n-i}(X)_{\mathbb{Q}}$. Of course, this conjecture has a motivic reformulation. Namely, if $\mathcal{M}_T(X)$ is the category of mixed Tate sheaves over X then the complex $\Gamma(X,n)\otimes \mathbb{Q}$ should represent $\mathrm{Hom}_{D_{\mathcal{M}_T(x)}}(\mathbb{Q}(0)_X, \mathbb{Q}(n)_X)$, where $D_{\mathcal{M}_T(X)}$ is the derived category.

c) Now let X be an arbitrary regular scheme, X^i the set of all points of X of codimension i, $F(x)$ the field of functions corresponding to a point $x \in X^i$. We define complexes $\Gamma(X;n)$ for $n \le 3$ as follows:

$\Gamma(X;0)$: $\mathbb{Z}$ placed in degree 0

$\Gamma(X;1)$: $F(X)^* \xrightarrow{\partial} \coprod_{x\in X^1} F(x)^*$

$$\Gamma(X;2):\quad \begin{array}{ccccc} \Lambda^2 F(X)^* & \xrightarrow{\ \partial\ } & \coprod_{x\in X^1} F(x)^* & \xrightarrow{\ \partial\ } & \coprod_{x\in X^2} \mathbb{Z} \\ \uparrow{\scriptstyle\delta} & & & & \\ B_2(F(X)) & & & & \end{array}$$

$$\Gamma(X;3):\quad \begin{array}{ccccccc} \Lambda^3 F(X)^* & \xrightarrow{\ \partial\ } & \coprod_{x\in X^1} \Lambda^2 F(x)^* & \xrightarrow{\ \partial\ } & \coprod_{x\in X^2} F(x)^* & \xrightarrow{\ \partial\ } & \coprod_{x\in X^3} \mathbb{Z} \\ {\scriptstyle\delta}\uparrow & & \uparrow{\scriptstyle\delta} & & & & \\ B_2(F(X))\otimes F(X)^* & \xrightarrow{\ \partial\ } & \coprod_{x\in X^1} B_2(F(x)) & & & & \\ {\scriptstyle\delta}\uparrow & & & & & & \\ B_3(F(X)) & & & & & & \end{array}$$

where $B_i(F(X))$ is placed in degree 1, coboundaries have degree $+1$, and $\Gamma(X;3)$ is the total complex associated with the bicomplex defined above.

The coboundary ∂ is defined as follows. Let $x \in X^k$ and $v_1(y),\dots,v_m(y)$ be all discrete valuations of $F(x)$ over a point $y \in X^{k+1}$, $y \in \bar{x}$. Then the homomorphism of complexes

$$\partial : B_{F(X)}(n) \longrightarrow \coprod_{x\in X^1} B_{F(x)}(n-1)[-1]$$

is defined setting $\partial = \coprod_{x\in X^1} \partial_x$. The definition of other coboundaries ∂ is a little bit more complicated. Let $x \in X^k$ and $v_1(y),\dots,v_m(y)$ be all discrete valuations of the field $F(x)$ over a point $y \in X^{k+1}$, $y \in \bar{x}$. Then $\overline{F(x)}_i := \overline{F(x)}_{v_i(y)} \supset F(y)$. (Note that if $\bar{x}$ is nonsingular at the point y, then $\overline{F(x)}_i = F(y)$ and $m = 1$.) Let us define a homomorphism of complexes

$$\partial_{x,y} : B_{F(x)}(j) \longrightarrow B_{F(y)}(j-1)\,[-1]$$

as the following composition $(j \le 2)$:

$$B_{F(x)}(j) \xrightarrow{\oplus \partial_{v_i(y)}} \bigoplus_{i=1}^{m} B_{\overline{F(x)}_i}(j-1)\,[-1] \xrightarrow{\oplus N_{\overline{F(x)}_i/F(y)}} B_{F(y)}(j-1)\,[-1] \qquad (1.48)$$

A priori the highly nontrivial feature here is the transfer homomorphism in the second arrow. However in our situation we need only the classical transfer $N_{K/k} : K^* \longrightarrow k^*$ for finite extensions $K \supset k$. Now we define

$$\partial : B_{F(x)}(j) \longrightarrow \coprod_{y \in X^{k+1}} B_{F(y)}(j-1)\,[-1]$$

as $\partial := \coprod_{y \in X} \partial_{x,y}$. Note that for the upper line in the bicomplex $\Gamma(X;3)$ ∂^2 coincides with ∂^2 in the Gersten resolution and so is equal to 0 .

Proposition 1.37. $H^i(\Gamma(X;n) \otimes \mathbb{Q}) = gr^{\gamma}_n K_{2n-i}(X)_{\mathbb{Q}}$ for $n \le 2$.

Proof. This is trivial for $n = 0$, well–known for $n=1$ and follows easily from Suslin's theorem and properties of the Gersten resolution for $n=2$.

Conjecture 1.38. $H^i(\Gamma(X;3) \otimes \mathbb{Q}) = gr^{\gamma}_3 K_{6-i}(X)_{\mathbb{Q}}$.

Another construction of weight 2 motivic complexes was given by S. Lichtenbaum. In fact he defines an integral version of motivic complexes (that is quite important!), but his definition uses essentially algebraic K–theory and is more complicated.

d) Now it is clear that the motivic complexes $\Gamma(X;n)$ for an arbitrary regular scheme X should be defined as the simple complex associated with the following bicomplex

$$\Gamma_{F(X)}(n) \xrightarrow{\partial} \bigoplus_{x \in X^1} \Gamma_{F(x)}(n-1)[-1] \xrightarrow{\partial} \dots \xrightarrow{\partial} \bigoplus_{x \in X^n} \Gamma_{F(x)}(0)[-n]$$

where $\partial := \bigoplus_{y \in \bar{x}} \partial_{x,y}$ and $\partial_{x,y}$ is defined as the composition (1.48) where the B–complexes are replaced by the Γ–complexes. The only difficulty that remains is to show the existence of the transfer. Note that in order to define the motivic complex $\Gamma(X;n)$ as an object in the derived category it is sufficient to define the transfer as a morphism in the derived category. This can be done assuming the homotopy invariance of $\Gamma(A^1_F;n)$ (see below) in complete analogy with the Bass–Tate definition of the transfer for $K^M_*(F)$.

More precisely, let $F = k(t)$; $v_\infty(f) = -\deg f$, $f \in k(t)$. The other discrete valuations v of F, trivial on k, are in 1–1 correspondence with the monic irreducible polynomials $\pi_v \in k[t]$. We have $k(v) = k[t]/(\pi_v)$. There is the canonical homomorphism of complexes $\Gamma_k(n) \longrightarrow \Gamma_F(n)$.

<u>Conjecture 1.39</u>. (The homotopy invariance.)

$$\Gamma_F(n)/\Gamma_k(n) \xrightarrow{(\partial_v)} \coprod_{v \neq v_\infty} \Gamma_{k(v)}(n-1)[-1]$$

is a quasi–isomorphism.

Assuming this conjecture we can define the following morphism in the derived category

$$\begin{array}{ccc} & \Gamma_F(n)/\Gamma_k(n) & \\ \coprod \partial_v \swarrow \wr & & \searrow \partial_{v_\infty} \\ \coprod_{v \neq v_\infty} \Gamma_{k(v)}(n-1)[-1] & & \Gamma_{k(v_\infty)}(n-1)[-1] \equiv \Gamma_k(n-1)[-1] \end{array}$$

It seems that it is possible to prove using ideas of [B–T] and [Ka] that the so defined transfer depends only on the extension $L \supset k$.

Note that to construct a homomorphism from $H^i\Gamma(X;n) \otimes \mathbb{Q}$ to $gr_n^\gamma K_{2n-i}(X) \otimes \mathbb{Q}$ it is sufficient to construct a map $H^i\Gamma_F(n) \otimes \mathbb{Q} \longrightarrow gr_n^\gamma K_{2n-i}(F)_\mathbb{Q}$ that commutes with the residue homomorphism and to use the Gersten resolution.

16. The groups $\mathcal{B}_n(F)$ and the scissors congruence groups of pairs of oriented polyhedra in P^n_F. In this section we define groups $B'_n(F)$ that hypothetically should be isomorphic to the groups $\mathcal{B}_n(F)$. More precisely, we define a map

$$\ell_n : \mathbb{Z}[P^1_F] \longrightarrow A_n(F)$$

where $A_n(F)$ is the scissors congruence group of pairs of oriented polyhedra in P^n_F ([BGSV], see also [BMSch]) and set $B'_n(F) :=$ the image of $\ell_n(\mathbb{Z}[P^1_F])$ in the quotient $A_n(F)/P_n(F)$, where $P_n(F)$ is the subgroup of "prisms" (see below). We state a conjecture describing the structure of the groups $A_n(F)$.

First of all let us recall the definition of groups $A_n(F)$ (see [BGSV], § 2). Call an n–simplex a family of $n+1$ hyperplanes $L = (L_0,\dots,L_n)$ in P^n_F. Say that an n–simplex is non–degenerate if the hyperplanes are in general position. Call a face of an n–simplex any non–empty intersection of hyperplanes from L. Call a pair of n–simplices (L,M) admissible, if L and M have no common faces.

Define the group $A_n(F)$ as the group with generators $(L;M)$, where (L,M) runs through all admissible pairs of simplices, and the following relations

(A1) If one of the simplices L or M is degenerate, then $(L;M) = 0$

(A2) Skew symmetry. For every permutation

$$\sigma : \{0,1,\dots,n\} \longrightarrow \{0,1,\dots,n\}$$

$$(\sigma L;M) = (L;\sigma M) = (-1)^{|\sigma|}(L;M)$$

where $\sigma L = (L_{\sigma(0)},\dots,L_{\sigma(n)})$, $|\sigma|$ is the parity of σ .

(A3) Additivity in L . For every family of hyperplanes $(L_0,\dots,L_{n+1})$ and any n–simplex M such that all pairs $(\hat{L}^j,M)$ are admissible

$$\sum_{j=0}^{n+1} (-1)^j(\hat{L}^j;M) = 0$$

where $\hat{L}^j = (L_0,\dots,\hat{L}_j,\dots,L_{n+1})$.

Additivity in M . For every family $(M_0,\dots,M_{n+1})$ and any simplex L such that all $(L,\hat{M}^j)$ are admissible

$$\sum_{j=0}^{n+1} (-1)^j(L;\hat{M}^j) = 0 \ .$$

(A4) Projective invariance. For every $g \in PGL_{n+1}(F)$

$$(gL;gM) = (L;M) \ .$$

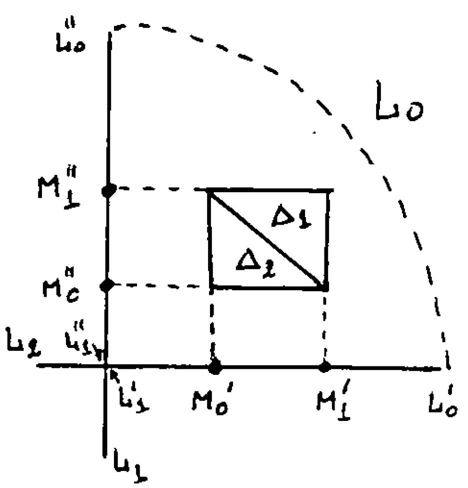
In the case $F = \mathbb{C}$ there is a canonical holomorphic differential form ω_L with logarithmic singularities on the hyperplanes L_i . If $x_i = 0$ is a homogeneous equation of L_i then $\omega_L = d\log(x_1/x_0) \wedge \dots \wedge d\log(x_n/x_0)$. Let Δ_M be an n–cycle representing a generator of the group $H_n(P^n_{\mathbb{C}}, \cup M_j)$. Then

$$a_n(L,M) = \int_{\Delta_M} \omega_L$$

is a multivalued analytic function – Aomoto's polylogarithm ([A]). This integral depends on the choice of Δ_M but does not change under continuous deformation.

There is a canonical isomorphism $r : A_1(F) \longrightarrow F^*$,
$r : (L_0,L_1;M_0,M_1) \longmapsto r(L_0,L_1,M_0,M_1)$.

Now let us define the subgroup of "prisms" $P_n(F)$. Let $(L';M') \subset P^{n'}$ and $(L'',M'') \subset P^{n''}$ be two admissible pairs of non–degenerate simplices, $(L_0,\dots,L_n)$ a non–degenerate simplex in P^n, $n = n' + n''$. Identify the affine space $P^n \backslash L_0$ with the product of the affine spaces $(P^{n'} \backslash L_0') \times (P^{n''} \backslash L_0'')$. Then the simplices M', M'' define the prism $M' \times M''$ in $P^n \backslash L_0$ and hence in P^n (see fig. 1.13 for the case $n' = n'' = 1$). A cutting of $M' \times M''$ into simplices, $M' \times M'' = \cup \Delta_j$, defines the element $\Sigma(L,\Delta_j) \in A_n$. (It does not depend on the choice of cutting.) Let us denote by $P_n(F)$ the subgroup of $A_n(F)$ generated by all prisms for all $n' \geq 1$, $n'' \geq 1$, $n' + n'' = n$.



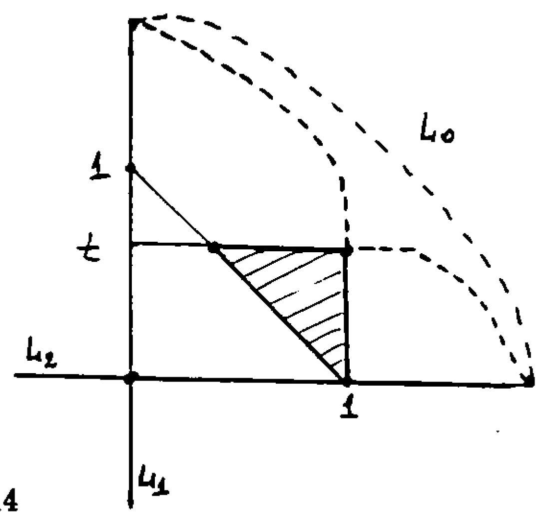


fig. 1.13 – 1.14

Let $x_1,\ldots,x_n$ be coordinates in the affine space $P^n\backslash L_0$ such that L_i is the hyperplane $x_i = 0$. Denote the hyperplanes $0 = 1-x_1$; $1-x_1 = x_2$; $x_2 = x_3;\ldots;x_{n-1} = x_n$; $x_n = t(t \in F^*)$ by $M_0,M_1,\ldots,M_{n-1},M_n(t)$. Set $(L;M(t)) = (L_0,\ldots,L_n;M_0,\ldots,M_n(t))$. Then (see fig. 1.14)

$$\ell_n(\{t\}) := (L,M(t)) , \ \ell_n(\{0\}) = \ell_n(\{\infty\}) = 0 \ .$$

$$B_n'(F) := \mathrm{Image}(\ell_n : \mathbb{Z}[P_F^1] \longrightarrow A_n(F)/P_n(F)) \ .$$

<u>Remark</u> $a_n(L,M(t)) = \mathrm{Li}_n(t)$.

<u>Conjecture 1.40</u> The groups $B_n'(F)$ and $\mathcal{B}_n(F)$ are canonically isomorphic.

More precisely, we conjecture that $R_n(F) = \mathrm{Ker}(\ell_n : \mathbb{Q}[P_F^1] \longrightarrow A_n(F)/P_n(F) \otimes \mathbb{Q})$.

Note that $B_1'(F) = \mathcal{B}_1(F) = B_1(F) = F^*$. It was proved in [BVGS] that $B_2'(F) = B_2(F)$.

It was conjectured in [BSGV], see also [BMSch] that $A(F)_{\mathbb{Q}} := \overset{\infty}{\underset{n=0}{\oplus}} A_n(F)_{\mathbb{Q}}$, $(A_0 = \mathbb{Z})$, can be equipped with the structure of a commutative graded algebra such that $A_n(F)_{\mathbb{Q}} = U(L(F)_\bullet)_{-n}^{\vee}$ (the dual to the subspace of degree $-n$ elements in the universal enveloping algebra of the Lie algebra $L(F)$.) This conjecture and Conjectures B and 1.40 imply the following striking pure geometrical conjecture describing the structure of the scissors congruence group of pairs of oriented polyhedra in P_F^n.

Conjecture 1.41 There is an isomorphism of $\mathbb{Q}$–vector spaces

$$A_n(F)_{\mathbb{Q}} = \bigoplus_{\substack{0\le k\le n,\ i_s\ge 2\\ i_1+\ldots+i_m=n-k}} S^k F^*_{\mathbb{Q}} \otimes B'_{i_1}(F)_{\mathbb{Q}} \otimes \ldots \otimes B'_{i_m}(F)_{\mathbb{Q}} .$$

Indeed, it is well–known that the universal enveloping algebra of the free Lie algebra generated by a vector space V is isomorphic to the tensor algebra $T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$.

The multiplication $\mu : A_{n'} \times A_{n''} \longrightarrow A_{n'+n''}$ should be defined by the formula $\mu((L';M'),(L'';M'')) = \Sigma(L,\Delta_j)$ – see the definition of "prisms" above. So by definition $\bigoplus_{n'+n''=n} \mu(A_{n'} \times A_{n''}) = P_n$. Therefore $A_n(F)/P_n(F) \otimes \mathbb{Q}$ should be isomorphic to $L(F)^{\vee}_{-n}$. In particular $A_3(F)/P_3(F) \overset{?}{=} B'_3(F)$,
$A_4(F)/P_4(F) \otimes \mathbb{Q} \overset{?}{=} B'_4(F)_{\mathbb{Q}} \oplus \Lambda^2 B_2(F)_{\mathbb{Q}}$. Note that by definition $B'_4(F)_{\mathbb{Q}}$ is a subgroup in $A_4(F)/P_4(F) \otimes \mathbb{Q}$ and the quotient should be canonically isomorphic to $\Lambda^2 B_2(F)_{\mathbb{Q}}$. The existence of the canonical embedding $\Lambda^2 B_2(F)_{\mathbb{Q}} \hookrightarrow A_4(F)/P_4(F) \otimes \mathbb{Q}$ is a very intriguing problem.

Note that $A_n(F)/_{P_n(F)+B'_n(F)} \otimes \mathbb{Q} \neq 0$ for $n \ge 4$. Geometrically this means that we cannot cut a generic pair of simplices in P^n_F, $n \ge 4$, to a sum of prisms and polylogarithmic simplices $(L,M(t))$. The reason is quite typical: $4=2+2$. More precisely, set $\mathfrak{L}(F)_{-n} := A_n(F)/_{P_n(F)} \otimes \mathbb{Q}$. Then $\mathfrak{L}(F)_{\bullet} := \bigoplus_{n=1}^{\infty} \mathfrak{L}(F)_{-n}$ is equipped with the structure of a graded Lie coalgebra. The coboundary $\delta : \mathfrak{L}(F)_{\bullet} \longrightarrow \Lambda^2 \mathfrak{L}(F)_{\bullet}$

is induced by the comultiplication $\Delta : A_{\bullet} \longrightarrow A_{\bullet} \otimes A_{\bullet}$. For example for $n=4$ we have $\delta : \tilde{L}(F)_{-4} \longrightarrow \begin{matrix} \tilde{L}(F)_{-3} \otimes F^{*}_{\mathbb{Q}} \\ \oplus \\ \Lambda^{2}\tilde{L}(F)_{-2} \end{matrix}$ and so on. Let $\pi : A_{n} \longrightarrow \tilde{L}^{\vee}_{-n}$ be the canonical projection. Then

$$\delta(\pi\, B'_{n}(F)) \subset \pi\, B'_{n-1}(F) \otimes F^{*} \subset L(F)^{\vee}_{-(n-1)} \otimes F^{*}_{\mathbb{Q}} .$$

In particular, all $\tilde{L}(F)^{\vee}_{-k_1} \wedge \tilde{L}(F)^{\vee}_{-k_2}$ – components of $\delta(\pi\, B'_{n}(F))$ are zero if $k_1 > 1$, $k_2 > 1$. But it is easy to construct an element $a \in A_{n}(F)$ such that the $A_{n-2} \otimes A_{2}$ – component of $\Delta(a)$ has a non–zero projection onto $\tilde{L}(F)^{\vee}_{-(n-2)} \wedge \tilde{L}(F)^{\vee}_{-2}$. (For example, for $n=4$ the last group is isomorphic to $\Lambda^{2}B_{2}(F)_{\mathbb{Q}}$).

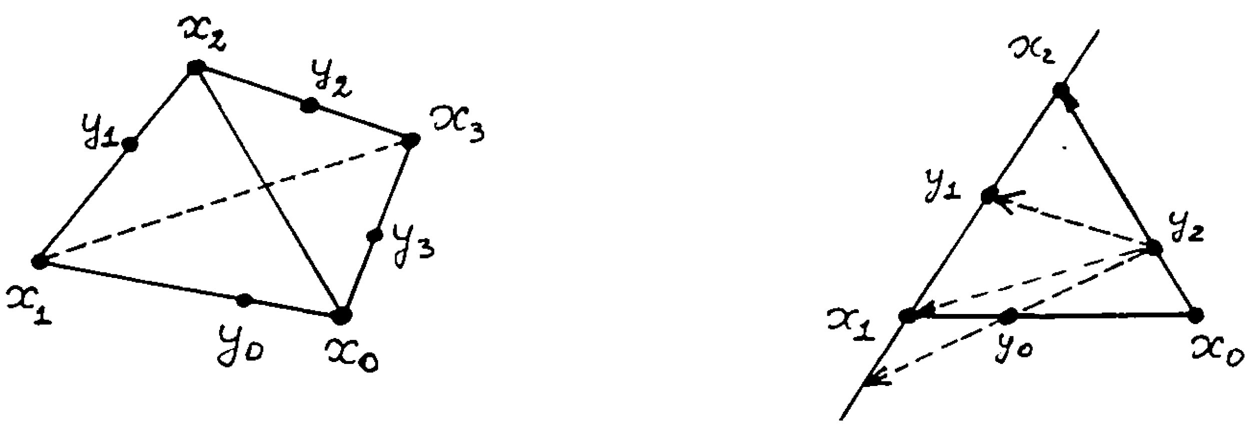


fig. 1.15, 1.16

Let me state another conjecture describing the $\mathbb{Q}$–vector spaces $L(F)^{\vee}_{-n}$ underlying the Lie coalgebra $L(F)^{\vee}_{\bullet}$. Let $\mathcal{G}_{n}(F)$ be the quotient of the free abelian group $C_{2n}(P^{n-1}_{F})$ generated by all possible configurations of $2n$ points $(\ell_{0},\dots,\ell_{2n-1})$

in P_F^{n-1} by the following relations

R0) (The skew–symmetry.) $(\ell_0,\dots,\ell_{2n-1}) = (-1)^{|\sigma|}(\ell_{\sigma(0)},\dots,\ell_{\sigma(2n-1)})$.

R1) $(\ell_0,\dots,\ell_{2n-1})$ is zero if there are $2k+2$ points in a k–dimensional plane among these points.

R2) (The 2n+1–term relation.) For any $2n+1$ points $(\ell_0,\dots,\ell_{2n})$ in P_F^{n-1}

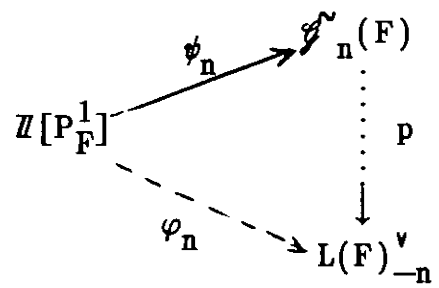
$$\sum_{i=0}^{2n} (-1)^i(\ell_0,\dots,\hat{\ell}_i,\dots,\ell_{2n}) = 0 \ .$$

<u>Conjecture 1.42</u> $L(F)^{\vee}_{-n} \otimes \mathbb{Q}$ is a quotient of $\mathcal{G}_n(F)_{\mathbb{Q}}$.

<u>Remark</u>. Relation R1) means that all semistable configurations are zero.

Note that in the case $n=3$ we have a mysterious relation R3) and its higher analogies is the main problem that remains.

We believe that there should be a canonical homomorphism $\varphi_n : \mathbb{Z}[P_F^1] \longrightarrow L(F)^{\vee}_{-n}$. Let us describe the canonical homomorphism $\psi_n : \mathbb{Z}[P_F^1] \longrightarrow \mathcal{G}_n(F)$ that should make the following diagram commutative (p is the canonical projection predicted by Conjecture 1.42.)



Let $x_0,\dots,x_{n-1}$ be points in generic position in P_F^{n-1} and $y_i \in \overline{x_i x_{i+1}}$ (the line generated by x_i and x_{i+1}, indices modulo n; see fig. 1.15, 1.16 for the cases $n = 4,3$). Now let $z = r(y_2y_3...y_{n-1} | x_0,x_1,y_0,y_1)$ (the cross–ratio of the configuration of 4 points on P^1_F obtained by the projection with center at the (n–3)–plane $\overline{y_2...y_{n-1}}$). Set

$$\psi_n : \{z\} \longmapsto (x_0,y_0,x_1,y_1,...,x_{n-1},y_{n-1})$$

Lemma 1.43

$$r(y_2...y_{n-1} | x_0,x_1,y_0,y_1) = r(y_3...y_{n-1}y_0 | x_1,x_2,y_1,y_2)$$

Proof. Projection of the configuration $(x_0,1,...,y_{n-1})$ with center at the point y_{n-1} gives a similar configuration in P^{n-2}_F. So we can assume $n=3$. In this case project the picture onto the line $\overline{x_1x_2}$ – see fig. 1.16. ■

This lemma shows the correctness of the definition.

The last conjecture tells us that there should be a canonical homomorphism

$$p_n : \mathcal{G}_n(F) \longrightarrow A_n(F)/P_n(F)$$

such that

$$p_n(x_0,y_0,...,x_{n-1},y_{n-1}) = \pi(L,M(z))$$

where $z = r(y_2y_3...y_{n-1} | x_0,x_1,y_0,y_1)$.

Let me emphasize that elements of $\mathcal{G}_n(F)$ are represented by configurations of 2n points in P_F^{n-1}, while elements of $A_n(F)$ are pairs of simplices in P_F^n, and hence produce configurations of $2n+2$ points in P_F^n. It is interesting that the best construction that I know of the element $(L,M(z)) \in A_n(F)$ uses the configuration $(x_0,y_0,\dots,x_n,y_n)$:

$$(L,M(z)) = (x_0,y_0,y_1,\dots,y_{n-1};x_1,x_2,\dots,x_n,y_n)$$

Remark 1.44 The configuration $(x_0,y_0,\dots,x_{n-1},y_{n-1})$ was independently considered by J. Dupont and S.M. Sah in their study of the homology of GL_n (I know this thanks to the beautiful lecture of J. Dupont given at the "Polylogarithmic conference" at the M.I.T., 1990) and by I.M. Gelfand and M.I. Graev in their theory of generic hypergeometric functions. (Note that hypergeometric functions also live on configurations of points in $\mathbb{C}P^n$; however it seems that they still live rather separately from polylogarithms: the only connection that I know is the observation of I.M. Gelfand that Aomoto polylogarithms are very special examples of hypergeometric functions.)

Conjecture 1.42 implies also that there should be an absolutely canonical R–valued function on configurations of $2n$ points in $P_{\mathbb{C}}^{n-1}$ (expected to be real–analytic on stable configurations and continuous on semistable ones.) The reason is that there should be a canonical realisation functor from the category of mixed motivic Tate sheaves over Spec C to the category of mixed Tate R–Hodge structures $\mathscr{H}_R$. Therefore there should be a canonical homomorphism of the corresponding mixed Tate Lie coalgebras $L(\mathbb{C})^\vee \longrightarrow L(\mathscr{H}_R)^\vee$. A.A. Beilinson and P. Deligne constructed a canonical homomorphism $\ell_n : L(\mathscr{H}_R)^\vee_{-n} \longrightarrow R$. So the composition

$$\mathscr{G}_n(\mathbb{C}) \longrightarrow L(\mathbb{C})^\vee_{-n} \longrightarrow L(\mathscr{H}_R)^\vee_{-n} \longrightarrow R \qquad (*)$$

gives a canonical function on configurations of $2n$ points in $P_{\mathbb{C}}^{n-1}$.

Recall the definition of $\ell_n : L(\mathscr{H}_R)^\vee_{-n} \longrightarrow R$ (see, for example, [D], § 2). By definition $\mathscr{H}_R$ is the tannakian category over R of mixed R–Hodge structures such that $h^{p,q} \neq 0$ only for $p=q$. An object $H \subset \mathscr{H}_R$ is a graded $\mathbb{C}$–vector space $H_{\mathbb{C}} = \oplus H_p$ together with a real structure H_R on $\oplus H_p$ such that the weight filtration

$$W_{-2p} := \underset{\ell \geq p}{\oplus} H_p$$

is defined over R; i.e. $(H_R \cap W_{-2p}) \otimes \mathbb{C} = W_{-2p}$. The Hodge filtration $F^{-p} := \underset{\ell \leq p}{\oplus} H_p$ is opposite to the weight filtration. The real structure H_R induces a real structure on $gr^W_{-2p} H_{\mathbb{C}} = H_p$. We have 2 different real structures on $H_{\mathbb{C}} = \oplus H_p$: the structure H_R and the structure $gr^W H_R$. Let $X \subset GL(H_{\mathbb{C}})$ be the subgroup of all transformations that preserve the weight filtration and induce the identity on graded quotients. Then there is a uniquely defined $n \in X/X(R)$ such that

$$H_R = n \cdot (\oplus H_{pR}) .$$

Set

$$b = n\, \bar{n}^{-1}, \;\; N = \tfrac{1}{2} \log b$$

Then $b\bar{b} = 1$, $\bar{N} = -N$, $N = \Sigma N_k$ where N_k has degree k. We have

$$N_{k_1 k_2}(H^1 \otimes H^2) = N_{k_1}(H^1) \otimes 1_{H^2} + 1_{H^1} \otimes N_{k_2}(H^2) .$$

Now let us recall the following construction of $L(\mathscr{H}_R)^\vee_{-n}$ (see § 2 of [BGSV] or ch. 2 of [BMS]). Let $H \in \mathcal{H}_R$, $W_{-2}H = \phi$, $W_{2n}H = H$. Say that H is framed if the isomorphisms $i_{-n} : R(-n) \xrightarrow{\sim} gr^W_{2n}H$, $i_0 : gr^W_0 H \longrightarrow R(0)$ are fixed. Consider the set of all such framed mixed R–Hodge structures. Introduce on this set the coarsest equivalence relation for which H^1 is equivalent to H^2 if there is a morphism of mixed Hodge structures $H_1 \longrightarrow H_2$ compatible with frames. Denote by $\mathcal{H}_n$ the set of equivalence classes. One may introduce on $\mathcal{H}_n$ a structure of an abelian group in complete analogy with the Baer sum on Ext groups. The multiplication

$$\mu : \mathcal{H}_k \otimes \mathcal{H}_\ell \longrightarrow \mathcal{H}_{k+\ell}$$

is induced by the tensor product of Hodge structures. It is commutative. Then we have the canonical isomorphism

$$L(\mathcal{H}_R)^\vee_{-n} = \mathcal{H}_n / \bigoplus_{k+\ell=n} \mu(\mathcal{H}_k \otimes \mathcal{H}_\ell)$$

<u>Lemma–Definition</u>. Let $H \in \mathcal{H}_n$. Then the "matrix coefficient"

$$i_0 N_n i_{-n} : R(-n) \longrightarrow R(0)$$

is a multiplication on $\ell(\mathcal{H})$. It is equal to zero on $\bigoplus_{k+\ell=n} \mu(\mathcal{H}_k \otimes \mathcal{H}_\ell)$ and hence defines the homomorphism

$$\ell_n : L(\mathcal{H}_R)^\vee_{-n} \longrightarrow R .$$

Note that according to a theorem of A.A. Beilinson $L(\mathcal{H}_R)$ is a <u>free</u> graded Lie algebra over R. Its space of degree $-n$ generators is isomorphic to $\mathbb{C}/(2\pi i)^n R$. The homomorphism ℓ_n gives an element of $L(\mathscr{H}_R)$. These elements generate the Lie algebra $L(\mathscr{H}_R)$. It is interesting that the canonical polylogarithmic function (*) on configurations of 2n points in $P^{n-1}_{\mathbb{C}}$, which coincides with the Bloch–Wigner function for n=2, can be expressed by the classical trilogarithm $\mathscr{L}_3(Z)$ for n=3 (this is one of the main results of this paper) and cannot be expressed by the classical n-logarithmic $\mathscr{L}_n(z)$ for $n \geq 4$ (because of the reasons that we discussed above). However, I suppose that the following conjecture is valid.

Conjecture 1.45. Let $\mathbb{Z}[C_{2n}(P^{n-1}_F)]$ be the free abelian group generated by stable configurations $(\ell_0,\dots,\ell_{2n-1})$ of 2n points in P^{n-1}_F. Then there exists a canonical homomorphism

$$P_n : \mathbb{Z}[C_{2n}(P^{n-1}_F)] \longrightarrow \mathbb{Z}[P^1_F]$$

(the generalised cross-ratio of 2n points in P^{n-1}) such that

a) for a generic configuration $(\ell_0,\dots,\ell_{2n})$ of 2n+1 points in P^{n-1}_F

$$P_n\Big(\sum_{i=0}^{2n} (-1)^i(\ell_0,\dots,\hat{\ell}_i,\dots,\ell_{2n})\Big) \subset \mathscr{R}_n(F)$$

b) for a generic configuration $(m_0,\dots,m_{2n})$ of 2n+1 points in P^n_F

$$P_n\Big(\sum_{i=0}^{2n} (-1)^i(m_i \,|\, m_0,\dots,\hat{m}_i,\dots,m_{2n})\Big) \subset \mathscr{R}_n(F)$$

c) for a special configuration $(x_0,y_0,\dots,x_{n-1},y_{n-1})$ (see above)

$$P_n((x_0,y_0,\ldots,x_{n-1},y_{n-1})) = \{r(y_2,y_3,\ldots,y_{n-1} \mid x_0,x_1,y_0,y_1)\} \ .$$

Now let $F = \mathbb{C}$, $x \in P_{\mathbb{C}}^{n-1}$ and $g_i \in GL_n(\mathbb{C})$. Then part a) of Conjecture 1.45 just means that the function

$$\mathcal{L}_n(P_n(g_0x,\ldots,g_{2n-1}x))$$

is a measurable (2n–1)–cocycle of $GL_n(\mathbb{C})$. So it defines a class in $H_{cts}^{2n-1}(GL_n(\mathbb{C}))$.
It can be proved that part c) of Conjecture 1.45 guarantees that this class coincides with the Borel class. Moreover, it can be shown that part b) of the conjecture provides us with an explicit construction of the cocycle representing the Borel class in $H_{cts}^{2n-1}(GL_N(\mathbb{C}))$ for all $N > n$ (see the forthcoming paper). So Conjecture 1.45 implies Zagier's conjecture, and in fact it is a way how to prove it.
Finally, I am sure that the mysterious subgroup $\mathcal{R}_n(F)$ coincides with the image of the (2n+1)–term relations a) and b) in Conjecture 1.45 under the homomorphism P_n.

Zagier's conjecture about $\zeta_F(n)$ follows immediately from Conjecture 1.45. In fact this conjecture is stronger than Zagier's.

§ 2 The value of the Dedekind zeta–function at the point $s = 2$.

1. The formulation of the theorem. Let F be an arbitrary algebraic number field, d_F the discriminant of F , r_1 and r_2 the numbers of real and complex places ($r_1 + 2r_2 = [F : \mathbb{Q}]$) , σ_j the set of all possible embeddings $F \hookrightarrow \mathbb{C}$ numbered in such a way that $\overline{\sigma_{r_1+k}} = \sigma_{r_1+r_2+k}$.

Recall that the Bloch–Suslin complex is defined as follows:

$$\mathbb{Z}[P^1_F \backslash 0,1,\infty] / R_2(F) \xrightarrow{\delta_2} \Lambda^2 F^*$$

$$[x] \longmapsto (1-x) \wedge x$$

Theorem 2.1. Let $\zeta_F(s)$ be the Dedekind zeta–function of F . Then there exist

$$y_1, \dots, y_{r_2} \in \operatorname{Ker} \delta_2 \subset \mathbb{Z}[P^1_F \backslash 0,1,\infty]$$

such that

$$\zeta_F(2) = q \cdot \pi^{2(r_1+r_2)} \cdot |d_F|^{-\frac{1}{2}} \cdot \det |D_2(\sigma_{r_1+j}(y_i))|$$

where $1 \le i,j \le r_2$ and q is some rational number.

We give a proof using only one hard result – the Borel theorem. The proof of the analogous result about $\zeta_F(3)$ follows the same scheme, but it is more complicated.

Recall that Theorem 2.1 was proved by D. Zagier [Z] using different methods; another proof follows immediately from results of A. Borel [Bl 1–2], S. Bloch [Bl1] and A. Suslin [S 3].

2. The Borel theorems. Set $R(n) = (2\pi i)^n \mathbb{R} \subset \mathbb{C}$ and $X_F := \mathbb{Z}^{\mathrm{Hom}(F,\mathbb{C})}$. Let us define the Borel regulator

$$r_m : K_{2m-1}(F) \longrightarrow X_F \otimes R(m-1)\,. \tag{2.2}$$

The Hurewicz map gives a canonical homomorphism

$$K_{2m-1}(F) := \pi_{2m-1}(BGL(F)^+) \longrightarrow H_{2m-1}(BGL(F)^+,\mathbb{Z}) = H_{2m-1}(GL(F),\mathbb{Z})\,. \tag{2.3}$$

For every embedding $\sigma : F \hookrightarrow \mathbb{C}$ we have a homomorphism

$$H_{2m-1}(GL(F),\mathbb{Z}) \longrightarrow H_{2m-1}(GL(\mathbb{C}),\mathbb{Z})\,. \tag{2.4}$$

There is a canonical pairing

$$H^{2m-1}(GL(\mathbb{C}),R(m-1)) \times H_{2m-1}(GL(\mathbb{C}),\mathbb{Z}) \xrightarrow{\langle,\rangle} R(m-1)\,. \tag{2.5}$$

Let us define a canonical element

$$b_{2m-1} \in H_{cts}^{2m-1}(GL(\mathbb{C}),R(m-1)) \subset H^{2m-1}(GL(\mathbb{C}),R(m-1))\,.$$

Recall that (cf. [Bo 1])

$$H^*_{cts}(GL(\mathbb{C}),R) \cong H^*_{top}(U,R)$$

where $H^*_{top}(U,R)$ is the cohomology of the infinite unitary group, considered as a topological space. Further,

$$H^*_{top}(U,\mathbb{Z}) = H^*(S^1 \times S^3 \times S^5 \times \ldots,\mathbb{Z}) = \Lambda^*_{\mathbb{Z}}(u_1,u_3,\ldots)$$

where u_i denotes the class of the sphere S^i.

Combining the above isomorphisms we get an isomorphism

$$\varphi : H^*_{cts}(GL(\mathbb{C}),R) \xrightarrow{\sim} \Lambda^*_{\mathbb{Z}}(u_1,u_3,\ldots) \otimes R . \qquad (2.6)$$

Set $b'_{2m-1} := 2\pi \cdot \varphi^{-1}(u_{2m-1})$ and

$$b_{2m-1} := (2\pi i)^{m-1} \cdot b'_{2m-1} \in H^*_{cts}(GL(\mathbb{C}),R(m-1)) .$$

So combining this with (2.3) – (2.5) we get

$$K_{2m-1}(F) \longrightarrow \bigoplus_{Hom(F,\mathbb{C})} K_{2m-1}(\mathbb{C}) \longrightarrow X_F \otimes R(m-1) .$$

It is known that if $\lambda \in H^d_{cont}(GL(\mathbb{C}),R)$ and c^* denotes the involution defined by complex conjugation c, then in (2.6)

$$c^*\varphi(\lambda) = (-1)^d \varphi(c^*\lambda) ,$$

where c acts also on $S^{2m-1} \subset \mathbb{C}^m$. Note, that

$$c^* u_{2m-1} = (-1)^m u_{2m-1} .$$

So we see that

$$r_m : K_{2m-1}(F) \longrightarrow [X_F \otimes R(m-1)]^+ = R^{d_m}$$

where

$$d_m = \begin{cases} r_1 + r_2 , & \text{if } m \text{ is odd} \\ r_2 , & \text{if } m \text{ is even} \end{cases}$$

and on the right–hand side stands the subspace of invariants of the action of c .

In fact, we construct a homomorphism

$$r_m^{(n)} : \operatorname{Prim} H_{2m-1}(GL_n(F),\mathbb{Z}) \longrightarrow [X_F \otimes R(m-1)]^+ .$$

For any lattice Λ of $(X_F \otimes R(m-1))^+$ define its (co)volume vol Λ by

$$\det(\Lambda) = \operatorname{vol}(\Lambda) \cdot \det [X_F \otimes R(m-1)]^+ .$$

Theorem 2.2 (Borel [Bo 1], [Bo 2]). For every $m \geq 2$ and sufficiently large n

a) $\operatorname{Im} r_m^{(n)}$ is a lattice in $(X_F \otimes R(m-1))^+$

b) $R_m := \mathrm{vol}(\mathrm{Im}\, r_m^{(n)}) \sim \mathbb{Q}^* \cdot \lim_{s \to 1-m} (s-1+m)^{-d_m} \zeta_F(s)$.

Here $a \sim \mathbb{Q}^* b$ means that $a = \kappa b$ for some $\kappa \in \mathbb{Q}^*$.

According to (§ 1) we can assume $n = 2m-1$. However, we will not use this result.

Remark 2.3. The functional equation for $\zeta_F(s)$ shows that

$$\zeta_F(s) \sim \mathbb{Q}^* \cdot \pi^{(r_1+2r_2)m - d_m} \cdot |d_F|^{-\frac{1}{2}} \cdot R_m .$$

3. The Grassmannian complex and the Bloch–Suslin complex. Let us say that n vectors in an m–dimensional vector space are in generic position, if every $k \le m$ of them generate a k–dimensional subspace. The notion of n points in P^m in generic position is defined in a similar way.

Definition 2.4. $C_m(n)$ (resp. $C_m(P_F^n)$) is the free abelian group, generated by configurations $(\ell_0, \dots, \ell_{m-1})$ of m vectors in an n–dimensional vector space V_n over a field F (resp. m points in P_F^n) in generic position.

Let us define the Grassmannian complex $C_F(2)$ as follows (see [S 1], [BMS] and [GGL])

$$\dots \xrightarrow{d} C_5(2) \xrightarrow{d} C_4(2) \xrightarrow{d} C_3(2)$$

$$d : (\ell_0, \dots, \ell_m) \longmapsto \sum_{i=0}^{m} (-1)^i (\ell_0, \dots, \hat{\ell}_i, \dots, \ell_m) .$$

Let us define a diagram

$$\begin{array}{ccc} C_4(2) & \xrightarrow{\quad d \quad} & C_3(2) \\ \downarrow f_1^{(2)} & & \downarrow f_0^{(2)} \\ C_4(P_F^1) & \overset{\delta_2}{- - \to} & \Lambda^2 F^* \end{array} .$$

Let $\omega \in \det V_2^*$ be a volume form in V_2. Set

$$\Delta(\ell_1, \ell_2) := < \omega, \ell_1 \wedge \ell_2 > , \quad \ell_i \in V_2 .$$

Put

$$f_0^{(2)} : (\ell_0, \ell_1, \ell_2) \longmapsto \Delta(\ell_0, \ell_1) \wedge \Delta(\ell_0, \ell_2) - \\ \Delta(\ell_0, \ell_1) \wedge \Delta(\ell_1, \ell_2) + \Delta(\ell_0, \ell_2) \wedge \Delta(\ell_1, \ell_2) . \tag{2.7}$$

<u>Lemma</u> 2.5. $f_0^{(2)}(\ell_0, \ell_1, \ell_2)$ does not depend on ω.

<u>Proof</u>. An easy direct calculation.

Lemma 2.6. Modulo 2–torsion

$$f_0^{(2)} \circ d : (\ell_0, \dots ,\ell_3) \longmapsto \frac{\Delta(\ell_0,\ell_1) \cdot \Delta(\ell_2,\ell_3)}{\Delta(\ell_0,\ell_2) \cdot \Delta(\ell_1,\ell_3)} \wedge \frac{\Delta(\ell_0,\ell_3) \cdot \Delta(\ell_1,\ell_2)}{\Delta(\ell_0,\ell_2) \cdot \Delta(\ell_1,\ell_3)} . \qquad (2.8)$$

Proof. Direct calculation $(a \wedge a = 0$ modulo 2–torsion$)$.

Corollary 2.7. $f_0^{(2)} \circ d(\ell_0, \dots ,\ell_3)$ does not depend on the "length" of the vectors ℓ_i:

$$f_0^{(2)} \circ d\,[(\ell_0, \dots ,\ell_i, \dots ,\ell_3) - (\ell_0, \dots ,\lambda\ell_i, \dots ,\ell_3)] = 0 ,$$

where $\lambda \in F^*$ and $0 \le i \le 3$.

The proof follows immediately from (2.8). So $f_0^{(2)} \circ d$ defines a homomorphism

$$\delta_2 : C_4(P_F^1) \longrightarrow \Lambda^2 F^* .$$

Every 4–tuple of distinct points $\overline{\ell}_0, \dots ,\overline{\ell}_3$ on P_F^1 is $PGL_2(F)$–equivalent to $(0,\infty,1,z)$, where

$$z = \frac{\Delta(\ell_0,\ell_3) \cdot \Delta(\ell_1,\ell_2)}{\Delta(\ell_0,\ell_2) \cdot \Delta(\ell_1,\ell_3)}$$

is the cross–ratio of $(\overline{\ell}_0, \dots ,\overline{\ell}_3)$ and it does not depend on the lifting of the points $\overline{\ell}_i$ to vectors ℓ_i.

Further, the identity

$$\Delta(\ell_0,\ell_2)\cdot\Delta(\ell_1,\ell_3)-\Delta(\ell_0,\ell_3)\cdot\Delta(\ell_1,\ell_2)=\Delta(\ell_0,\ell_1)\cdot\Delta(\ell_2,\ell_3)$$

shows that $\delta_2 : (0,\infty,1,z) \longmapsto (1-z)\wedge z$.

Set

$$\delta_2 : C_5(P^1_F) \longrightarrow C_4(P^1_F)$$

$$\delta_2 : (\overline{\ell}_0, \dots ,\overline{\ell}_4) \longmapsto \sum_{i=0}^{4} (-1)^i(\overline{\ell}_0, \dots ,\hat{\overline{\ell}}_i, \dots ,\overline{\ell}_4) .$$

Lemma 2.8. $\delta_2(\delta_2(\overline{\ell}_0, \dots ,\overline{\ell}_4)) = 0$.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc} C_5(2) & \xrightarrow{d} & C_4(2) & \xrightarrow{d} & C_3(2) \\ \downarrow f_2^{(2)} & & \downarrow f_1^{(2)} & & \downarrow f_0^{(2)} \\ C_5(P^1_F) & \xrightarrow{\delta_2} & C_4(P^1_F) & \xrightarrow{\delta_2} & \Lambda^2 F^* \end{array}$$

where $f_2^{(2)}$ is the projectivisation and $d^2 = 0$. ■

So we have constructed a homomorphism of complexes

$$\begin{array}{ccccc} C_5(2) & \xrightarrow{d} & C_4(2) & \xrightarrow{d} & C_3(2) \\ \downarrow & & \downarrow f_1^{(2)} & & \downarrow f_0^{(2)} \\ 0 & \longrightarrow & B_2(F) & \xrightarrow{\delta_2} & \Lambda^2 F^* \end{array} \qquad (2.9)$$

4. The 5–term functional equation for the Bloch–Wigner function.

Lemma 2.9. $dD_2(z) = -\log|1-z|\ d\arg z + \log|z| \cdot d\arg(1-z)$.

Proof.

$$dD_2(z) = \mathrm{Im}\,[d(Li_2(z) + \log(1-z) \cdot \log|z|)]$$

$$= \mathrm{Im}\,[-\log(1-z)\ d\log z + \log(1-z)\ d\log|z| + \log|z|\ d\log(1-z)]$$

$$= -\log|1-z|\ d\arg z + \log|z|\ d\arg(1-z)\ .$$

Proposition 2.10. Let $F = \mathbb{C}$, then

$$D_2(\delta_2(x_0, \ldots, x_4)) = 0\ .$$

Proof. It follows from Lemmas 2.8 and 2.9 that

$$d(D_2(\delta_2(x_0, \ldots, x_4))) = 0$$

where $D_2(\delta_2(x_0, \dots, x_4))$ is considered as a function on the manifold of configurations of 5 points in $P^1_{\mathbb{C}}$. So $D_2(\delta_2(x_0, \dots, x_4)) = \text{const}$. Recall that $D_2(z)$ is continuous on $P^1_{\mathbb{C}}$ and $D_2(0) = D_2(\infty) = D_2(1) = 0$. So the specialisation to the configuration (x,x,y,y,z) shows that this constant is equal to zero. ■

5. <u>Explicit formula for the regulator</u> $r_2^{(2)} : H_3(GL_2(\mathbb{C}),R) \longrightarrow R$. Let $\mathcal{C}_m(n)$ be the free abelian group generated by the m–tuples of vectors in generic position in an n–dimensional vector space. Let us define a differential $d : \mathcal{C}_m(n) \longrightarrow \mathcal{C}_{m-1}(n)$ setting

$$d : (\ell_0, \dots, \ell_{m-1}) \longmapsto \sum_{i=0}^{m-1} (-1)^i (\ell_0, \dots, \hat{\ell}_i, \dots, \ell_{m-1}) \tag{2.10}$$

<u>Lemma</u> 2.11. The following complex $\mathcal{C}_*(n)$

$$\xrightarrow{d} \mathcal{C}_3(n) \xrightarrow{d} \mathcal{C}_2(n) \xrightarrow{d} \mathcal{C}_1(n)$$

($\mathcal{C}_i(n)$ placed in degree i–1) is acyclic in degree > 0.

<u>Proof</u> ([Bl 1]). Let $\sum_{i=1}^{N} n_i(\ell_0^{(i)}, \dots, \ell_k^{(i)})$ be a cycle in $C_*(n)$. Choose a vector v in a generic position with $\ell_j^{(i)}$. Then

$$d(\sum_{i=1}^{N} n_i(v, \ell_0^{(i)}, \dots, \ell_k^{(i)})) = \sum_{i=1}^{N} n_i(\ell_0^{(i)}, \dots, \ell_k^{(i)}) . \quad ■$$

Note that $\tilde{C}_1(n)/d\tilde{C}_2(n) = \mathbb{Z}$, so $\tilde{C}_*(n)$ is a resolution of the trivial $GL_n(F)$–module. By definition $C_m(n) = \tilde{C}_m(n)/GL_n(F)$. So we have a canonical homomorphism

$$H_*(GL_n(F),\mathbb{Z}) \longrightarrow H_*(C_*(n)) .$$

In particular

$$H_3(GL_2(F),\mathbb{Z}) \longrightarrow H_3(C_*(2)) .$$

Combining with the homomorphism of complexes $C_*(2) \longrightarrow B(2)$ (see (2.9)) we get a canonical homomorphism

$$g^{(2)} : H_3(GL_2(F),\mathbb{Z}) \longrightarrow \text{Ker}\ \delta_2 \subset B_2(F) .$$

According to Proposition 2.10 in the case $F = \mathbb{C}$ the function $D_2(z)$ defines a homomorphism $B_2(\mathbb{C}) \longrightarrow \mathbb{R}$. So we obtain a homomorphism

$$D_2 \circ g^{(2)} : H_3(GL_2(\mathbb{C}),R) \longrightarrow \mathbb{R} \qquad (2.11)$$

i.e., an element in $H^3(GL_2(\mathbb{C}),R)$.

If $x \in P^1_{\mathbb{C}}$ then

$$\mathcal{D}_2(g_0,g_1,g_2,g_3) := D_2(g_0x,g_1x,g_2x,g_3x) \qquad (2.12)$$

is a cocycle, representing the constructed cohomology class. (The cocycle condition is just the 5–term functional equation for the Bloch–Wigner function $D_2(z)$) .

Notice that this cocycle is <u>not</u> continuous near the identity. However, the corresponding cohomology class lies in

$$\mathrm{Im}(H^3_{cts}(GL_2(\mathbb{C}),R) \longrightarrow H^3(GL_2(\mathbb{C}),R)) . \qquad (2.13)$$

To see this we repeat an argument of J. Dupont [D 1]. Recall that $\mathcal{D}_2(g_0, \dots ,g_3)$ is equal to $2/3$ times the volume $V(g_0x, \dots ,g_3x)$ of the "ideal" tetrahedron in the Lobachevsky space H^3 with vertices at $g_0x, \dots ,g_3x$ on the <u>absolute</u> $\partial H^3 \cong \mathbb{C}P^1$. If $h \in H^3$, then the volume $V(g_0h, \dots ,g_3h)$ of the <u>regular</u> tetrahedron is also a continuous cocycle and it is cohomologous (actually in a canonical way) to $V(g_0x, \dots ,g_3x)$, $x \in \partial H^3$.

On the other hand, it is not hard to see ([D 2]) that the cocycle $V(g_0h, \dots ,g_3h)$ represents the class $(2\pi)^2 \cdot \varphi^{-1}(u_3)$ (see 2.6). So formula (2.11) defines the Borel regulator $r_2^{(2)} : H_3(GL_2(\mathbb{C})) \longrightarrow R$.

Note that there is an easier way to see that the cohomology class of cocycle (2.12) lies in (2.13). Indeed, the function $D_2(z)$ is continuous on $P^1(\mathbb{C})$ and so is bounded. Hence cocycle (2.12) is bounded and as a result its cohomology class lies in (2.13) – see [Gu]. So the only problem is to check that the constructed cohomology class coincides up to a rational number with the one constructed by Borel.

In order to construct explicitly the Borel regulator $r_2 : H_3(GL(\mathbb{C}),R) \longrightarrow R$ we will study in the next section some bicomplex $C^m_*(n)$ which will also be useful in § 4.

6. The bicomplex $C^m_*(n)$. Let us define a differential $d^{(k)} : \tilde{C}_p(n) \longrightarrow \tilde{C}_{p-1}(n)$ as follows:

$$d^{(k)} : (\ell_1, \dots, \ell_p) \longmapsto \sum_{i=1}^{p-k} (-1)^{i-1} (\ell_1, \dots, \hat{\ell}_{k+i}, \dots, \ell_p) .$$

Note, that $d^{(0)} \equiv d$ – see 2.10.

Lemma 2.12. The following complex is acyclic $(k > 0)$:

$$\dots \longrightarrow \tilde{C}_{k+2}(n) \xrightarrow{d^{(k)}} \tilde{C}_{k+1}(n) \xrightarrow{d^{(k)}} C_k(n) .$$

The proof is in a complete analogy with the one of Lemma 2.11.

Let $\mathrm{Sym}_k : \tilde{C}_p(n) \longrightarrow \tilde{C}_p(n)$ be the symmetrisation of the first k vectors:

$$\mathrm{Sym}_k : (\ell_1, \dots, \ell_p) \longmapsto \sum_{\sigma \in S_k} \frac{1}{k!} (x_{\sigma(1)}, \dots, x_{\sigma(k)}, x_{k+1}, \dots, x_p) .$$

Define a homomorphism $\lambda^{(k)} : \tilde{C}_p(n) \longrightarrow \tilde{C}_p(n)$ as follows:

$$\lambda^{(k)} : (\ell_1, \dots, \ell_p) \longmapsto$$

$$\sum_{i=1}^{p-k} (-1)^{i-1} \mathrm{Sym}_{k+1}(\ell_1, \dots, \ell_k, \ell_{k+i}, \ell_{k+2}, \dots, \hat{\ell}_{k+i}, \dots, \ell_p) .$$

<u>Lemma</u> 2.13. $d^{(k+1)} \circ \lambda^{(k)} = \lambda^{(k)} \circ d^{(k)}$.

<u>Proof</u>. It is obvious for the homomorphism $\mathfrak{X}^{(k)}$ that is defined by the same formula as $\lambda^{(k)}$, but without symmetrisation.

It remains to symmetrise the first $k+1$ vectors. ∎

<u>Lemma</u> 2.14. $\lambda^{(k+1)} \circ \lambda^{(k)} = 0$.

<u>Proof</u>. Straightforward. (Note, that $\mathfrak{X}^{(k+1)} \circ \mathfrak{X}^{(k)} \neq 0$.) ∎

Therefore we get the following bicomplex $\mathfrak{C}^m_*(n)$

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathfrak{C}_4(n) & \xrightarrow{d} & \mathfrak{C}_3(n) & \xrightarrow{d} & \mathfrak{C}_2(n) & \xrightarrow{d} & \mathfrak{C}_1(n) \\
& \downarrow \lambda^{(1)} & & \downarrow -\lambda^{(1)} & & \downarrow \lambda^{(1)} & & \downarrow -\lambda^{(1)} \\
\cdots \longrightarrow & \mathfrak{C}_4(n) & \xrightarrow{d^{(1)}} & \mathfrak{C}_3(n) & \xrightarrow{d^{(1)}} & \mathfrak{C}_2(n) & \xrightarrow{d^{(1)}} & \mathfrak{C}_1(n) \\
& \downarrow \lambda^{(2)} & & \downarrow -\lambda^{(2)} & & \downarrow \lambda^{(2)} & & \\
\cdots \longrightarrow & \mathfrak{C}_4(n) & \xrightarrow{d^{(2)}} & \mathfrak{C}_3(n) & \xrightarrow{d^{(2)}} & \mathfrak{C}_2(n) & & \\
& \downarrow \lambda^{(3)} & & \downarrow -\lambda^{(3)} & & & & \\
\cdots \longrightarrow & \mathfrak{C}_4(n) & \xrightarrow{d^{(3)}} & \mathfrak{C}_3(n) & & & & \\
& \vdots & & & & & & \\
& \downarrow & & & & & & \\
\cdots \longrightarrow & \mathfrak{C}_{m-1}(n) & & & & & &
\end{array}
\qquad (2.14)
$$

<u>Remark</u> 2.15. The bicomplex $C^2_*(3)$ was considered by A.A. Suslin in § 3 of [S 3].

Let $(\tilde{\mathcal{D}}_*^m(n),\partial)$ be a complex, associated with the bicomplex $\tilde{C}_*^m(n)$. It is placed at degrees -1 , 0 , $+1$, ...

Lemma 2.16. $$H^i(\tilde{\mathcal{D}}_*^m(n)) = \begin{cases} \mathbb{Z} , & i = 0 \\ 0 , & i \neq 0 \end{cases} .$$

The proof follows immediately from Lemmas 2.11 and 2.12.

The group $GL_n(F)$ acts naturally on the complex $\tilde{\mathcal{D}}_*^m(n)$. Let us denote complex $\tilde{\mathcal{D}}_*^m(n)_{GL_n(F)}$ as $\mathcal{D}_*^m(n)$. Lemma 2.16 implies that there is a canonical homomorphism.

$$H_*(GL_n(F),\mathbb{Z}) \longrightarrow H_*(\mathcal{D}_*^m(n)) . \qquad (2.15)$$

Our next problem will be to construct a homomorphism φ of complexes

$$\begin{array}{ccccccc} \xrightarrow{\partial} & \mathcal{D}_4^{(n-2)}(n) & \xrightarrow{\partial} & \mathcal{D}_3^{(n-2)}(n) & \longrightarrow & \\ & \downarrow \varphi & & \downarrow \varphi & & \\ 0 \longrightarrow & B_2(F) & \xrightarrow{\delta_2} & \Lambda^2 F^* & \longrightarrow 0 \end{array} \qquad (2.16)$$

We will often use the following notations. Let $(\ell_1, \dots ,\ell_k, \dots ,\ell_m)$ be a configuration of m vectors in a vector space V . Denote by $< \ell_1, \dots ,\ell_k >$ the subspace generated by the vectors $\ell_1, \dots ,\ell_k$. Then let us denote by $(\ell_1, \dots ,\ell_k | \ell_{k+1}, \dots ,\ell_m)$ the configuration of $m-k$ vectors in the vector space $V/< \ell_1, \dots ,\ell_k >$, obtained by the projection of the vectors $\ell_{k+1}, \dots ,\ell_m$, and by $(\ell_1, \dots ,\ell_k | \bar{\ell}_{k+1}, \dots ,\bar{\ell}_m)$ the corresponding configuration of points in the projective space $P(V/< \ell_1, \dots ,\ell_k >)$.

Let us define the homomorphism $p : C_m(n) \longrightarrow C_{m-1}(n-1)$ by the formula

$$p : (\ell_1, \dots, \ell_m) \longmapsto \sum_{i=1}^{m} (-1)^{i-1} (\ell_i \,|\, \ell_1, \dots, \hat{\ell}_i, \dots, \ell_m) . \qquad (2.17)$$

Then we get the bicomplex

$$\begin{array}{ccccc} & & C_5(3) & \xrightarrow{d} & C_4(3) \\ & & \downarrow p & & \downarrow p \\ C_5(2) & \xrightarrow{d} & C_4(2) & \xrightarrow{d} & C_3(2) \ . \end{array} \qquad (2.18)$$

Let us define a homomorphism f from the associated complex to the Bloch–Suslin complex $B_F(2)$ in the following way: it coincides with the above constructed homomorphism (2.9) on the subcomplex $C_*(2)$ and is zero in other places:

$$\begin{array}{ccccc} & & C_5(3) & \xrightarrow{d} & C_4(3) \\ & & \downarrow -p & \swarrow 0 & \downarrow p \\ C_5(2) & \longrightarrow & C_4(2) & \xrightarrow{d} & C_3(2) \\ & & \downarrow f_1^{(2)} & & \downarrow f_0^{(2)} \\ 0 & \longrightarrow & B_2(F) & \xrightarrow{\delta_2} & \Lambda^2 F^* \end{array} \ . \qquad (2.19)$$

The correctness of this definition is provided by the following lemmas:

<u>Lemma</u> 2.17. $p \circ f_0^{(2)} = 0$.

Proof. Let ω_3 be a volume form in a 3–dimensional vector space V_3 and $(\ell_1, \dots, \ell_4) \in C_4(V_3)$. Then $\Delta_{\omega_3}(\ell_i, \cdot, \cdot)$ is a volume form in $V_3/<\ell_i>$. So

$$p \circ f_0^{(2)}(\ell_1, \dots, \ell_4) = \Delta(\ell_1,\ell_2,\ell_3) \wedge \Delta(\ell_1,\ell_2,\ell_4) + \dots .$$

It is easy to check that the right hand side is zero.

See also the proof of Lemma 3.1. ■

The assertion that $f_1^{(2)} \circ p = 0$ is an immediate consequence of the following useful fact.

Lemma 2.18. Let $x_1, \dots, x_5$ be 5 points in generic position in P_F^2. Then

$$\sum_{i=1}^{5} (-1)^i [r(x_1, \dots, \hat{x}_i, \dots, x_5)] = 0 \text{ in } B_2(F) \tag{2.20}$$

Proof. There is a (exactly one) conic passing through the points $x_1, \dots, x_5$. Choose an isomorphism from this conic to P_F^1. Let y_i be a point in P_F^1 corresponding to x_i by this isomorphism, then

$$(x_i \mid x_1, \dots, \hat{x}_i, \dots, x_5) = (y_1, \dots, \hat{y}_i, \dots, y_5) .$$

So (2.20) corresponds just to the 5–term relation in $B_2(F)$. ■

Now in order to construct a homomorphism of the complexes (2.16) we define a homomorphism $\tilde{\varphi}$ from the bicomplex (2.14) to the following one

$$
\begin{array}{ccccccccccc}
\vdots & & \vdots & & & & & & \vdots & & \\
\cdots\cdots & & \cdots\cdots & & \cdots\cdots & & \cdots\cdots & & C_1(4) & \xrightarrow{d} & \mathbb{Z} \\
\downarrow p & & \downarrow -p & & \downarrow p & & \downarrow -p & & \downarrow p & & \downarrow -p \\
\to C_5(3) & \xrightarrow{d} & C_4(3) & \xrightarrow{d} & C_3(3) & \xrightarrow{d} & C_2(3) & \xrightarrow{d} & C_1(3) & \xrightarrow{d} & \mathbb{Z} \\
\downarrow p & & \downarrow -p & & \downarrow p & & \downarrow -p & & \downarrow p & & \\
\to C_4(2) & \xrightarrow{d} & C_3(2) & \xrightarrow{d} & C_2(2) & \xrightarrow{d} & C_1(2) & \xrightarrow{d} & \mathbb{Z} & &
\end{array}
\qquad (2.21)
$$

Namely, if $(\ell_1, \dots, \ell_m) \in C_m(n)$ is placed at the level k in the bicomplex (2.14) (this means that we apply to $(\ell_1, \dots, \ell_m)$ the horizontal differential $d^{(k)}$) , then we set

$$\varphi : (\ell_1, \dots, \ell_m) \longmapsto (\ell_1, \dots, \ell_k \mid \ell_{k+1}, \dots, \ell_m) \in C_{m-k}(n-k) .$$

It is clear that φ is a homomorphism of bicomplexes.

Finally, we define a homomorphism from the complex associated with the bicomplex 2.21 to the Bloch–Suslin complex as a prolongation by zero of the homomorphism f – see (2.19), i.e. it is zero for all groups $C_m(n)$ different from $C_4(2)$ and $C_3(2)$. So we construct the desired homomorphism of the complexes (2.16) and, in particular, a homomorphism

$$g^{(n)} : H_3(GL_n(F),\mathbb{Z}) \longrightarrow \operatorname{Ker} \delta_2 \subset B_2(F) . \qquad (2.22)$$

Lemma 2.19. The restriction of this homomorphism to the subgroup $GL_2(F) \subset GL_n(F)$ coincides with the one

$$g^{(2)} : H_3(GL_2(F),\mathbb{Z}) \longrightarrow \operatorname{Ker} \delta_2 \subset B_2(F)$$

constructed in § 2.5.

Indeed, choose $n-2$ linear independent vectors $v_1, \dots ,v_{n-2}$ in an n–dimensional vector space V_n and a 2–dimensional complementary subspace $V_2 : V_n = \langle v_1, \dots ,v_{n-2} \rangle \oplus V_2$. Then there is a homomorphism of complexes

$$\psi : C_*(V_2) \longrightarrow \mathcal{D}_*^{n-2}(V_n)$$

where $\psi(C_*(V_2))$ lies in the lowest line of the bicomplex (2.14) and ψ is defined by the formula

$$\psi : (\ell_1, \dots ,\ell_k) \longmapsto (v_1, \dots ,v_{n-2},\ell_1, \dots ,\ell_k) .$$

From the definitions it is clear that we get a commutative diagram (the left arrow was constructed in § 2.3)

$$\begin{array}{ccc} C_*(2) & \xrightarrow{\ \psi\ } & \mathcal{D}_*^{n-2}(n) \\ & \searrow \quad \swarrow \varphi & \\ & B(2) & \end{array}$$

■

Finally, let us consider the composition

$$D_2 \circ g^{(n)} : H_3(GL_n(\mathbb{C}),R) \longrightarrow R . \tag{2.23}$$

The same arguments as in § 2.5 show that it defines an element in $H^3_{cont}(GL_n(\mathbb{C}),R)$. By Lemma 2.19 its restriction to the subgroup $GL_2(\mathbb{C})$ coincides with the Borel class. But $\dim_R H^3_{cont}(GL_n(\mathbb{C}),R) = 1$. So the map (2.23) is just the Borel regulator $r_2^{(n)}$. Q.E.D.

Finally, let me note that if we are interested only in the proof of Theorem 2.1, then section 6 can be cancelled if we are ready to use Suslin's results about homology of GL(F) ([S1]) and finiteness of $K_3^M(F)$ for number fields. Indeed, $H_3(GL_3(F)) = H_3(GL(F))$ and $H_3(GL_3(F))/H_3(GL_2(F)) = K_3^M(F)$ up to 2–torsion. So in the case of number fields $H_3(GL_2(F),\mathbb{Q}) \xrightarrow{\sim} H_3(GL(F),\mathbb{Q})$.

However, section 6 is necessary for the construction of characteristic classes from K–groups of an arbitrary field to the cohomology of the motivic complexes $B_F(2)$ and $B_F(3)$. (In fact the stabilisation trick that we used in s. 6 for the case $B_F(2)$ is based on the same idea as Suslin's trick in § 3 of [S3]). Our proof of Zagier's conjecture about $\zeta_F(3)$ also used constructions from s. 6, but in a more complicated situation.

§ 3. The trilogarithmic complex: generic configurations

1. Our plans. From now on we will work up to 6–torsion.

Let $C_6(P^2_F)$ be the free abelian group generated by all possible configurations $(\ell_1, \dots, \ell_6)$ of 6 points in P^2_F.

Definition 3.1. $\tilde{\mathcal{G}}_3(F)$ is the quotient of the group $C_6(P^2_F)$ by the following relations

R1) $(\ell_1, \dots, \ell_6) = 0$, if 2 of the points coincide or 4 lie on a line

R2) (The 7–term relation) For any 7 points $(\ell_1, \dots, \ell_7)$ in P^2_F

$$\sum_{i=1}^{7} (-1)^i (\ell_1, \dots, \hat{\ell}_i, \dots, \ell_7) = 0 .$$

Note that the configurations from the relation R1) are just the unstable ones in the sense of D. Mumford [Mu].

Lemma 3.2. (The skew–symmetry relation). In the group $\tilde{\mathcal{G}}_3(F)$

$$(\ell_1, \dots, \ell_6) = (-1)^{|\sigma|} (\ell_{\sigma(1)}, \dots, \ell_{\sigma(6)})$$

where $|\sigma|$ is the sign of the permutation $\sigma \in S_6$.

Proof. Let us apply the 7–term relation for a 7–tuple $(\ell_1, \dots, \ell_7)$ such that $\ell_i = \ell_{i+2}$ $(i \le 5)$. Then R1) implies that we get just the skew–symmetry relation for the transposition $(i, i+1)$. ∎

Let $\widetilde{\mathcal{B}}_3^{0}(F)$ be a subgroup of $\widetilde{\mathcal{B}}_3(F)$, generated by the configurations in generic position. In this section we will define the following commutative diagram

$$\begin{array}{ccccc} C_6(3) & \xrightarrow{\ d\ } & C_5(3) & \xrightarrow{\ d\ } & C_4(3) \\ \downarrow f_2^{(3)} & & \downarrow f_1^{(3)} & & \downarrow f_0^{(3)} \\ \widetilde{\mathcal{B}}_3^{0}(F) & \overset{\delta}{\dashrightarrow} & B_2(F)\otimes F^* & \xrightarrow{\ \delta\ } & \Lambda^3 F^* \end{array} \qquad (3.1)$$

and prove that $f_1^{(3)} \circ d(\ell_1, \dots, \ell_6)$ does not depend on the "length" of the vectors ℓ_i (see proposition 3.9). Hence we define δ on the generators of the group . Further,

$$\delta\Big(\sum_{i=0}^{6} (-1)^i (\bar{\ell}_0, \dots, \hat{\bar{\ell}}_i, \dots, \bar{\ell}_6)\Big) = f_1^{(3)} \circ d \circ d(\ell_0, \dots, \ell_6) = 0 .$$

Hence we get a correctly defined homomorphism

$$\delta : \widetilde{\mathcal{B}}_3^{0}(F) \longrightarrow B_2(F) \otimes F^* .$$

Recall that the homomorphism $\delta : B_2(F) \otimes F^* \longrightarrow \Lambda^3 F^*$ is defined by the formula $\delta : [x] \otimes y \longmapsto (1-x) \wedge x \wedge y$.

The property $\delta \circ \delta = 0$ follows immediately from the commutativity of (3.1).

In § 4 we define δ on degenerate configurations and hence get a definition of the homomorphism

$$\widetilde{\mathcal{B}}_3(F) \longrightarrow (B_2(F) \otimes F^*)_{\mathbb{Q}} .$$

Then we do the second crucial step: compute δ for a configuration $(m_0, \dots, m_5)$ as in fig. 1.8 and prove that, in a rather miraculous way, $\delta(m_0, \dots, m_5)$ is a linear combination of the expressions $[x] \otimes x$. More precisely

$$\delta(m_0, \dots, m_5) = \frac{1}{3}\delta \sum_{i=0}^{4} (-1)^i L_3'\{r(m_5 | m_0, \dots, \hat{m}_i, \dots, m_4)\}$$

where the homomorphism $L_3' : \mathbb{Z}[P_F^1 \backslash 0,1,\infty] \longrightarrow C_6(P_F^2)$ was defined on p. 20 of § 1.

So, if we define $\mathcal{G}_3(F)$ as the factor of the group $\tilde{\mathcal{G}}_3(F)$ by the following relations

$$\text{R3)} \quad (m_0, \dots, m_5) = \frac{1}{3} \sum_{i=0}^{4} (-1)^i L_3'\{r(m_5 | m_0, \dots, \hat{m}_i, \dots, m_4)\} + \frac{1}{3}\eta_3$$

then we get a complex

$$\mathcal{G}_3(F) \xrightarrow{\delta} (B_2(F) \otimes F^*)_{\mathbb{Q}} \xrightarrow{\delta} (\Lambda^3 F^*)_{\mathbb{Q}} .$$

In § 5 we prove that there is a canonical isomorphism

$$M_3 : \mathcal{G}_3(F) \longrightarrow \mathbb{Z}[P_F^1 \backslash 0,1,\infty] / R_3 =: B_3(F)$$

commuting with δ (where R_3 is the subgroup generated by the functional equations for the trilogarithm which were defined in (1.3).

Finally we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
\longrightarrow C_7(3) & \xrightarrow{d} & C_6(3) & \xrightarrow{d} & C_5(3) & \xrightarrow{d} & C_4(3) & \longrightarrow & \\
\downarrow & & \downarrow f_2^{(3)} & & \downarrow f_1^{(3)} & & \downarrow f_0^{(3)} & & \\
0 & \longrightarrow & \mathcal{G}_3(F) & \xrightarrow{\delta} & (B_2(F)\otimes F^*)_{\mathbb{Q}} & \xrightarrow{\delta} & (\Lambda^3 F^*)_{\mathbb{Q}} & & \\
 & & \downarrow \int & & \| & & \| & & \\
 & & B_3(F) & \xrightarrow{\delta} & (B_2(F)\otimes F^*)_{\mathbb{Q}} & \xrightarrow{\delta} & (\Lambda^3 F^*)_{\mathbb{Q}} & & .
\end{array}$$

Let us denote by $B_F(3)$ the complex in the lowest line of this diagram; the group $B_3(F)$ is placed in degree 1.

Therefore we construct the homomorphisms

$$\begin{aligned}
c_1^{(3)} &: H_5(GL_3(F),\mathbb{Z}) \longrightarrow H_5(C_*(3)) \longrightarrow H^1(B_F(3)) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.2) \\
c_2^{(3)} &: H_4(GL_3(F),\mathbb{Z}) \longrightarrow H_4(C_*(3)) \longrightarrow H^2(B_F(3)) .
\end{aligned}$$

A stabilisation trick with the bicomplex $C_*^{n-3}(n)$ permits to construct the homomorphisms

$$c_1^{(n)} : H_5(GL_n(F),\mathbb{Z}) \longrightarrow H^1(B_F(3))$$

$$c_2^{(n)} : H_4(GL_n(F),\mathbb{Z}) \longrightarrow H^2(B_F(3))$$

which restricted to $GL_3(F)$ coincide with the homomorphisms (3.2). So we get the canonical maps

$$c_1 : K_5^{[2]}(F)_{\mathbb{Q}} \longrightarrow H^1(B_F(3))$$

$$c_2 : K_4^{[1]}(F)_{\mathbb{Q}} \longrightarrow H^2(B_F(3))\ .$$

In the case $F = \mathbb{C}$ the function $\mathcal{L}_3(z)$ is identically zero on the subgroup R_3 so it defines a homomorphism $\mathcal{L}_3 : B_3(\mathbb{C}) \longrightarrow R$. We prove that the composition

$$H_5(GL(\mathbb{C}),R) \xrightarrow{c_1} \operatorname{Ker} \delta_1 \subset B_3(\mathbb{C}) \xrightarrow{\mathcal{L}_3} \mathbb{R}$$

is just the Borel class in $H^5_{cts}(GL(\mathbb{C}),R)$. This fact together with the Borel theorem [Bo 2] implies Theorem 1.

Now let us begin to realise this plan.

2. <u>The homomorphism</u> $f_0^{(m)} : C_{m+1}(m) \longrightarrow \Lambda^m F^*$. Let V_m be a vector space of dimension m and $\omega \in \det V_m^*$. Set

$$\Delta_\omega(\ell_1, \dots ,\ell_m) := < w,\ell_1 \wedge \dots \wedge \ell_m > .$$

Often we will write simply $\Delta(\ell_1, \dots ,\ell_m)$. Set

$$f_0^{(m)} : (\ell_0, \dots ,\ell_m) := \sum_{i=0}^{m} (-1)^i \bigwedge_{\substack{j=0\\ j\neq i}}^{m} \Delta(\ell_0, \dots ,\hat{\ell}_j, \dots ,\ell_m) \in \Lambda^m F^*\ .$$

For example $f_0^{(2)}$ is given by formula (2.7) and

$$f_0^{(3)}(\ell_0,\ell_1,\ell_2,\ell_3) = \Delta(\ell_0,\ell_2,\ell_3) \wedge \Delta(\ell_0,\ell_1,\ell_3) \wedge \Delta(\ell_0,\ell_1,\ell_2)$$

$$- \Delta(\ell_1,\ell_2,\ell_3) \wedge \Delta(\ell_0,\ell_1,\ell_3) \wedge \Delta(\ell_0,\ell_1,\ell_2)$$

$$+ \Delta(\ell_1,\ell_2,\ell_3) \wedge \Delta(\ell_0,\ell_1,\ell_3) \wedge \Delta(\ell_0,\ell_1,\ell_2)$$

$$- \Delta(\ell_1,\ell_2,\ell_3) \wedge \Delta(\ell_0,\ell_1,\ell_3) \wedge \Delta(\ell_0,\ell_1,\ell_3) .$$

Lemma 3.3. $f_0^{(m)}(\ell_0, \dots ,\ell_m)$ does not depend on ω.

Proof. It is not hard to prove it directly. However we give another proof that might clarify the situation.

Let $S^{0,m}(\ell_0, \dots ,\ell_m)$ be a simplex with vertices $S(0), \dots ,S(m)$ that (formally) correspond to the vectors $\ell_0, \dots \ell_m$. We will denote by $S(i_0, \dots ,i_k)$ its k–dimensional face with vertices $S(i_0), \dots ,S(i_k)$. Centers of codimension 1 faces are vertices of the dual simplex $\hat{S} = S^{m,0}(\ell_0, \dots ,\ell_m)$. For example, $\hat{S}(i)$ is the center of the face $S(0, \dots ,\hat{i}, \dots ,m)$.

Let $\hat{S}(i_0, \dots ,i_\ell)$ be an ℓ–dimensional face of $\hat{S}(0, \dots ,m)$ with vertices at $\hat{S}(i_0), \dots ,\hat{S}(i_\ell)$. Denote by $C_\ell(\hat{S}^{m,0}(\ell_0, \dots ,\ell_m),\mathbb{Z})$ the group of ℓ–chains of this simplex.

Let us consider a homomorphism

$$\varphi_\omega : C_\ell(\widehat{S}^{m,0}(\ell_0, \dots, \ell_m), \mathbb{Z}) \longrightarrow \Lambda^\ell F^*$$

that takes $\widehat{S}(i_0, \dots, i_\ell)$ to

$$\Delta(\ell_0,\dots,\widehat{\ell}_{i_0},\dots,\ell_m) \wedge \Delta(\ell_0,\dots,\widehat{\ell}_{i_1},\dots,\ell_m) \wedge \dots \wedge \Delta(\ell_0,\dots,\widehat{\ell}_{i_\ell},\dots,\ell_m) \in \Lambda^\ell F^* .$$

Then by definition

$$f_0^{(m)}(\ell_0, \dots, \ell_m) = \varphi_\omega(\partial\widehat{S}(0, \dots, m)) \qquad (3.3)$$

where ∂ is the differential in the chain complex $C_*(\widehat{S})$.

Now let $\omega^1 = \lambda\omega$, $\lambda \in F^*$. Then

$$(\varphi_{\lambda\omega} - \varphi_\omega)(\widehat{S}(i_0, \dots, i_\ell)) = \lambda \wedge \varphi_\omega(\partial\widehat{S}(i_0, \dots, i_\ell)) . \qquad (3.4)$$

Now the property $\partial^2 = 0$ and formulae (3.3) and (3.4) prove Lemma 3.3. ■

<u>Remark</u> 3.4. The symmetric group S_{m+1} acts naturally on $C_{m+1}(m)$. For $\sigma \in S_{m+1}$ and $c \in C_{m+1}(m)$ we have $f_0^{(m)}(\sigma c) = (-1)^{|\sigma|} f(c)$.

<u>Example</u> 3.5 (Compare [S1]). If $\ell_0 = \sum_{i=1}^{m} a_i\ell_i$ and $\Delta(\ell_1, \dots, \ell_m) = 1$, then

$$f_0^{(m)}(\ell_0, \dots, \ell_m) = a_1 \wedge (-a_2) \wedge \dots \wedge ((-1)^{m-1} a_m) .$$

Lemma 3.6. The composition

$$C_{m+2}(m+1) \xrightarrow{\ p\ } C_{m+1}(m) \xrightarrow{f_0^{(m)}} \Lambda^m F^*$$

is equal to zero.

Proof. Let $S(0, \dots ,m+1)$ be a simplex with vertices corresponding (formally) to the vectors $\ell_0, \dots ,\ell_{m+1}$. Then

$$f_0^{(m)} \circ p = \varphi_\omega \partial(\partial \hat{S}(0, \dots ,m+1)) = 0 .$$ ■

3. The homomorphism $f_1^{(3)} : C_5(3) \longrightarrow B_2(F) \otimes F^*$. Let $(\ell_0, \dots ,\ell_4)$ be a configuration of 5 vectors in generic position in a 3–dimensional space V_3. Set

$$\Delta(\hat{\ell}_i,\hat{\ell}_j) := \Delta(\ell_0, \dots ,\hat{\ell}_i, \dots ,\hat{\ell}_j, \dots ,\ell_4)$$

$$f_1^{(3)}(\ell_0, \dots ,\ell_4) := -\frac{1}{3} \sum_{i=0}^{4} (-1)^i [r(\ell_i \mid \ell_0, \dots ,\hat{\ell}_i, \dots ,\ell_4)] \otimes \prod_{j \neq i} \Delta(\hat{\ell}_i,\hat{\ell}_j) \tag{3.5}$$

$$\in (B_2(F) \otimes F^*) \otimes \mathbb{Z}\left[\tfrac{1}{3}\right] .$$

Recall that $\Delta(\hat{\ell}_i,\hat{\ell}_j)$ is defined using a volume form $\omega \in \det V_3^*$.

Proposition 3.7. $f_1^{(3)}(\ell_0, \dots ,\ell_4)$ does not depend on ω.

<u>Proof</u>. The difference of the elements $f_1^{(3)}(\ell_0, \dots, \ell_4)$ defined using the volume forms $\lambda \cdot \omega$ and $\omega\,(\lambda \in F^*)$ is equal to

$$-\frac{1}{3}\sum_{i=0}^{4} (-1)^i \,[r(\ell_i \,|\, \overline{\ell}_0, \dots, \widehat{\overline{\ell}_i}, \dots, \overline{\ell}_4)] \otimes \lambda^4 \,.$$

It remains to use Lemma 2.18 . ■

<u>Proposition</u> 3.8. The following diagram (defined modulo 6–torsion)

$$\begin{array}{ccc} C_5(3) & \xrightarrow{\quad d \quad} & C_4(3) \\ \downarrow{\scriptstyle f_1^{(3)}} & & \downarrow{\scriptstyle f_0^{(3)}} \\ B_2(F)\otimes F^* & \xrightarrow{\quad \delta \quad} & \Lambda^3 F^* \end{array}$$

is commutative modulo 6–torsion.

We prove the proposition by direct calculation. Here we indicate the main steps. First of all, using (2.8) we get

$$\delta : (\ell_0 \,|\, \overline{\ell}_1, \overline{\ell}_2, \overline{\ell}_3, \overline{\ell}_4) \longmapsto \frac{\Delta(\ell_0,\ell_1,\ell_2) \cdot \Delta(\ell_0,\ell_3,\ell_4)}{\Delta(\ell_0,\ell_1,\ell_3) \cdot \Delta(\ell_0,\ell_2,\ell_4)} \wedge \frac{\Delta(\ell_0,\ell_1,\ell_4) \cdot \Delta(\ell_0,\ell_2,\ell_3)}{\Delta(\ell_0,\ell_1,\ell_2) \cdot \Delta(\ell_0,\ell_3,\ell_4)} \,.$$

Then we compute $\delta \circ f_1^{(3)}(\ell_0, \dots, \ell_4)$ using this formula. The skew–symmetry relation in the group $B_2(F)$ implies that

$$f_1^{(3)}(\ell_{\sigma(0)}, \dots, \ell_{\sigma(4)}) = (-1)^{|\sigma|} f_1^{(3)}(\ell_0, \dots, \ell_4) .$$

Any summand in $\delta \circ f_1^{(3)}(\ell_0, \dots, \ell_4)$ can be transformed by some permutation of the vectors ℓ_i to one of the following expressions:

a) $\Delta(\ell_0,\ell_2,\ell_3) \wedge \Delta(\ell_0,\ell_1,\ell_3) \wedge \Delta(\ell_0,\ell_1,\ell_2)$

b) $\Delta(\ell_0,\ell_2,\ell_3) \wedge \Delta(\ell_0,\ell_1,\ell_3) \wedge \Delta(\ell_0,\ell_1,\ell_4)$

c) $\Delta(\ell_0,\ell_2,\ell_3) \wedge \Delta(\ell_0,\ell_1,\ell_3) \wedge \Delta(\ell_1,\ell_2,\ell_4)$

A simple computation shows that the first expression appears in $\delta \circ f_1^{(3)}(\ell_0, \dots, \ell_4)$ with coefficient 1, and the second and third with coefficient 0.

The computation of $f_0^{(3)} \circ d(\ell_0, \dots, \ell_4)$ gives the same result. ■

<u>Proposition</u> 3.9. The composition

$$C_6(3) \xrightarrow{d} C_5(3) \xrightarrow{f_1^{(3)}} (B_2(F) \otimes F^*) \otimes \mathbb{Z}\left[\frac{1}{6}\right]$$

does not depend on the length of the vectors ℓ_i, i.e.

$$f_1^{(3)} \circ d[(\ell_0, \dots, \ell_5) - (\lambda_0\ell_0, \dots, \lambda_5\ell_5)] = 0 , \ (\lambda_i \in F^*) .$$

<u>Proof</u>. It is sufficient to consider the case when $\lambda_1 = \dots = \lambda_5 = 1$, $\lambda_0 = \lambda$. Recall that

$$d(\ell_0, \dots, \ell_5) = \sum_{j=0}^{5} (-1)^j (\ell_0, \dots, \hat{\ell}_j, \dots, \ell_5) .$$

The first summand $(\ell_1, \dots, \ell_5)$ does not give a contribution to the difference

$$f_1^{(3)} \circ d[(\ell_0, \dots, \ell_5) - (\lambda\, \ell_0, \dots, \ell_5)] . \tag{3.6}$$

The contribution of the second summand $-(\ell_0, \ell_2, \ell_3, \ell_4, \ell_5)$ is equal to

$$\begin{aligned} &\tfrac{1}{3}\, [r(\ell_2 \,|\, \overline{\ell}_0 \overline{\ell}_3 \overline{\ell}_4 \overline{\ell}_5)] - [r(\ell_3 \,|\, \overline{\ell}_0, \overline{\ell}_2, \overline{\ell}_4, \overline{\ell}_5)] + [r(\ell_4 \,|\, \overline{\ell}_0, \overline{\ell}_2, \overline{\ell}_3, \overline{\ell}_5)] \\ &- [r(\ell_5 \,|\, \overline{\ell}_0, \overline{\ell}_3, \overline{\ell}_4, \overline{\ell}_5)] \otimes \lambda^3 \in (B_2(F) \otimes F^*) \otimes \mathbb{Z}\left[\tfrac{1}{6}\right] . \end{aligned} \tag{3.7}$$

Applying Lemma 2.18 to the 5–tuple $(\overline{\ell}_0, \overline{\ell}_2, \dots, \overline{\ell}_5)$ of points in P_F^2 we see that (3.7) is equal to $[r(\ell_0 \,|\, \overline{\ell}_2, \overline{\ell}_3, \overline{\ell}_4, \overline{\ell}_5)] \otimes \lambda$. Analogously the contribution of the summand $(-1)^j (\ell_0, \dots, \hat{\ell}_j, \dots, \ell_5)$ in (3.6) is

$$(-1)^{j-1} [r(\ell_0 \,|\, \overline{\ell}_1, \dots, \hat{\overline{\ell}}_j, \dots, \overline{\ell}_5)] \otimes \lambda .$$

Summarising we see that (3.6) is equal to

$$\sum_{j=1}^{5} (-1)^{j-1} [r(\ell_0 \,|\, \overline{\ell}_1, \dots, \hat{\overline{\ell}}_j, \dots, \overline{\ell}_5)] \otimes \lambda \tag{3.8}$$

But the left factor is just a 5–term relation in $B_2(F)$, so (3.8) is 0 . ■

Let $f_2^{(3)}$ be the projectivisation map:

$$f_2^{(3)} : (\ell_0, \dots, \ell_5) \longmapsto (\bar{\ell}_0, \dots, \bar{\ell}_5) .$$

Now the commutative diagram (3.1) is constructed.

§ 4. The trilogarithmic complex: degenerate configurations.

1. The homomorphism $\delta : \tilde{\mathcal{B}}_3(F) \longrightarrow B_2(F) \otimes F^*$. Let $C'_m(3)$, $(m \geq 5)$, be the free abelian group generated by the configurations of m vectors in the space V_3 such that no 4 lie in a plane.

First of all let us define a skew–symmetric (with respect to permutations) homomorphism $f_1^{(3)} : C'_5(3) \longrightarrow (B_2(F) \otimes F^*) \otimes \mathbb{Z}\left[\frac{1}{6}\right]$. On the subgroup $C_5(3)$ of generic configurations it was already defined in § 3.

Up to permutations there are exactly two types of degenerate configurations in $C'_5(3)$ – see fig. 4.1 where the corresponding configurations of the points in P^2 are presented.

fig. 4.1

By definition $f_1^{(3)}$ takes the configurations of the second type in fig. 4.1 to zero.

Now let $(\ell_0, \ell_1, \ell_2, \ell_3, \ell_4)$ be the configuration of the first type such that ℓ_0, ℓ_1, ℓ_2 are in the same plane.

Denote by $\tilde{\ell}_3$ and $\tilde{\ell}_4$ the projections of the vectors ℓ_3 and ℓ_4 onto the 1–dimensional space $V_3/<\ell_0,\ell_1,\ell_2>$. Let us define $v(\tilde{\ell}_4/\tilde{\ell}_3) \in F^*$ as follows:

$$\tilde{\ell}_4 = v(\tilde{\ell}_4/\tilde{\ell}_3) \cdot \tilde{\ell}_3 .$$

Put

$$f_1^{(3)}(\ell_0, \dots ,\ell_4) := [r(\ell_4 | \ell_0, \dots ,\ell_3)] \otimes v(\tilde{\ell}_4/\tilde{\ell}_3) \in B_2(F) \otimes F^* . \qquad (4.1)$$

It is clear that

$$f_1^{(3)}(\ell_0, \dots ,\ell_4) := \frac{1}{3} [r(\ell_4 | \ell_0, \dots ,\ell_3)] \otimes \frac{\Delta(\ell_0,\ell_1,\ell_4)\cdot\Delta(\ell_0,\ell_2,\ell_4)\cdot\Delta(\ell_1,\ell_2,\ell_4)}{\Delta(\ell_0,\ell_1,\ell_3)\cdot\Delta(\ell_0,\ell_2,\ell_3)\cdot\Delta(\ell_1,\ell_2,\ell_3)} . \qquad (4.2)$$

Remark that $(\ell_4 | \ell_0,\ell_1,\ell_2,\ell_3) \equiv (\ell_3 | \ell_0\ell_1\ell_2\ell_4)$, and (4.2) can be considered as a "regularisation" of the formula (3.5) for the homomorphism $f_1^{(3)}$. Namely, we removed from (3.5) all factors $\Delta(\ell_0,\ell_1,\ell_2)$ which are zero in our case.

<u>Proposition</u> 4.1. Let $(\ell_0, \dots ,\ell_6) \in C_6(3)$. Then

$$(f_1^{(3)} \circ d)[(\ell_0, \dots ,\ell_5) - (\lambda_0\ell_0, \dots ,\lambda_5\ell_5)] = 0 .$$

The proof is in complete analogy with the one for Proposition 3.9, and even simpler.

Now let us define the homomorphism δ on generators of the group $\mathcal{B}_3(F)$ as follows: δ is zero for the configurations satisfying the condition R1) and $\delta = f_1^{(3)} \circ d$ in the opposite case.

2. Computation of the homomorphism δ for degenerate configurations. We begin with the remembrance of some notations.

If $\ell_0, \dots, \ell_3$ are vectors in V_2 and $\bar{\ell}_0, \dots, \bar{\ell}_3$ – corresponding points in $P^*(V_2)$, then

$$r(\bar{\ell}_0, \dots, \bar{\ell}_0) := \frac{\Delta(\ell_0,\ell_3)\cdot\Delta(\ell_1,\ell_2)}{\Delta(\ell_0,\ell_2)\cdot\Delta(\ell_1,\ell_3)} \in F^* .$$

Sometimes we will omit bars and write $r(\ell_0,\ell_1,\ell_2,\ell_3)$ instead of $r(\bar{\ell}_0,\bar{\ell}_1,\bar{\ell}_2,\bar{\ell}_3)$. We use the symbol [.] only for denoting elements of the group $B_2(F)$.

All possible combinatorial types of the configurations of 6 points in P^2, where no 4 lie on a line, are presented in fig. 4.2.

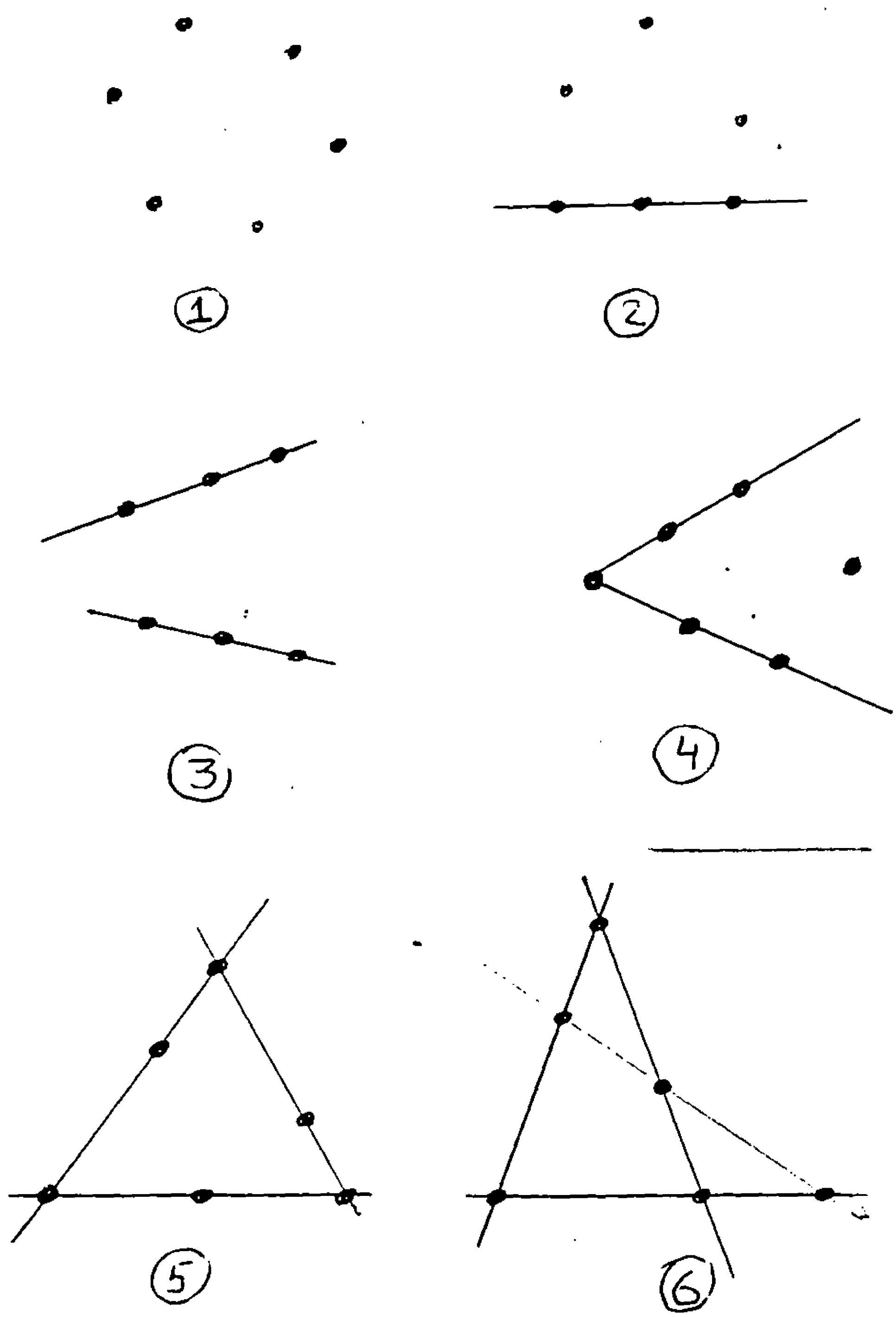


fig. 4.2

<u>Theorem</u> 4.2. For a configuration $(\mathfrak{C}_0, \dots, \mathfrak{C}_5)$ as in fig. 4.3

$$\delta(\bar{\ell}_0, \dots, \bar{\ell}_5) = +\tfrac{1}{3} \sum_{i=0}^{4} (-1)^i \gamma\{\bar{\ell}_5 \mid \bar{\ell}_0, \dots, \bar{\ell}_i, \dots, \bar{\ell}_4\} \qquad (4.3)$$

where $\gamma\{r(\bar{x}_0,\bar{x}_1,\bar{x}_2,\bar{x}_3)\} = -\delta\{r(\bar{x}_0,\bar{x}_1,\bar{x}_2,\bar{x}_3)\} - 2\delta\{r(\bar{x}_0,\bar{x}_2,\bar{x}_1,\bar{x}_3)\}$.

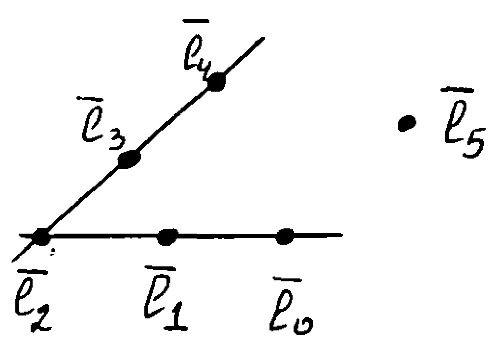


fig. 4.3

<u>Proof</u>. Let $(\ell_0, \dots, \ell_4)$ be a configuration of vectors in generic position in V_3 . Denote by W the plane generated by ℓ_0 and ℓ_1 .

Let $(m_0, \dots, m_4)$ be a configuration of covectors in W defined as follows (the dual configuration – see § 7).

Let $\tilde{m}_2,\tilde{m}_3,\tilde{m}_4$ be functionals dual to the basis ℓ_2,ℓ_3,ℓ_4 in V_3 , and let m_i be the restriction of $\tilde{m}_i$ to the plane W $(i = 2,3,4)$. Further, let m_0 , m_1 be the basis in W^* , dual to ℓ_0 , ℓ_1 .

Projectivisation of the kernel of the functional m_j is a point on the line $P(W)$. We denote it by $\overline{m}_j$ – see fig. 4.4.

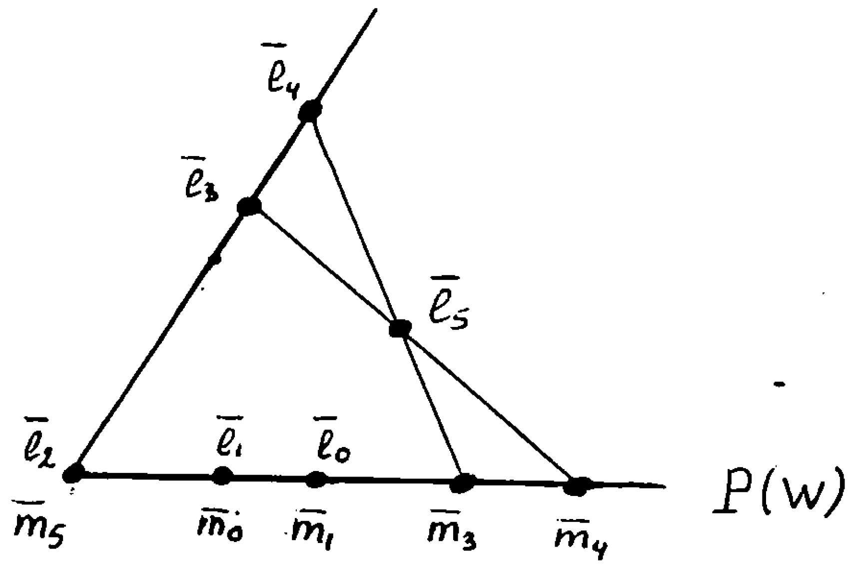


fig. 4.4

Let us fix the volume elements in V_3 and W^*. Then $\Delta(\ell_i,\ell_j,\ell_k)$ and $\Delta(m_i,m_j)$ are defined. We will denote them $(\ell_i\ell_j\ell_k)$ and (m_im_j) for short.

<u>Proposition</u> 4.3.

$$f_1^{(3)}(\ell_0, \dots, \ell_4) = \frac{1}{3} \sum_{i=0}^{4} (-1)^i [r(\overline{m}_0, \dots, \hat{\overline{m}}_i, \dots, \overline{m}_4)] \otimes \prod_{\substack{0 \le j_1 < j_2 \le 4 \\ i \notin \{j_1, j_2\}}} (m_{j_1} m_{j_2}) . \quad (4.4)$$

<u>Proof</u>. By definition

$$f_1^{(3)}(\ell_0, \dots, \ell_4) = \frac{1}{3} \sum_{i=0}^{4} (-1)^i [r(\overline{\ell}_i, \dots, \widehat{\overline{\ell}_i}, \dots, \overline{\ell}_4)] \otimes \prod_{\substack{0 \le j_1 < j_2 < j_3 \le 4 \\ i \notin \{j_1, j_2, j_3\}}} (m_{j_1} m_{j_2}) . \quad (4.5)$$

Lemma 4.4.

$$[r(\overline{\ell}_i \,|\, \overline{\ell}_0, \dots, \widehat{\overline{\ell}_i}, \dots, \overline{\ell}_4)] = [r(\overline{m}_0, \dots, \widehat{\overline{m}_i}, \dots, \overline{m}_4)] . \qquad (4.6)$$

It can also be proved looking at fig. 4.4. For example, projecting the points $\overline{\ell}_1, \dots, \overline{\ell}_4$ with center at the point $\overline{\ell}_0$ onto the line $\overline{\overline{\ell}_2\overline{\ell}_3}$ and then projecting the obtained points with center at $\overline{\ell}_4$ onto the line $\overline{\overline{m}_0\overline{m}_1}$, we obtain the configuration $(\overline{m}_4, \overline{m}_3, \overline{m}_2, \overline{m}_1) \equiv (\overline{m}_1, \overline{m}_2, \overline{m}_3, \overline{m}_4)$ and so on . ■

Lemma 4.5.

$$(\ell_0\ell_1\ell_2) = \lambda(m_3m_4) \; ; \; (\ell_0\ell_1\ell_3) = \lambda(m_2m_4) \; ; \dots ; (\ell_2\ell_3\ell_4) = \lambda(m_0m_1)$$

where $\lambda \in F^*$.

Using (4.6) and Lemma 4.5 we can rewrite (4.5) as

$$-\frac{1}{3} \sum_{i=0}^{4} (-1)^i [r(\overline{m}_0, \dots, \widehat{\overline{m}_i}, \dots, \overline{m}_4)] \otimes \prod_{\substack{j=0 \\ j \ne i}} (m_i m_j) -$$

$$-\frac{1}{3} \Big(\sum_{i=0}^{4} (-1)^i [r(\overline{m}_0, \dots, \widehat{\overline{m}_i}, \dots, \overline{m}_4)] \Big) \otimes \lambda^4 . \qquad (4.7)$$

The second term is 0 .

Finally, substracting 4.4 from 4.7 we get

$$-\frac{1}{3}\Big(\sum_{i=0}^{4} (-1)^i [r(\overline{m}_0, \dots, \hat{\overline{m}}_i, \dots, \overline{m}_4)] \otimes \prod_{0\le k_1<k_2\le 4} (m_{k_1} m_{k_2}) = 0 . \qquad \blacksquare$$

Now let us compute the homomorphism δ for a configuration $(\ell_0, \dots, \ell_5)$ as in fig. 4.3.

Let (m_0,m_1,m_3,m_4,m_5) be a configuration of covectors in the plane $W = < \ell_0,\ell_1 >$, dual to a configuration of vectors $(\ell_0,\ell_1,\ell_3,\ell_4,\ell_5)$ in V_3 . Note that $\overline{\ell}_2 = \overline{m}_5$ (see fig. 4.4.).

Using (4.4), Lemma 4.5 and looking at fig. 4.4 we obtain:

$$f_1^{(3)}(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5) = -f_1^{(3)}(\ell_2,\ell_3,\ell_4,\ell_1,\ell_5) = -\,[r(\overline{\ell}_5 \,|\, \overline{\ell}_2,\overline{\ell}_3,\overline{\ell}_4,\overline{\ell}_1)] \otimes v(\overline{\ell}_5/\overline{\ell}_1) =$$

$$-\,[r(\overline{m}_0,\overline{m}_3,\overline{m}_4,\overline{m}_5)] \otimes \frac{(\ell_3\ell_4\ell_5)}{(\ell_3\ell_4\ell_1)} = -\,[r(\overline{m}_0,\overline{m}_3,\overline{m}_4,\overline{m}_5)] \otimes \frac{(m_0m_1)}{(m_0m_5)} . \qquad (4.8)$$

Similarly

$$-f_1^{(3)}(\ell_0,\ell_2,\ell_3,\ell_4,\ell_5) = [r(\overline{m}_1,\overline{m}_3,\overline{m}_4,\overline{m}_5)] \otimes \frac{(m_0m_1)}{(m_1m_5)} \qquad (4.9)$$

$$-f_1^{(3)}(\ell_0,\ell_1,\ell_2,\ell_4,\ell_5) = [r(\overline{m}_0,\overline{m}_1,\overline{m}_3,\overline{m}_5)] \otimes \frac{(m_3m_4)}{(m_3m_5)} \quad (4.10)$$

$$f_1^{(3)}(\ell_0,\ell_1,\ell_2,\ell_3,\ell_5) = [r(\overline{m}_0,\overline{m}_1,\overline{m}_4,\overline{m}_5)] \otimes \frac{(m_3m_4)}{(m_4m_5)} \quad (4.11)$$

$$-f_1^{(3)}(\ell_0,\ell_1,\ell_2,\ell_3,\ell_4) = 0\,.$$

Using the 5–term relation in the group B_2 we can rewrite the sum of (4.8) – (4.11) as follows:

$$(-[r(\overline{m}_0,\overline{m}_1,\overline{m}_4,\overline{m}_5)] + [r(\overline{m}_0,\overline{m}_1,\overline{m}_3,\overline{m}_5)] - [r(\overline{m}_0,\overline{m}_1,\overline{m}_3,\overline{m}_4)]) \otimes (m_0m_1) +$$

$$(-[r(\overline{m}_0,\overline{m}_1,\overline{m}_3,\overline{m}_4)] + [r((m_0\, m_3\, m_4\, m_5)] - [r((m_1\, m_3\, m_4\, m_5)]) \otimes (m_3m_4)$$

$$(4.12)$$

$$+ [r(\overline{m}_0,\overline{m}_3,\overline{m}_4,\overline{m}_5)] \otimes (m_0m_5) - [r(\overline{m}_1,\overline{m}_3,\overline{m}_4,\overline{m}_5)] \otimes (m_1m_5)$$

$$+ [r(\overline{m}_0,\overline{m}_1,\overline{m}_3,\overline{m}_5)] \otimes (m_3m_5) - [r(\overline{m}_0,\overline{m}_1,\overline{m}_4,\overline{m}_5)] \otimes (m_4m_5)\,.$$

Note that for a configuration (x_0,x_1,x_2,x_3) of 4 vectors in V_2

$$\gamma\{r(\bar{x}_0,\bar{x}_1,\bar{x}_2,\bar{x}_3)\} = [r(\bar{x}_0,\bar{x}_4,\bar{x}_2,\bar{x}_3)] \otimes \frac{(x_0x_2)(x_0x_3)(x_1x_2)(x_1x_3)}{(x_0x_1)^2 \cdot (x_2x_3)^2}\,. \quad (4.13)$$

$\delta(\ell_0, \dots ,\ell_5)$ is equal to the sum of the right–hand side (4.14) and (4.15). Using (4.16), after some arithmetical calculations we obtain:

$$\delta(\ell_0, \dots ,\ell_5) = \tfrac{1}{3}\, \gamma(\{r(\overline{m}_0,\overline{m}_5,\overline{m}_4,\overline{m}_3)\} - \{r(m_1,m_5,m_4,m_3)\} +$$

$$+ \{r(\overline{m}_1,\overline{m}_0,\overline{m}_4,\overline{m}_3)\} - \{r(\overline{m}_1,\overline{m}_0,\overline{m}_5,\overline{m}_3)\} + \{r(\overline{m}_1,\overline{m}_0,\overline{m}_5,\overline{m}_4)\}) .$$

It remains to note that (see fig. 4.4) the configuration $(\overline{m}_1,\overline{m}_0,\overline{m}_5,\overline{m}_4,\overline{m}_3)$ which seems rather awkward in this notation, coincides with the configuration $(\ell_5 | \ell_0\ell_1\ell_2\ell_3\ell_4)$. Theorem 4.2 is proved completely. ■

Remark 4.6. Looking at (4.15) and (4.4) we see that every term of these formulae depends on only 4 points $\overline{m}_i$. So it is not too surprising that after some computations we get that $\delta(\ell_0, \dots ,\ell_5)$ lies in the subgroup $\delta(\mathbb{Z}[P^1_F \backslash 0,1,\infty])$ of $B_2 \otimes F^*$.

Lemma 4.7. For a configuration $(\ell_0, \dots ,\ell_5)$ represented in fig. 4.5.

$$\delta(\ell_0, \dots ,\ell_5) = \delta\{r(\ell_5 | \ell_0,\ell_1,\ell_3,\ell_4)\} .$$

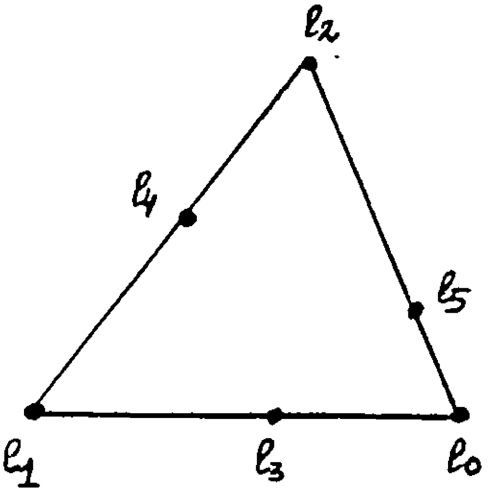


fig. 4.5

Proof. Let $(\ell_0, \dots, \ell_5)$ be some configuration of the vectors corresponding to the configuration of points $(\overline{\ell}_0, \dots, \overline{\ell}_5)$. By definition

$$f_1^{(3)}(\ell_0, \dots, \hat{\ell}_{3+i}, \dots, \ell_5) = 0 \ , \ \ i \geq 0,1,2 \ .$$

According to 4.1

$$f_1^{(3)}(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5) = f_1^{(3)}(\ell_1,\ell_4,\ell_2,\ell_3,\ell_5) = [r(\ell_5 \,|\, \overline{\ell}_1,\overline{\ell}_4,\overline{\ell}_2,\overline{\ell}_3)] \otimes \frac{(\ell_1\ell_4\ell_5)}{(\ell_1\ell_4\ell_3)} =$$

$$- [r(\overline{\ell}_5 \,|\, \overline{\ell}_0,\overline{\ell}_1,\overline{\ell}_3,\overline{\ell}_4)] \otimes \frac{(\ell_1\ell_4\ell_5)}{(\ell_1\ell_4\ell_3)} \, .$$

Similarly

$$-f_1^{(3)}(\ell_0,\ell_2,\ell_3,\ell_4,\ell_5) = [r(\ell_5 \,|\, \overline{\ell}_0\overline{\ell}_1\overline{\ell}_3\overline{\ell}_4)] \otimes \frac{(\ell_0\ell_5\ell_4)}{(\ell_0\ell_5\ell_3)}$$

$$f_1^{(3)}(\ell_0,\ell_1,\ell_3,\ell_4,\ell_5) = [r(\ell_5 \,|\, \overline{\ell}_0,\overline{\ell}_1,\overline{\ell}_3,\overline{\ell}_4)] \otimes \frac{(\ell_1\ell_3\ell_5)}{(\ell_1\ell_3\ell_4)} \, .$$

Adding, we get

$$\delta(\ell_0, \dots, \ell_5) = [r(\ell_5 \,|\, \overline{\ell}_0\overline{\ell}_1\overline{\ell}_3\overline{\ell}_4)] \otimes \frac{(\ell_0\ell_4\ell_5)(\ell_1\ell_3\ell_5)}{(\ell_1\ell_4\ell_5)(\ell_0\ell_3\ell_5)} =$$

$$\delta \, \{r(\overline{\ell}_5 \,|\, \ell_0,\ell_1,\ell_3,\ell_4)\} \, .$$

■

We will see below that the configurations of type 5 in fig. 4.2 correspond to the classical trilogarithm.

Note that there are two natural numerations of points of this configurations corresponding to different orientations of the triangle $(\ell_0\ell_1\ell_2)$ – see fig. 4.5 and 4.6.

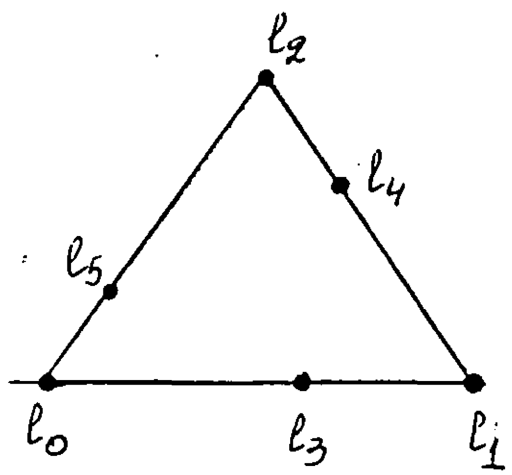


fig. 4.6

Let us denote by $\check{\ell}_3$ the intersection point of the lines $\ell_0\ell_1$ and $\ell_4\ell_5$. For similarly defined points $\check{\ell}_4$ and $\check{\ell}_5$ – see fig. 4.7.

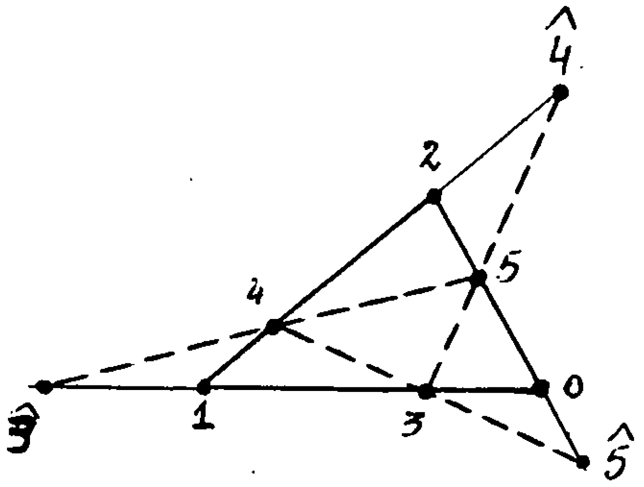


fig. 4.7

Lemma 4.8.

$$r(\ell_0,\ell_1,\ell_3,\check{\ell}_3) = r(\ell_1,\ell_2,\ell_4,\check{\ell}_4) = r(\ell_2,\ell_3,\ell_5,\check{\ell}_5)\,.$$

Proof. Looking at fig. 4.7 we see that

$$(\ell_5\,|\,\ell_0,\ell_1,\ell_3,\check{\ell}_3) \equiv (\ell_5\,|\,\ell_2,\ell_1,\check{\ell}_4,\ell_4)\,.$$

But $r(\ell_5\,|\,\ell_2,\ell_1,\check{\ell}_4,\ell_4) = r(\ell_5\,|\,\ell_1,\ell_2,\ell_4,\check{\ell}_4)$, so we get the first equality. Analogously projecting from ℓ_3 we get the second one. ■

Inversely, for every $x \in P^1_F\backslash\{0,1,\infty\}$ we can construct a configuration $C_3(x)$ as on fig. 4.5 with $r(\ell_5\,|\,\ell_0,\ell_1,\ell_3,\ell_4) = x$. In fact, we can do this canonically, namely, choose 4 points $\ell_0,\ell_1,\ell_3,\check{\ell}_3$ on a line L_2 in P^2 such that $r(\ell_0,\ell_1,\ell_3,\check{\ell}_3) = x$, and add the fifth point ℓ_2 not on the line L_2. Then the configuration in fig. 4.8 is constructed uniquely.

The configuration $C_3(x)$ defines an element of the group $\mathcal{B}_3(F)$ that we have denoted $L_3(x)$. So we construct the canonical homomorphism

$$L_3 : \mathbb{Z}[P^1_F\backslash 0,1,\infty] \longrightarrow \mathcal{B}_3(F) .$$

Remark that although the configurations $C_3(x)$ and $C_3(x^{-1})$ are different, the second one can be obtained from the first by permutation of the vertices $\sigma : (1,2,3,4,5,6) \longrightarrow (2,1,3,4,6,5)$. If this permutation is even, then

$$L_3(\{x\} - \{x^{-1}\}) = 0 . \tag{4.14}$$

Recall that a configuration of type 6 in fig. 4.2 is denoted by η_3.

<u>Lemma</u> 4.9. For every $x \in P^1_F\backslash\{0,1,\infty\}$

$$L_3(\{x\} + \{1-x\} + \{1-x^{-1}\}) = \eta_3 . \tag{4.15}$$

<u>Proof</u>. Let us write down the 7–term relation for a configuration $(\ell_0, \dots ,\ell_6)$ on fig. 4.8:

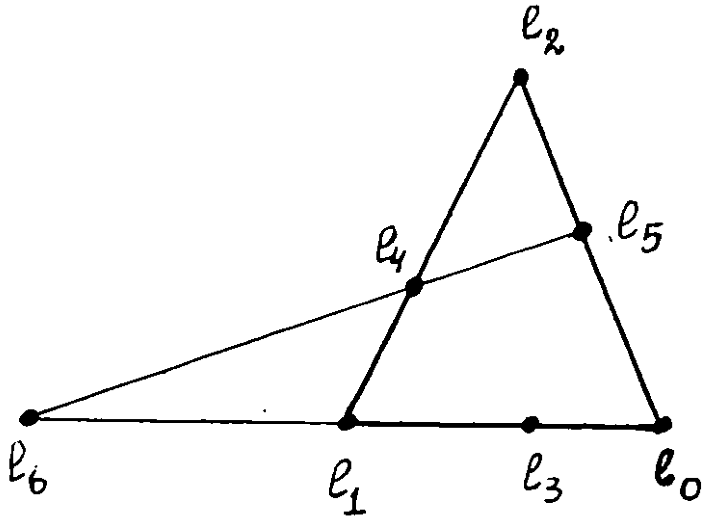


fig. 4.8

$$(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5,\ell_6) - (\ell_0,\ell_2,\ell_3,\ell_4,\ell_5,\ell_6) + (\ell_0,\ell_1,\ell_2,\ell_3,\ell_4,\ell_5) =$$

$$= (\ell_0,\ell_1,\ell_2,\ell_4,\ell_5,\ell_6) .$$

(The other 3 summands are zero according to the relation R1). Using the skew–symmetry relation in the group $\mathcal{G}_3(F)$, we have

$$(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5,\ell_6) = (\ell_1,\ell_6,\ell_4,\ell_3,\ell_5,\ell_2) = L_3\{r(\ell_1,\ell_6,\ell_3,\ell_0)\} =$$

$$= L_3\{r(\ell_0,\ell_3,\ell_1,\ell_6)\}$$

$$-(\ell_0,\ell_2,\ell_3,\ell_4,\ell_5,\ell_6) = (\ell_0,\ell_6,\ell_5,\ell_3,\ell_4,\ell_2) = L_3\{(r(\ell_0,\ell_6,\ell_3,\ell_4)\}$$

$$(\ell_0,\ell_1,\ell_2,\ell_3,\ell_4,\ell_5) = L_3\{r(\ell_0,\ell_1,\ell_3,\ell_6)\}$$

$$(\ell_0,\ell_1,\ell_2,\ell_4,\ell_5,\ell_6) = (\ell_0,\ell_1,\ell_2,\ell_6,\ell_4,\ell_5) = \eta_3 .$$

Therefore

$$L_3(\{r(\ell_0,\ell_1,\ell_3,\ell_6)\} + \{r(\ell_0,\ell_6,\ell_1,\ell_3)\} + \{r(\ell_0,\ell_3,\ell_6,\ell_1)\}) = \eta_3 . \quad (4.16)$$

If $r(\ell_0,\ell_1,\ell_3,\ell_6) = x$ we get (4.15). ■

Set

$$\zeta_3(x) := \{x\} + \{1-x\} + \{1-x^{-1}\} .$$

It is easy to check that $\delta(\zeta_3(x)) = 0$.

Lemma 4.10. $\delta(\eta_3) = 0$.

Proof. Using Lemmas 4.9 and 4.7 we have

$$\delta(\eta_3) = \delta L_3(\zeta_3(x)) = L_3\delta(\zeta_3(x)) = 0 .$$ ■

Note that according to Lemma 4.9

$$L_3(\zeta_3(x)) - L_3(\zeta_3(y)) = 0 .$$

So we get the following commutative diagram

$$\begin{array}{ccc} \dfrac{\mathbb{Z}[P^1_F\setminus 0,1,\infty]}{\{x\}-\{x\}^{-1},\zeta_3(x)-\zeta_3(y)} & \xrightarrow{\quad\delta\quad} & B_2(F)\otimes F^* \\ L_3 \big\downarrow & & \big\downarrow \mathrm{id} \\ \tilde{\mathcal{B}}_3(F) & \xrightarrow{\quad\delta\quad} & (B_2(F)\otimes F^*)\otimes \mathbb{Z}[\tfrac{1}{6}] \end{array} \tag{4.17}$$

3. <u>The group</u> $\mathcal{B}_3(F)$.

<u>Definition</u> 4.11. $\mathcal{B}_3(F)$ is the quotient of the group $\tilde{\mathcal{B}}_3(F)$ by the following relations

R3) If $(\ell_0, \dots ,\ell_5)$ is a configuration as in fig. 4.3 then

$$(\ell_0, \dots ,\ell_5) = \frac{1}{3} \sum_{i=0}^{5} (-1)^i L_3'\{r(\ell_5 \mid \ell_0, \dots ,\hat{\ell}_i, \dots ,\ell_4)\} + \frac{1}{3}\eta_3 \tag{4.18}$$

where $L_3'\{x\} := -L_3\{x\} - 2L_3\{1-x\}$.

Of course, this relation is motivated by Theorem 2.

Let us explain the reason for the summand $\frac{1}{3}\eta_3$ in formula (4.18).

Consider the skew–symmetry relation

$$(\ell_0,\ell_1,\ell_2,\ell_3,\ell_4,\ell_5) + (\ell_1,\ell_0,\ell_2,\ell_3,\ell_4,\ell_5) = 0 \tag{4.19}$$

Let us express each term of (4.19) using a slightly modified formula (4.18), where η_3 is taken with an undetermined coefficient λ . Then using (4.14) we get

$$L_3(-\tfrac{2}{3}\,[\{r(\ell_5\,|\,\ell_0,\ell_1,\ell_3,\ell_4)\} + \{r(\ell_5\,|\,\ell_0,\ell_4,\ell_1,\ell_3)\} + \{r((\ell_5\,|\,\ell_0,\ell_3,\ell_4,\ell_1)\}]$$

$$+\tfrac{2}{3}\,[\{r(\ell_5\,|\,\ell_0,\ell_1,\ell_2,\ell_4)\} + \{r(\ell_5\,|\,\ell_0,\ell_4,\ell_1,\ell_2)\} + \{r(\ell_5\,|\,\ell_0,\ell_2,\ell_4,\ell_1)\}]$$

$$+\tfrac{2}{3}\,[\{r(\ell_5\,|\,\ell_0,\ell_1,\ell_2,\ell_4)\} + \{r(\ell_5\,|\,\ell_0,\ell_2,\ell_1,\ell_3)\} +$$

$$\{r(\ell_5\,|\,\ell_0,\ell_3,\ell_1,\ell_2)\}]) + 2\,\lambda\,\eta_3 = 0\,.$$

Applying (4.16), we get $-\frac{2}{3}\eta_3 + 2\lambda\eta_3 = 0$ and $\lambda = \frac{1}{3}$. It is clear, that the permutation of the points ℓ_3 and ℓ_4 gives the same result.

Other permutations lead us to new configurations (that do not satisfy the condition $\ell_2 = \overline{\ell_0\ell_1} \cap \overline{\ell_3\ell_4}$) . In fact, we need the skew–symmetry relation in order to express them as linear combinations of elements $L_3\{x\}$.

Proposition 4.12. For a configuration $(\ell_0, \dots ,\ell_5)$ as in fig. 4.9 we have in the group $\mathcal{B}_3(F)$

$$(\ell_0, \dots ,\ell_5) = -\frac{1}{3} \sum_{0\le i\,,\,j\le 2} (-1)^{i+j} L_3(1+2\chi)\circ$$

$$\{r(\ell_{2+j}\,|\,\ell_0, \dots ,\hat{\ell}_i, \dots ,\ell_2,\ell_3, \dots ,\ell_{2+j}, \dots ,\ell_5)\} + \frac{1}{3}\eta_3 \tag{4.20}$$

where, by definition

$$\chi\circ(\ell_1,\ell_2,\ell_3,\ell_4) := (\ell_1,\ell_3,\ell_2,\ell_4)\,. \tag{4.21}$$

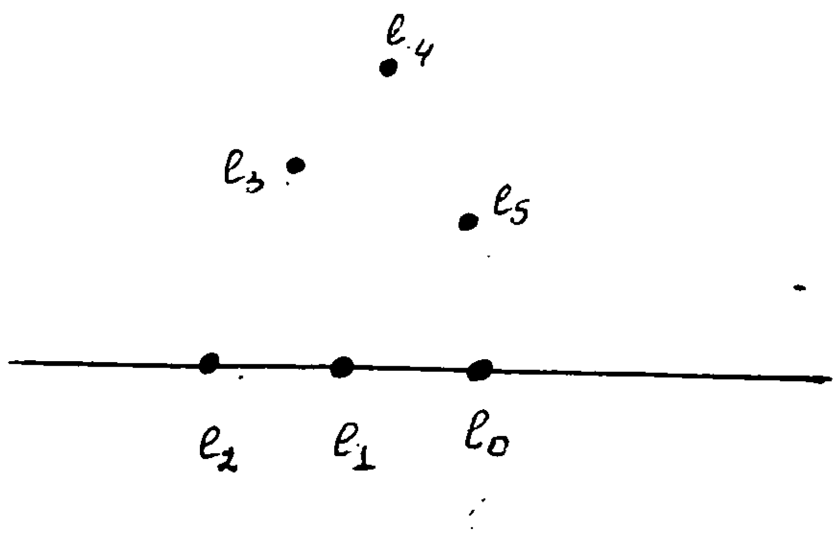


fig. 4.9

Proof. Apply the 7–term relation to a configuration as in fig. 4.10 and use the relations R1) and R3). ∎

Proposition 4.13. A configuration of type 3 as in fig. 4.2 defines the zero element in $\tilde{\mathcal{G}}_3(F)$.

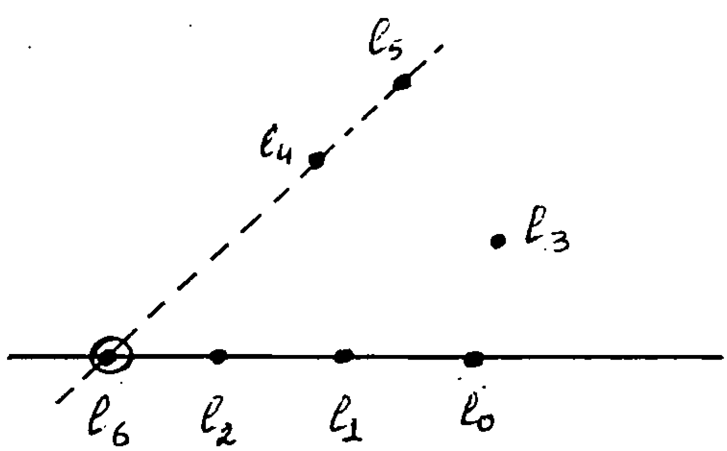


fig. 4.10

Proof. Consider the 7–term relation for a configuration as in fig. 4.11 and use the relation R1).

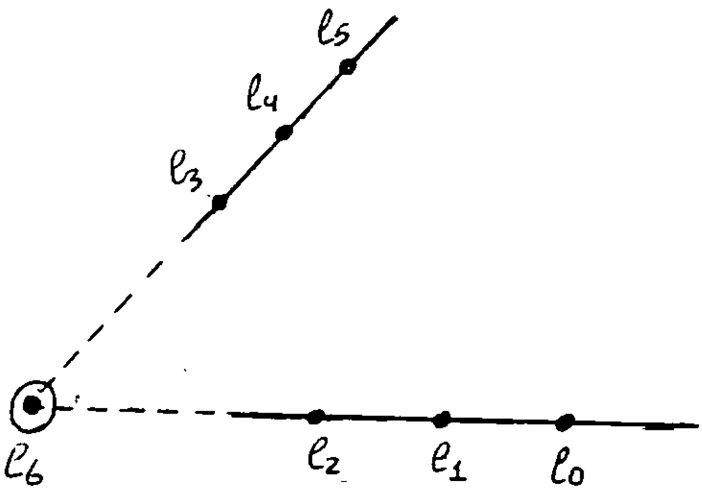


fig. 4.11

Theorem 4.14. The homomorphism

$$L_3 : \mathbb{Z}[P^1_F \backslash 0,1,\infty] \longrightarrow \mathcal{B}_3(F)$$

$$\{x\} \longmapsto C_3(x)$$

is surjective.

Proof. According to the definition of the homomorphism L_3, Lemma 4.9, Relation R3) and Propositions 4.12, 4.14, elements of the group $\tilde{\mathcal{B}}_3(F)$ corresponding to all degenerate configurations (presented in fig. 4.2) lie in $\mathrm{Im}\ L_3(\mathbb{Z}[P^1_F \backslash 0,1,\infty])$.

The 7–term relation for a configuration as in fig. 4.12 shows that the same is valid for the generic configurations. ■

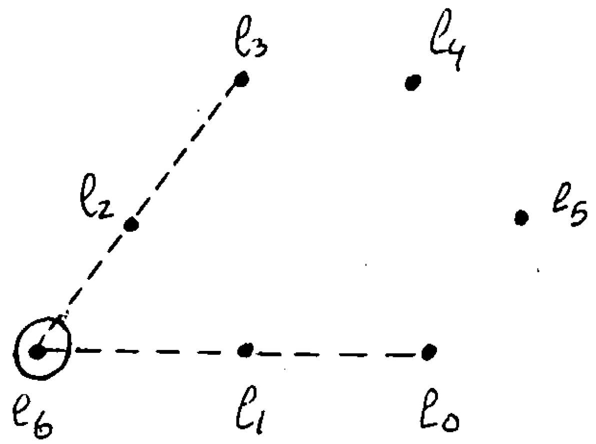


fig. 4.12

4. It follows from Theorem 4.2, Lemma 4.7 and Lemma 4.10 that the Relation R3 lies in a kernel of the above defined homomorphism $\delta : \tilde{\mathcal{B}}_3(F) \longrightarrow (B_2(F) \otimes F^*) \otimes \mathbb{Z}[\frac{1}{6}]$. So we get the homomorphism

$$\delta : \mathcal{B}_3(F) \longrightarrow (B_2(F) \otimes F^*) \otimes \mathbb{Z}[\tfrac{1}{6}] .$$

Moreover, there is the following commutative diagram

$$\begin{array}{ccccc}
\mathbb{Z}[P^1_F\backslash 0,1,\infty] \Big/ \begin{smallmatrix}\{x\} = \{x^{-1}\} \\ \zeta_3(x) = \zeta_3(y)\end{smallmatrix} & \xrightarrow{\ \delta\ } & (B_2(F)\otimes F^*)_{\mathbb{Q}} & \xrightarrow{\ \delta\ } & (\Lambda^3 F^*)_{\mathbb{Q}} \\
\big\downarrow L_3 & & \| & & \| \\
\mathcal{B}_3(F) & \xrightarrow{\ \delta\ } & (B_2(F)\otimes F^*)_{\mathbb{Q}} & \xrightarrow{\ \delta\ } & (\Lambda^3 F^*)_{\mathbb{Q}}
\end{array} .$$

<u>Corollary</u> 4.15. $\operatorname{Im}\delta(\mathcal{B}_3(F))$ lies in a subgroup generated by the expressions $[x]\otimes x$. ■

§ 5. Functional equations for the trilogarithm

1. Computations. Let $(\ell_0, \dots, \ell_5, z)$ be a configuration of 7 points in P^2 represented in fig. 5.1.

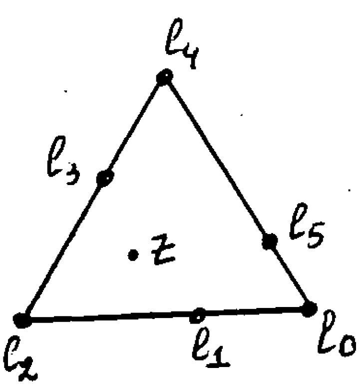


fig. 5.1

In this § we will denote for brevity the element $\{r(x_0 | x_1, x_2, x_3, x_4)\}$ as $\{x_0 | x_1, x_2, x_3, x_4\}$, omitting r. Consider the following element $R_3(\ell_0, \dots, \ell_5, z)$ of the group $\mathbb{Z}[P^1 \backslash 0,1,\infty]$:

$$R_3(\ell_0, \dots, \ell_5, z) := \sum_{0 \le i \le j \le 5} (-1)^{i+j} \chi \circ \{z | \ell_0, \dots, \hat{\ell}_i, \dots, \hat{\ell}_j, \dots, \ell_5\} -$$

$$- \sum_{i=0}^{5} \{z | \ell_i, \ell_{i+1}, \ell_{i+2}, \ell_{i+3}\} + \{\ell_5 | \ell_0, \ell_2, \ell_1, \ell_3\} +$$

$$+ \{\ell_1 | \ell_3, \ell_5, \ell_0, z\} + \{\ell_3 | \ell_5, \ell_1, \ell_2, z\} + \{\ell_5 | \ell_1, \ell_3, \ell_4, z\} - \tag{5.1}$$

$$- \{\ell_1 | \ell_3, \ell_5, \ell_4, z\} - \{\ell_3 | \ell_5, \ell_1, \ell_0, z\} - \{\ell_5 | \ell_1, \ell_3, \ell_2, z\}$$

where by definition $\chi \circ \{x_0,x_1,x_2,x_3\} := \{x_0,x_2,x_1,x_3\}$. We consider all indices modulo 6.

Theorem 5.1.

a) $L_3(R_3(\ell_0, \dots ,\ell_5,z)) + 3\eta_3 = 0$

b) $\delta \circ R(\ell_0, \dots ,\ell_5,z) = 0$

It is clear that b) follows immediately from a), Lemma 4.10 and the commutativity of diagram 4.17.

Proof. a) We will demonstrate that using equality (4.16) we can identify $L_3(R(\ell_0, \dots ,\ell_5,z)) + 3\eta_3$ with the 7–term relation for the configuration $(\ell_0, \dots ,\ell_5,z)$ multiplied by (-1) .

Taking into account the skew symmetry of the elements $(x_0, \dots ,x_6) \in \mathscr{G}_3(F)$ with respect to the permutations of the points x_i , we can rewrite the 7–term relation in the following form:

$$(\ell_0,\ell_1,\ell_2,\ell_3,\ell_4,z) + (\ell_2,\ell_3,\ell_4,\ell_5,\ell_0,z) + (\ell_4,\ell_5,\ell_0,\ell_1,\ell_2,z)$$

$$- (\ell_0,\ell_1,\ell_2,\ell_3,\ell_5,z) - (\ell_2,\ell_3,\ell_4,\ell_5,\ell_1,z) - (\ell_4,\ell_5,\ell_0,\ell_1,\ell_3,z) \qquad (5.2)$$

$$+ (\ell_0,\ell_2,\ell_4,\ell_1,\ell_3,\ell_5) = 0 .$$

Note that the terms in the first line of (5.2) have type 3 and in the second one type 5 in the list of all possible combinatorial types of configurations in fig. 4.2.

All lines in (5.2) are invariant under a cyclic transformation

$$\tau : \ell_i \longmapsto \ell_{i+2} \quad (\tau^3 = \mathrm{id}\ ;\ \tau \circ z = z)\ .$$

Roughly speaking τ is a "rotation" of the picture 5.1 on $2\pi/3$ in a direction given by an orientation of the triangle $(\ell_0 \ell_2 \ell_4)$.

Let us set

$$\tau\{\ell_{i_1} \big| \ell_{i_2}, \ell_{i_3}, \dots\} = \{\ell_{i_1+2} | \ell_{i_2+2}, \ell_{i_3+2}, \dots\}$$

and so on. Then (5.2) can be written as

$$(1 + \tau + \tau^2) \circ [(\ell_0,\ell_1,\ell_2,\ell_3,\ell_4,z) - (\ell_0,\ell_1,\ell_2,\ell_3,\ell_5,z)] + \qquad (5.3)$$

$$+ (\ell_0,\ell_2,\ell_4,\ell_1,\ell_3,\ell_5) = 0\ .$$

Applying Proposition 4.12 and Relation R3) from the Definition 4.11 to this formula we obtain:

$$\tfrac{1}{3}(1 + \tau + \tau^2) \circ (1 + 2\chi) \circ L_3 \left[\sum_{i=0}^{4} (-1)^{i-1} \{z | \ell_0, \dots, \widehat{\ell}_i, \dots, \ell_4\} + \right. \qquad (5.4a)$$

$$+ \{z | \ell_1,\ell_2,\ell_3,\ell_5\} - \{z | \ell_0,\ell_2,\ell_3,\ell_5\} + \{z | \ell_0,\ell_1,\ell_3,\ell_5\} \qquad (5.4b)$$

$$- \{\ell_5 | \ell_1,\ell_2,\ell_3,z\} + \{\ell_5 | \ell_0,\ell_2,\ell_3,z\} - \{\ell_5 | \ell_0,\ell_1,\ell_3,z\} \tag{5.4c}$$

$$\left. + \{\ell_3 | \ell_1,\ell_2,\ell_5,z\} - \{\ell_3 | \ell_0,\ell_2,\ell_5,z\} + \{\ell_3 | \ell_0,\ell_1,\ell_5,z\}\right]$$

$$+ \{\ell_5 | \ell_0,\ell_2,\ell_1,\ell_3\} . \tag{5.4d}$$

Note that there is no term η_3 in this formula. It is clear that

$$(1 + \tau + \tau^2) \circ (1 + 2\chi) \circ \left[\{\ell_3 | \ell_0,\ell_1,\ell_5,z\} - \{\ell_5 | \ell_2,\ell_3,\ell_1,z\}\right] = 0 .$$

Therefore (see the first and the last term in 5.4c):

$$\tfrac{1}{3}(1 + \tau + \tau^2) \circ (1 + 2\chi) \circ L_3\left[- \{\ell_5 | \ell_1,\ell_2,\ell_3,z\} + \{\ell_3 | \ell_0,\ell_1,\ell_5,z\}\right] =$$

$$= \tfrac{1}{3}(1 + \tau + \tau^2) \circ L_3\Big[- \{\ell_5 | \ell_1,\ell_2,\ell_3,z\} - 2\{\ell_5 | \ell_1,\ell_3,\ell_2,z\} + \tag{5.5}$$

$$\left. + \{\ell_5 | \ell_2,\ell_3,\ell_1,z\} + 2\{\ell_5 | \ell_2,\ell_1,\ell_3,z\}\right] .$$

Using the equalities

$$L_3\{\ell_5 | \ell_1,\ell_2,\ell_3,z\} = L_3\{\ell_5 | \ell_2,\ell_1,\ell_3,z\}$$

and

$$L_3\left[\{\ell_5 | \ell_1,\ell_3,\ell_2,z\} + \{\ell_5 | \ell_2,\ell_3,\ell_1,z\} + \{\ell_5 | \ell_2,\ell_1,\ell_3,z\}\right] = \eta_3$$

we obtain that (5.5) is equal to

$$-(1+\tau+\tau^2)\circ L_3\{\ell_5\,|\,\ell_1,\ell_3,\ell_2,z\} + \eta_3\,. \tag{5.6}$$

Analogously the contribution of the third and fourth term in (5.4c) is equal to

$$\tfrac{1}{3}(1+\tau+\tau^2)\circ(1+2\chi)\circ L_3\Big[-\{\ell_5\,|\,\ell_0,\ell_1,\ell_3,z\} + \{\ell_5\,|\,\ell_3,\ell_4,\ell_1,z\}\Big]\,. \tag{5.7}$$

Note that $(\ell_5\,|\,\ell_3,\ell_0,\ell_1,z) \equiv (\ell_5\,|\,\ell_3,\ell_4,\ell_1,z)$.

So (5.7) is equal to

$$\begin{aligned}&\tfrac{1}{3}(1+\tau+\tau^2)\circ L_3[-\ell_5\,|\,\ell_4,\ell_1,\ell_3,z\} - 2\{\ell_5\,|\,\ell_4,\ell_3,\ell_1,z\} + \\ &+ \{\ell_5\,|\,\ell_3,\ell_4,\ell_1,z\} + 2\{\ell_5\,|\,\ell_3,\ell_1,\ell_4,z\}] = \\ &= (1+\tau+\tau^2)\circ L_3\{\ell_5\,|\,\ell_3,\ell_1,\ell_4,z\} - \eta_3\,.\end{aligned} \tag{5.8}$$

the sum of (5.6) and (5.8) coincides with the last two lines in (5.1).

Further note that (see pic. 5.1)

$$\begin{aligned}&(\ell_5\,|\,\ell_0,\ell_2,\ell_3,z) \equiv (\ell_5\,|\,\ell_4,\ell_2,\ell_3,z) \equiv \\ &\equiv (z\,|\,\ell_4,\ell_2,\ell_3,\ell_5) \equiv \tau\cdot(z\,|\,\ell_2,\ell_0,\ell_1,\ell_3)\,.\end{aligned}$$

So the sum of the second term in (5.4c) and the fifth in (5.4a) is equal to

$$\tfrac{1}{3}(1+\tau+\tau^2)\circ L_3\Big[\{z\,|\,\ell_2,\ell_0,\ell_1,\ell_3\} + 2\{z\,|\,\ell_2,\ell_1,\ell_0,\ell_3\} - \{z\,|\,\ell_0,\ell_1,\ell_2,\ell_3\} -$$

$$-2\{z\,|\,\ell_0\ell_2\ell_1\ell_3\}\Big] = (1+\tau+\tau^2)\circ L_3\{z\,|\,\ell_2,\ell_1,\ell_0,\ell_3\} - \eta_3 = \tag{5.9}$$

$$= -(1+\tau+\tau^2)\circ L_3\Big[\{z\,|\,\ell_0,\ell_2,\ell_1,\ell_3\} + \{z\,|\,\ell_0,\ell_1,\ell_2,\ell_3\}\Big] - 2\eta_3\,.$$

Similar computations for the fifth term in (5.4c) give the following result:

$$-(1+\tau+\tau^2)\circ L_3\Big[\{z\,|\,\ell_1,\ell_3,\ell_2,\ell_4\} + \{z\,|\,\ell_1,\ell_2,\ell_3,\ell_4\}\Big]\,. \tag{5.10}$$

Note that the sum of (5.9) and (5.10) is equal to

$$-(1+\chi)\circ L_3\sum_{i=0}^{5}\{z\,|\,\ell_i,\ell_{i+1},\ell_{i+2},\ell_{i+3}\}\,. \tag{5.11}$$

Now let us consider the first and third term in (5.4b). Accounting that $(z\,|\,\ell_1,\ell_2,\ell_3,\ell_5) \equiv \tau\circ(z\,|\,\ell_5,\ell_0,\ell_1,\ell_3)$, we have:

$$\tfrac{1}{3}(1+\tau+\tau^2)\circ L_3\Big[\{z\,|\,\ell_5,\ell_0,\ell_1,\ell_3\} + 2\{z\,|\,\ell_5,\ell_1,\ell_0,\ell_3\} +$$

$$\{z\,|\,\ell_0,\ell_1,\ell_3,\ell_5\} + 2\{z\,|\,\ell_0,\ell_3,\ell_1,\ell_5\}\Big] = (1+\tau+\tau^2)\circ L_3\{z\,|\,\ell_0,\ell_3,\ell_1,\ell_2\} + \eta_3\,. \tag{5.12}$$

Similar computations can be produced with the second term in (5.4b) and the third in (5.4a), and also with the second and fourth term in (5.4a).

We obtain the following results:

$$(1+\tau+\tau^2)\circ L_3\{z\,|\,\ell_0,\ell_3,\ell_1,\ell_4\}+\eta_3 \tag{5.13}$$

and

$$(1+\tau+\tau^2)\circ L_3\{z\,|\,\ell_0,\ell_3,\ell_2,\ell_4\}-\eta_3\,. \tag{5.14}$$

Finally,

$$(\ell_0,\ell_2,\ell_4,\ell_1,\ell_3,\ell_5\}=L_3\{\ell_5\,|\,\ell_0,\ell_2,\ell_1,\ell_3\}\,. \tag{5.15}$$

Adding (5.6), (5.8), (5.11) – (5.15), we obtain the right–hand side of (5.1).

Many alternative forms of (5.1) can be obtained using the relation (4.16). I give an expression with the minimal possible number of terms.

Let $(x_1,x_2,x_3,y_1,y_2,y_3,z)$ be a configuration of 7 points as in fig. 5.2. Put

$$R_3(x_i,y_i,z):=(1+\tau+\tau^2)\circ\Big[\{y_1\,|\,y_2,y_3,x_3,z\}+\{y_1\,|\,y_2,y_3,x_1,z\}+$$

$$+\{z\,|\,x_1,y_3,x_2,y_2\}+\{z\,|\,y_3,y_1,x_2,y_2\}+\{y_1,x_3,x_2,y_2\}+\{x_3,x_1,x_2,y_2\} \tag{5.16}$$

$$-\{z\,|\,x_1,y_1,x_2,y_2\})\Big]+\{y_1\,|\,y_2,y_3,x_1,x_3\}$$

where $\tau:x_i\longmapsto x_{i+1}\,,\ y_i\longmapsto y_{i+1}\,.$

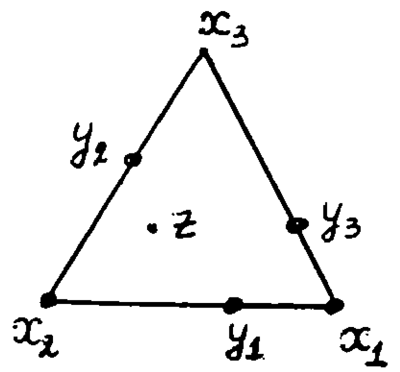


fig. 5.2

Lemma 5.2. If the configurations $(x_1,x_2,x_3,y_1,y_2,y_3,z)$ and $(\ell_0,\ell_2,\ell_4,\ell_1,\ell_3,\ell_5,z)$ coincide then

$$L_3R_3(x_i,y_i,z) = L_3R_3(\ell_0, \dots ,\ell_5;z) + 6\eta_3 .$$

Proof. Transform the third and fifth terms in (5.16) using (4.16). ■

Therefore in the group $\mathcal{B}_3(F)$

$$L_3R_3(x_i,y_i,z) - 3\eta_3 = 0 .$$

The remarkable feature of the formulae 5.1 and (5.16) is that all their terms have coefficients ± 1 .

Every term in (5.16) is obtained by the projection of 4 points from (x_i,y_i,z) with the center in a fifth. Consider all possible configurations of 4 different points of P^1

obtained in this way. Let us say that the two configurations are equivalent, if they differ only by a permutation of points. It turns out that every such equivalence class is represented by just one term in (5.16).

Let us emphasize that some configurations can be written in different forms, for example: $(x_1\,|\,y_3,y_2,y_1,z) = (z\,|\,x_2,x_3,y_1,x_1)$; $(y_3\,|\,x_1y_1x_3z) = (z\,|\,x_2y_1x_3y_3)$ and so on.

2. The main theorem. Let us recall that $\zeta_3(x) := \{x\} + \{1-x\} + \{1-x^{-1}\}$. Set

$$B_3(F) := \mathbb{Z}[P^1\backslash 0,1,\infty] \Big/ \begin{array}{l} \{x\} = \{x^{-1}\} \\ \zeta_3(x) = \zeta_3(y) \\ R_3(\ell_i;z) + 3\zeta_3(x) = 0 \end{array} \qquad (5.17)$$

where $(\ell_0, \dots ,\ell_5,z)$ is any configuration of 7 distinct points in P^2 such that (see fig. 5.1): a) the point ℓ_{2i+1} lies on the side L_{2i} of the nondegenerate triangle $(\ell_0\ell_2\ell_4)$, $0 \le i \le 2$ $(\ell_{2i} \notin L_{2i})$; b) z is in generic position with respect to $\ell_0, \dots ,\ell_5$, c) ℓ_1 , ℓ_3 and ℓ_5 do not lie on the same line.

Note that the main relation in (5.17) can be rewritten in such a way that the relations $\{x\} = \{x^{-1}\}$ and $\zeta_3(x) = \zeta_3(y)$ can be deduced from it. But we do not need this result.

Let us denote by ζ_3 the image of the element $\zeta_3(x)$ in $B_3(F)$.

Now we begin with the construction of the homomorphism

$$M_3 : \mathcal{G}_3(F) \longrightarrow B_3(F)$$

which is inverse to the epimorphism L_3.

Let us define the homomorphism M_3 on the generators of the group $\mathcal{G}_3(F)$.

a) Set $M_3(\eta_3) = \zeta_3$.

b) Put

$$M_3(\ell_0, \dots, \ell_5) := \{\ell_5 \mid \ell_0, \ell_1, \ell_3, \ell_4\}$$

for a configuration $(\ell_0, \dots, \ell_5)$ as in fig. 5.3a).

c) Put

$$M_3(\ell_0, \dots, \ell_5) := -\tfrac{1}{3} \sum_{i=0}^{4} (-1)^i (1 + 2\chi) \circ \{\ell_5 \mid \ell_0, \dots, \hat{\ell}_i, \dots, \ell_4\} + \tfrac{1}{3} \zeta_3$$

for a configuration $(\ell_0, \dots, \ell_5)$ as in fig. 5.3b).

d) Put

$$M_3(\ell_0, \dots, \ell_5) := -\tfrac{1}{3} \sum_{0 \le i, j \le 2} (-1)^{i+j} (1 + 2\chi) \circ$$

$$\{\ell_0, \dots, \hat{\ell}_i, \dots, \ell_2, \ell_3, \dots, \hat{\ell}_{2+j}, \dots, \ell_5\} + \tfrac{1}{3} \zeta_3$$

for a configuration $(\ell_0, \dots, \ell_5)$ as in fig. 5.3c).

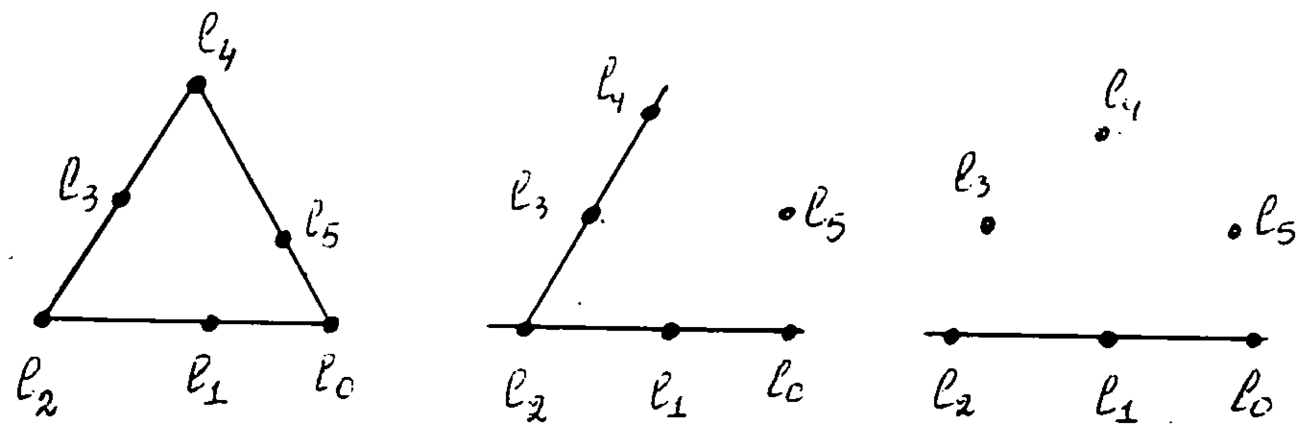


fig. 5.3 a), b), c)

The skew–symmetry property in the group $\mathcal{B}_3(F)$ provided the definition of the homomorphism M_3 on the configurations that differ from the ones considered above ones by some permutation of points. Lemma 4.8 and the considerations after the Definition 4.11 proved the correctness of this definition.

e) Now let $(\ell_0, \dots, \ell_5)$ be a configuration of the generic position. Denote by a the intersection point of the lines $\overline{\ell_0\ell_1}$ and $\overline{\ell_4\ell_5}$. Set

$$M_3(\ell_0, \dots, \ell_5) := \sum_{i=0}^{5} (-1)^i M_3(\ell_0, \dots, \hat{\ell}_i, \dots, \ell_5, a) .$$

All terms on the right–hand side were defined above.

Let $\{0, \dots ,5\} = \{i_1,i_2,i_3\} \cup \{j_1,j_2,j_3\}$. Set $x := \overline{\ell_{i_1}\ell_{i_2}} \cap \overline{\ell_{j_1}\ell_{j_2}}$ and

$$M_3^{(x)}(\ell_0, \dots ,\ell_5) := \sum_{i=0}^{5} (-1)^i(\ell_0, \dots ,\hat{\ell}_i, \dots ,\ell_5,x) .$$

<u>Proposition</u> 5.3. $(M_3^{(x)} - M_3)(\ell_0, \dots ,\ell_5) = 0$.

<u>Proof</u>. Set $y = \overline{\ell_{i_1}\ell_{i_2}} \cap \overline{\ell_{j_1}\ell_{j_3}}$ and

$$M_3^{(y)}(\ell_0, \dots ,\ell_5) := \sum_{i=0}^{5} (-1)^i(\ell_0, \dots ,\hat{\ell}_j, \dots ,\ell_5,y) .$$

<u>Lemma</u> 5.4. $(M_3^{(x)} - M_3^{(y)})(\ell_0, \dots ,\ell_5) = 0$.

<u>Proof</u>.

$$(M_3^{(x)} - M_3^{(y)})(\ell_0, \dots ,\ell_5) =$$

$$\sum_{i=0}^{5} (-1)^i M_3\Big[(\ell_0, \dots ,\hat{\ell}_i, \dots ,\ell_5 x) - (\ell_0, \dots ,\hat{\ell}_i, \dots ,\ell_5,x)\Big] .$$

In order to investigate the right–hand side of this equality we need the following lemma.

<u>Lemma</u> 5.5. Let $(x_0, \dots ,x_6)$ be 7 points in P^2 such that there are 4 points on a line among them. Then

$$\sum_{i=0}^{6} (-1)^i M_3(x_0, \dots, \hat{x}_i, \dots, x_6) = 0 . \tag{5.18}$$

Note that all configurations $(x_0, \dots, \hat{x}_i, \dots, x_6)$ are non–generic, so all terms in (5.18) were defined above.

The proof of this important lemma will be given below.

If $i \notin \{i_1, i_2\}$, then $\ell_{i_1}, \ell_{i_2}, x, y$ are four points among $\ell_0, \dots, \hat{\ell}_i, \dots, \ell_5, x, y$, belonging to the same line. So applying Lemma 5.5 we get

$$M_3\Big[(\ell_0, \dots, \hat{\ell}_i, \dots, \ell_5, x) - (\ell_0, \dots, \hat{\ell}_i, \dots, \ell_5, y)\Big] = \tag{5.19}$$

$$= \sum_{\substack{j=0 \\ j \neq i}}^{5} \pm M_3(\ell_0, \dots, \hat{\ell}_i, \dots, \hat{\ell}_j, \dots, \ell_5, x, y) .$$

The same result is true when $i \in \{i_1, i_2\}$. Indeed, the configuration $(\ell_0, \dots, \hat{\ell}_i, \dots, \ell_5, x, y)$ has the same combinatorial type as a configuration from fig. 5.1 (see fig. 5.4). So the image of the corresponding 7–term relation by the homomorphism M_3 is a relation in the group $B_3(F)$.

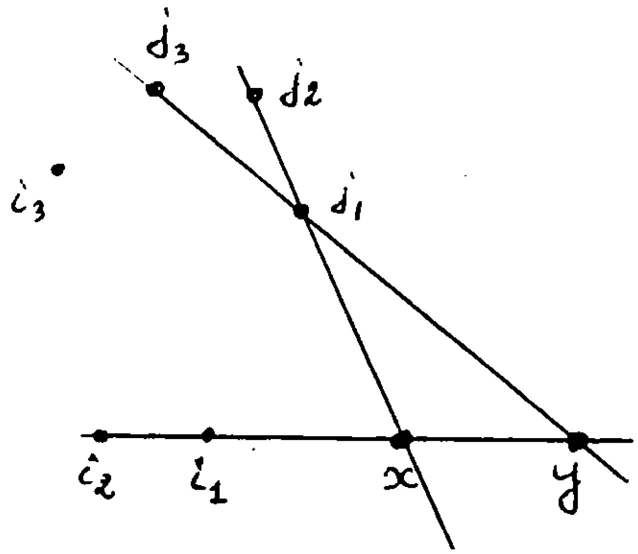


fig. 5.4

It is easy to see that the sum of the right hand side of (5.19) from $i \geq 0$ to $i \geq 5$ is zero. Lemma 5.4 is proved. ∎

Similarly the homomorphism defined with the help of the pairs (i_1,i_3) & (j_1,j_2) also coincides with $M_3^{(x)}$. Changing several times an index in one of this pairs, we can transfer the pairs (i_1,i_2) & (j_1,j_2) to (01) & (45), and as a result prove the proposition 5.3. ∎

<u>Proof of Lemma 5.5</u>. All combinatorial types of configurations of 7 distinct points in P^2 containing 4 points on the same line are shown in fig. 5.5.

The equality (5.18) for configurations of type 1 in fig. 5.4 follows from the definition d) of the homomorphism M_3.

For configurations of type 2 and 3 all terms in (5.18) are zero according to Lemma 4.13.

Equality (5.18) for the configurations of type 4 and 6 is an easy consequence of the definitions (see Proposition 4.12).

Finally, let $(\ell_0, \dots, \ell_6)$ be a configuration of type 5, as shown in fig. 5.6.

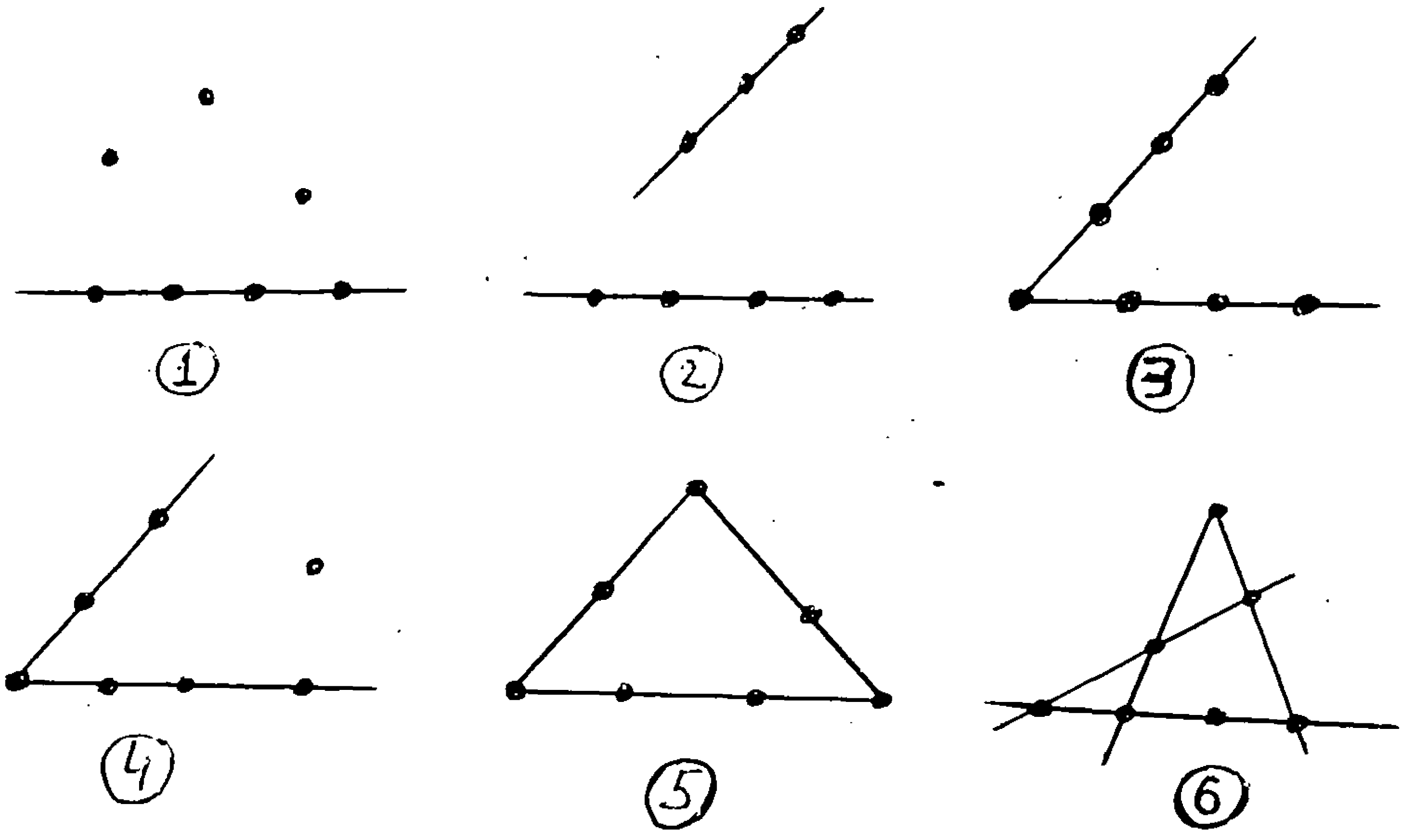


fig. 5.5

Then

$$M_3(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5,\ell_6) = -\tfrac{1}{3}(1 + 2\chi) \circ \Big[\{\ell_6 \,|\, \ell_2,\ell_3,\ell_4,\ell_0\} - \{\ell_6 \,|\, \ell_1,\ell_3,\ell_4,\ell_0\} +$$

$$+ \{\ell_6 | \ell_1, \ell_2, \ell_4, \ell_0\} - \{\ell_6 | \ell_1, \ell_2, \ell_3, \ell_0\} + \{\ell_0 | \ell_1, \ell_2, \ell_3, \ell_4\} +$$

$$\frac{1}{3}\eta_3 - M_3(\ell_0, \ell_1, \ell_2, \ell_4, \ell_5, \ell_6) = -\frac{1}{3}(1 + 2\chi) \circ \Big[- \{\ell_4 | \ell_6, \ell_0, \ell_1, \ell_2\} +$$

$$\{\ell_4 | \ell_3, \ell_0, \ell_1, \ell_2\} - \{\ell_4 | \ell_3, \ell_6, \ell_1, \ell_2\} + \{\ell_4 | \ell_3, \ell_6, \ell_0, \ell_2\}$$

$$- \{\ell_4 | \ell_3, \ell_6, \ell_0, \ell_1\} - \frac{1}{3}\eta_3 .$$

Let us add the right–hand sides of these formulae and group the i–th term of the first one with the (i + 3)–rd term of the second (indexes modulo 5). Further, remark that $(\ell_6 | \ell_{i_1}, \ell_{i_2}, \ell_{i_3}, \ell_4) \equiv (\ell_4 | \ell_{i_1}, \ell_{i_2}, \ell_{i_3}, \ell_6)$ if $0 \le i_1, i_2, i_3 \le 3$, $(\ell_4 | \ell_5, \dots) \equiv (\ell_4 | \ell_3, \dots)$ and $(\ell_6 | \ell_5, \dots) \equiv (\ell_6 | \ell_0, \dots)$. Then an easy computation shows that we obtain

$$\{\ell_6 | \ell_1, \ell_4, \ell_3, \ell_0\} - \{\ell_0 | \ell_3, \ell_0, \ell_4, \ell_2\} =$$

$$- M_3 [- (\ell_0, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6) + (\ell_0, \ell_1, \ell_3, \ell_4, \ell_5, \ell_6)] .$$

According to the relation R1 in the group $\mathcal{G}_3(F)$, the other terms in (5.18) are zero for the configuration from fig. 5.6. ■

So we define the homomorphism M_3 on generators of the group $\mathcal{G}_3(F)$. It follows immediately from the definitions that M_3 transfers the Relations R1 and R3 in the group $\mathcal{G}_3(F)$ to zero. We proved that the Relation R2 for 7 points in P^2 is also mapped to zero, if among these points there are 2 coinciding or 4 lying on the same line. The other combinatorial types of the configurations of 7 points in P^2 are shown in fig.

5.7, where all lines containing more than 2 points of the configuration are also distinguished.

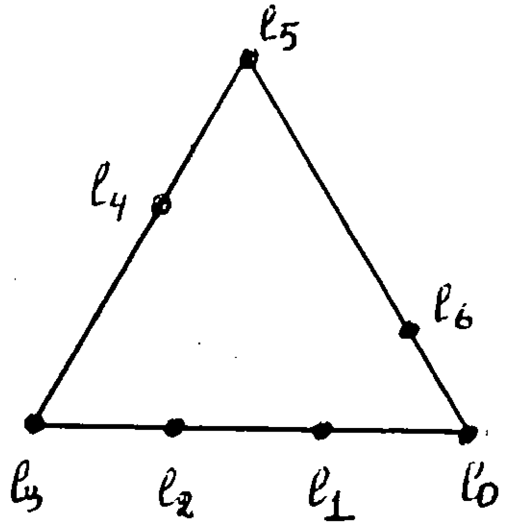


fig. 5.6

Proposition 5.6. The homomorphism M_3 annihilates the 7–term relations for all configurations represented in fig. 5.7.

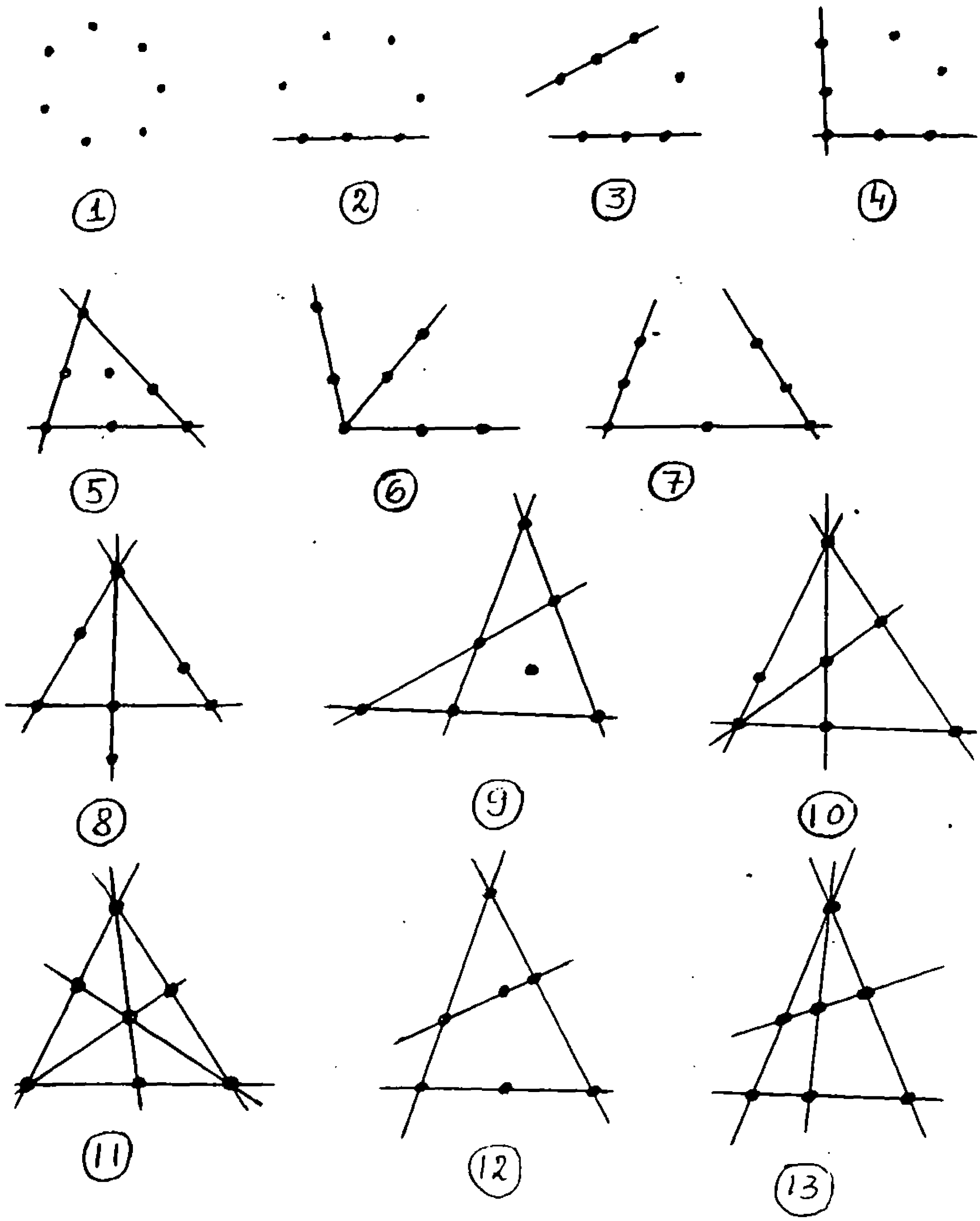


fig. 5.7

We need the following very simple lemma.

Lemma 5.7. Let $\ell_0, \dots, \ell_7$ be 8 points in P^2 and let it be known that the homomorphism M_3 annihilates Relation R2 for the 7 points, obtained by removing the point ℓ_i, where $i = 0, \dots, 6$. Then the same is true for $i = 7$. ■

Proof of Proposition 5.6. We will refer to the points of our configuration as "distinguished points". There are two distinguished lines in the configurations 3, 7, 12 and 13 in fig. 5.7 such that its intersection point is not distinguished. Let us add this point. Then after removing any other distinguished point there are 4 distinguished points lying on the same line. It remains to apply Lemmas 5.5. and 5.7.

Proposition 5.6 for the configurations of type 4 in fig. 5.7 follows from the definition of the homomorphism M_3 for the configuration of 6 points in generic position, obtained by removing the intersection point of two distinguished lines.

We will write Ⓜ ⇒ Ⓝ if the validity of Proposition 5.6 for the configurations of type m in fig. 5.7 implies the one for a configuration of type n.

Applying Lemmas 5.7 and 5.5 to the configurations in fig. 5.8 a) and 5.8 b) we obtain that ④ ⇒ ② and ④ & ② ⇒ ⑦.

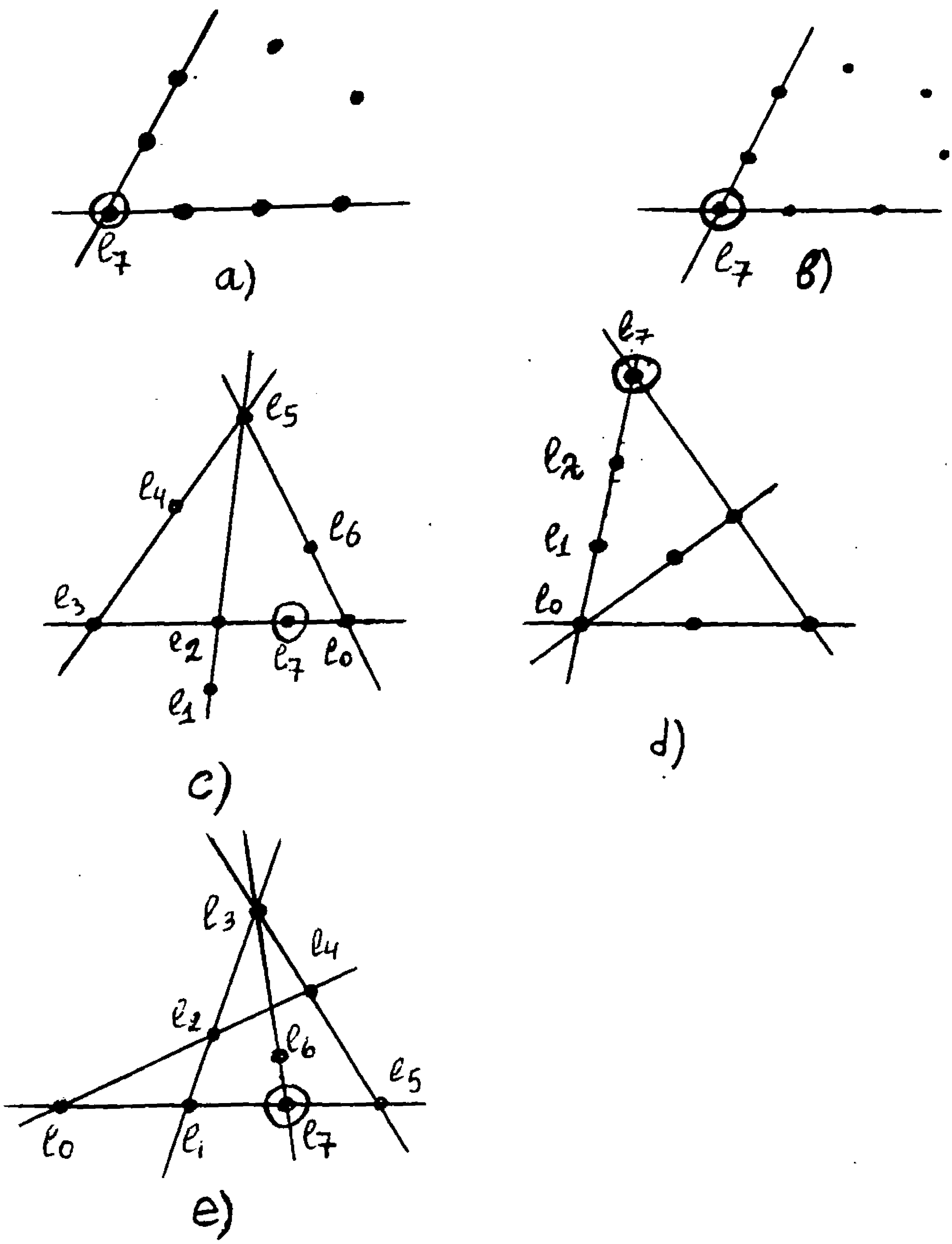


fig. 5.8

Theorem 5.1 claims that the 7–term relations for a configuration of type 5 in fig. 5.7 transfer to a (basic) relation in the group $B_3(F)$.

Applying the Lemmas 5.7 and 5.5 to a configuration of 8 points in fig. 5.8 c) we get that ④ & ⑤ ⇒ ⑧ , because after removing the points ℓ_0 and ℓ_3 we obtain a configuration of type 4, and after removing the point ℓ_2 one of type 5.

Similar considerations for configurations in fig. 5.8 d) show that ② & ⑧ ⇒ ⑥ (after removing the points ℓ_1 and ℓ_2 we get a configuration of type 8, and after removing ℓ_0 one of type 2) .

Removing points ℓ_1 or ℓ_5 from a configuration in fig. 5.8 c) we obtain a configuration of type 12, and removing ℓ_0 one of type 8 . Hence ⑫ & ⑧ ⇒ ⑨ .

Finally, let us add a generic point to a configuration of type 10 or 11. Then removing any other point of the obtained configuration we can get neither a configuration of type 10, nor 11, because every point of these configurations lies on some distinguished line.

So all possible cases were considered and hence we have proved Proposition 5.6. ■

Theorem A. For an arbitrary field F containing sufficiently many elements:

a) The groups $\mathcal{B}_3(F)$ and $B_3(F)$ are canonically isomorphic.

b) There is a canonical isomorphism of the complexes

$$\begin{array}{ccccc} \mathcal{B}_3(F)_{\mathbb{Q}} & \xrightarrow{\delta} & (B_2(F) \otimes F^*)_{\mathbb{Q}} & \xrightarrow{\delta} & (\Lambda^3 F^*)_{\mathbb{Q}} \\ \wr\big\downarrow M_3 & & \| & & \| \\ B_3(F)_{\mathbb{Q}} & \xrightarrow{\delta} & (B_2(F) \otimes F^*)_{\mathbb{Q}} & \xrightarrow{\delta} & (\Lambda^3 F^*)_{\mathbb{Q}} \end{array} \qquad (5.20)$$

Remark. Diagram (5.20) exists and is commutative even if we consider all groups only modulo 6–torsion instead of $\otimes \mathbb{Q}$.

3. The homomorphism M_3 and the specialisation. Let $\tilde{B}_3(F)$ be the quotient of $\mathbb{Z}[P^1_F]$ by the subgroup generated by the following elements

$$\{0\}\ ,\ \{\infty\}\ ,\ \{1\} - \zeta_3(x)\ ,\ \{x\} - \{x^{-1}\}\ ,\ R_3(\ell_0, \dots ,\ell_5,z)$$

where $(\ell_0, \dots ,\ell_5,z)$ is a configuration as in fig. 5.1.

There is a canonical isomorphism

$$f : \tilde{B}_3(F) \xrightarrow{\sim} B_3(F)$$

$$f : \{0\}\ ,\ \{\infty\} \longmapsto 0\ ;\ \ f : \{x\} \longmapsto \{x\}\ ,\ \ x \in P^1_F \backslash \{0,1,\infty\}\ .$$

Note that if just 2 among 4 points $\ell_0, \dots ,\ell_3$ on a line coincide, then

$$r(\ell_0, \dots, \ell_3) = \begin{cases} 0 & \text{if } \ell_0 = \ell_3 \text{ or } \ell_1 = \ell_2 \\ 1 & \text{if } \ell_0 = \ell_1 \text{ or } \ell_2 = \ell_3 \\ \infty & \text{if } \ell_0 = \ell_2 \text{ or } \ell_1 = \ell_3 \ . \end{cases} \tag{5.21}$$

Let us define the homomorphism $\tilde{M}_3 : \mathcal{G}_3(F) \longrightarrow \tilde{B}_3(F)$ as follows.

First of all, it is zero on semistable and unstable configurations (i.e. configurations satisfying condition R1).

Now let $(\ell_0, \dots, \ell_5)$ be a stable configuration such that ℓ_0 , ℓ_1 , ℓ_2 are on a line. Set

$$\tilde{M}_3(\ell_0, \dots, \ell_5) := -\tfrac{1}{3} \sum_{0 \le i,j \le 2} (-1)^{i+j}(1 + 2\chi) \circ$$
$$\{r(\ell_{2+j} \,|\, \ell_0, \dots, \hat{\ell}_i, \dots, \ell_2, \ell_3, \dots, \hat{\ell}_{2+j}, \dots, \ell_5)\} + \tfrac{1}{3}\{1\} \ . \tag{5.22}$$

Let us emphasize that we can compute all degenerate terms in this formula using 5.21 because $\ell_i \neq \ell_j$ and there are no 4 points on a line among $(\ell_0, \dots, \ell_5)$.

The definition of the homomorphism $\tilde{M}_3$ for generic configurations coincides with the definition of M_3 .

Lemma 5.8. The composition $\mathcal{G}_3(F) \xrightarrow{\tilde{M}_3} \tilde{B}_3(F) \xrightarrow{\sim} B_3(F)$ coincides with M_3 .

Proof. It is sufficient to check the lemma for the configurations in fig. 5.3 b), 5.3 a) and η_3

a) Applying formula 5.22 to a configuration 5.3 b) we get

$$-\tfrac{1}{3}(1+2\chi)\circ\ [\{r(\ell_5\,|\,\ell_0,\ell_1,\ell_3,\ell_4)\} - \{r(\ell_5\,|\,\ell_0,\ell_2,\ell_3,\ell_4\} + \{r(\ell_5\,|\,\ell_1,\ell_2,\ell_3,\ell_4)\} -$$

$$(5.23)$$

$$-\{r(\ell_4\,|\,\ell_0,\ell_1,\ell_3,\ell_5)\} + \{r(\ell_3\,|\,\ell_0,\ell_1,\ell_4,\ell_5)\} + \tfrac{1}{3}\{1\}\,.$$

Note that

$$(\ell_4\,|\,\ell_0,\ell_1,\ell_3,\ell_5) = (\ell_5\,|\,\ell_0,\ell_1,\ell_2,\ell_4)$$

$$(\ell_3\,|\,\ell_0,\ell_1,\ell_4,\ell_5) = (\ell_5\,|\,\ell_0,\ell_1,\ell_2,\ell_5)\,.$$

So (5.23) coincides with $M_3(\ell_0, \ldots, \ell_5)$.

b) For a configuration in fig. 5.3 a) formula (5.22) gives

$$-\tfrac{1}{3}\ [\{r(\ell_5\,|\,\ell_0,\ell_1,\ell_2,\ell_3)\} + 2\{r(\ell_5\,|\,\ell_0,\ell_2,\ell_1,\ell_3)\} + \{r(\ell_5\,|\,\ell_1,\ell_2,\ell_3,\ell_4)\} +$$

$$2\{r(\ell_5\,|\,\ell_1,\ell_3,\ell_2,\ell_4)\}] + \tfrac{1}{3}\{1\}\,.$$

Taking into account $(\ell_5\,|\,\ell_4, \ldots) \equiv (\ell_4\,|\,\ell_5, \ldots)$, we can rewrite this formula as

$$-\tfrac{1}{3}\ [\{r(\ell_5\,|\,\ell_0,\ell_1,\ell_2,\ell_3)\} + 2\{r(\ell_5\,|\,\ell_0,\ell_2,\ell_1,\ell_3)\} + \{r(\ell_5\,|\,\ell_0,\ell_3,\ell_1,\ell_2)\} +$$

$$3\{r(\ell_5\,|\,\ell_0,\ell_2,\ell_1,\ell_3)\}] + \tfrac{1}{3}\{1\} = \{r(\ell_5\,|\,\ell_0,\ell_2,\ell_1,\ell_3)\} = M_3(\ell_0, \dots ,\ell_5)\,.$$

c) For the configuration η_3 (ℓ_1,ℓ_3 and ℓ_5 in fig. 5.3 a) lie on a line) formula 5.22 gives $\{r(\ell_5\,|\,\ell_0,\ell_2,\ell_1,\ell_3)\} = \{1\}$ according to 5.21.

So in order to compute in the group $\mathbb{Z}[P^1_F]/(\{x\} - \{x^{-1}\}\ ,\ \{0\}\ ,\ \{\infty\}\ ,$ $\{1\} - \zeta_3(x))$ the image of the 7–term relation corresponding to the degenerations of a configuration in fig. 5.2 under the homomorphism M_3 it is sufficient to specialize formula (5.16) using (5.21).

In particular the Spence–Kummer functional equation for the trilogarithm is just the image of the 7–term relation for a configuration in fig. 1.3 under the homomorphism $\tilde{M}_3$ – see formulae (1.12) and (1.13).

On the other hand we have proved in s. 2 of this § that the Spence–Kummer relation is a linear combination of 3 generic relations $R_3(\ell_0, \dots ,\ell_5,z)$.

§ 6 $\underline{H}^i(\underline{B}_F(3))$ <u>and the algebraic K–theory of a field F</u>.

Recall (see s. 6 of § 2) that $\mathcal{A}_*^m(n)$ is a complex associated with the bicomplex $C_*^m(n)$. It is placed in degrees $-1, 0, 1, \ldots$. There is a canonical homomorphism

$$H_*(GL_n(F),\mathbb{Z}) \longrightarrow H_*(\mathcal{A}_*^m(n),\mathbb{Z})\,. \tag{6.1}$$

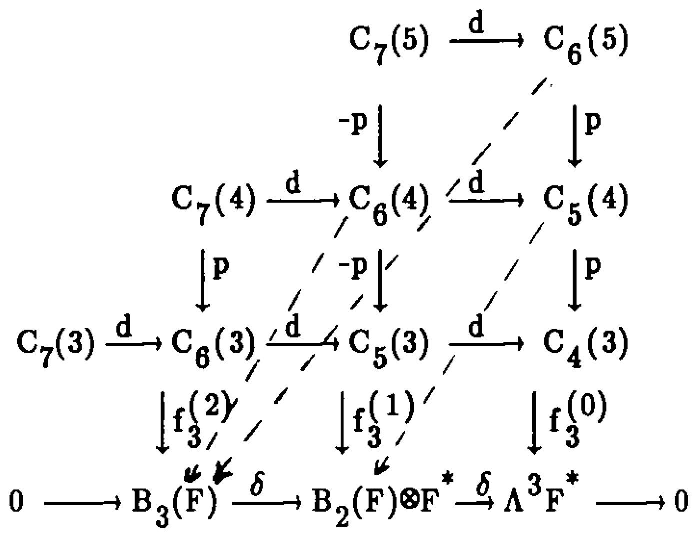
In this § using the results of § 3–5 we will construct a homomorphism of complexes (modulo 6–torsion)

$$\begin{array}{ccccccccc} \longrightarrow & \mathcal{A}_6^{(n-3)}(n) & \longrightarrow & \mathcal{A}_5^{(n-3)}(n) & \longrightarrow & \mathcal{A}_4^{(n-3)}(n) & \longrightarrow & & \\ & \downarrow \varphi_3 & & \downarrow \varphi_3 & & \downarrow \varphi_3 & & & \\ 0 \longrightarrow & B_3(F) & \longrightarrow & B_2(F)\otimes F^* & \longrightarrow & \Lambda^3 F^* & \longrightarrow & 0 & . \end{array} \tag{6.2}$$

Let us consider the following bicomplex

$$\begin{array}{ccccccc} & & & & C_7(5) & \xrightarrow{d} & C_6(5) \\ & & & & \downarrow -p & & \downarrow p \\ & & C_7(4) & \xrightarrow{d} & C_6(4) & \xrightarrow{d} & C_5(4) \\ & & \downarrow p & & \downarrow -p & & \downarrow p \\ C_7(3) & \xrightarrow{d} & C_6(3) & \xrightarrow{d} & C_5(3) & \xrightarrow{d} & C_4(3) \end{array} \tag{6.3}$$

Then there is a homomorphism f of the complex, associated with this bicomplex to the complex $B_F(3)$ that is defined (modulo 6–torsion) in the following way



(6.4)

where all dotted arrows are zero.

Theorem 6.1. f is a well–defined homomorphism of complexes.

Proof. By Lemma 3.6 $f_3^{(0)} \circ p = 0$.

Lemma 6.2. The composition $f_3^{(1)} \circ p : C_6(4) \longrightarrow B_2(F) \otimes F^*$ is zero (modulo 6–torsion).

Proof. Let us prove that

$$f_3^{(1)}(\ell_0, \dots, \ell_4) = \frac{1}{3} \sum (-1)^i [r(\ell_i \mid \ell_0, \dots, \hat{\ell}_i, \dots, \ell_4)] \otimes \prod_{\substack{j_1 \neq i \\ j_2 \neq i}} \Delta(\ell_i, \ell_{j_1}, \ell_{j_2}) . \qquad (6.5)$$

Indeed, according to Lemma 2.18 $\sum_{i=0}^{4} (-1)^i [r(\ell_i \mid \ell_0, \dots, \hat{\ell}_i, \dots, \ell_4)] = 0$. So

$$\frac{1}{3}\Big(\sum_{i=0}^{4}(-1)^i[r(\ell_i\,|\,\ell_0,\dots,\hat{\ell}_i,\dots,\ell_4)]\Big)\otimes\prod_{0\le j_1<j_2<j_3\le 4}\Delta(\ell_{j_1},\ell_{j_2},\ell_{j_3})=0\,. \qquad (6.6)$$

But the sum of (6.6) and (3.5) is just the formula (6.5). ■

Theorem 6.3. The composition $f_3^{(2)}\circ p$ is equal to zero.

This theorem follows immediately from Corollary 7.6 and Theorem 8.1 that will be proved in § 7 – 8.

Theorem 6.1 follows from the Lemmas 3.6, 6.2 and Theorem 6.3. ■

Now in order to construct a homomorphism of the complexes (6.2) we define a homomorphism $\tilde{\varphi}$ of the bicomplex $C_*^{(n-3)}(n)$ to the following one

$$\begin{array}{ccccccccccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \vdots \\
\to & C_8(5) & \to & C_7(5) & \to & C_6(5) & \to & C_5(5) & \to & C_4(5) & \to & C_3(5) & \to & C_2(5) & \to & C_1(5) & \to \mathbb{Z} \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
\to & C_7(4) & \to & C_6(4) & \to & C_5(4) & \to & C_4(4) & \to & C_3(4) & \to & C_2(4) & \to & C_1(4) & \to & \mathbb{Z} & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
\to & C_6(3) & \to & C_5(3) & \to & C_4(3) & \to & C_3(3) & \to & C_2(3) & \to & C_1(3) & \to & \mathbb{Z} & & &
\end{array} \qquad (6.7)$$

Namely, if $(\ell_1,\dots,\ell_m)\in C_m(n)$ is placed at the level k in the bicomplex $C_*^{n-3}(n)$, i.e. we apply to $(\ell_1,\dots,\ell_m)$ the horizontal differential $d^{(k)}$, (see 2.14

where the bicomplex $\tilde{C}^m_*(n)$ is presented) then we set

$$\tilde{\varphi} : (\ell_1, \dots, \ell_m) \longmapsto (\ell_1, \dots, \ell_k \mid \ell_{k+1}, \dots, \ell_m) \in C_{m-k}(n-k)\,.$$

The composition of this homomorphism and the homomorphism f gives the desired homomorphism of complexes (6.2). Therefore we get the canonical homomorphisms

$$H_{6-i}(GL_n(F),\mathbb{Z}) \longrightarrow H^i(B_F(3))\,. \qquad (6.8)$$

(Recall that $B_F(3)$ is placed in degrees 1, 2, 3) .

In particular

$$c_1^{(n)} : H_5(GL_n(F),\mathbb{Z}) \longrightarrow H^1(B_F(3)) \qquad (6.9\text{ a})$$

$$c_2^{(n)} : H_4(GL_n(F),\mathbb{Z}) \longrightarrow H^2(B_F(3)) \qquad (6.9\text{ b})$$

In § 3 – 5 we have constructed the homomorphism of complexes

$$\begin{array}{ccccccc} C_7(3) \longrightarrow & C_6(3) & \longrightarrow & C_5(3) & \longrightarrow & C_4(3) & \\ & \downarrow f_3^{(2)} & & \downarrow f_3^{(1)} & & \downarrow f_3^{(0)} & \\ 0 \longrightarrow & B_3(F) & \longrightarrow & B_2(F)\otimes F^* & \longrightarrow & \Lambda^3 F^* & . \end{array} \qquad (6.10)$$

So using Lemma 2.11 we get the canonical homomorphisms

$$c_1^{(3)} : H_5(GL_3(F),\mathbb{Z}) \longrightarrow H^1(B_F(3)) \qquad (6.11\text{ a})$$

$$c_2^{(3)} : H_4(GL_3(F),\mathbb{Z}) \longrightarrow H^2(B_F(3)) \ . \qquad (6.11\text{ b})$$

Lemma 6.4. The restriction of the homomorphism (6.9 a) (respectively (6.9 b)) to the subgroup $GL_3(F) \subset GL_n(F)$ coincides with the one of (6.11 a) (respectively (6.11 b)).

The proof is in complete analogy with the one of Lemma 2.19. ■

Finally, the restriction of the homomorphism (6.11) to the subgroup $GL_2(F) \subset GL_3(F)$ is equal to zero, because the resolution $\breve{C}_*(3)$ of the trivial $GL_3(F)$ – module $\mathbb{Z}$ has a $GL_2(F)$ – invariant section

$$\begin{array}{ccccc} & & & & \mathbb{Z} \\ & & & & \downarrow \\ \cdots & \longrightarrow & \breve{C}_2(3) & \longrightarrow & \breve{C}_1(3) \ . \end{array}$$

(Namely, if $V_3 = V_2 \oplus < v >$, $\dim V_i = i$, then the map $n \longmapsto n \cdot (v) \in \breve{C}_1(V_3)$ defines a $GL(V_2)$ – invariant section $\mathbb{Z} \longrightarrow \breve{C}_*(V_3)$) .

So we have constructed canonical homomorphisms (see § 1)

$$c_1 : K_5^{[2]}(F)_{\mathbb{Q}} \longrightarrow H^1(B_F(3) \otimes \mathbb{Q})$$

$$c_2 : K_4^{[1]}(F)_{\mathbb{Q}} \longrightarrow H^2(B_F(3) \otimes \mathbb{Q})$$

§ 7. The duality of the configurations

1. Generic part of a Grassmannian and the configurations. Let W be a vector space over a field F with basis $e_1, \dots, e_n$. Let us denote by $f^1, \dots, f^n$ the dual basis in W^* and by h_j the hyperplane $f^j = 0$ in W $(f^j(e_i) = \delta_{ij})$.

Let $\hat{G}_m(n)$ be the manifold of all m–dimensional subspaces in W transversal to the coordinate hyperplanes.

R. MacPherson defined a canonical isomorphism between $\hat{G}_m(n)$ and the generic configurations of n vectors in V_m ([M], [GM]). One of the possible constructions is as follows: the restriction of the functionals f^j to a subspace $S \in \hat{G}_m(n)$ defines an n–tuple of vectors in generic position in S^*.

Sometimes another definition is more convenient: Let $s_1, \dots, s_m$ be a basis in S, then

$$s_j = \sum_{i=1}^{n} a_j^i e_i\,, \quad j = 1, \dots, m\,.$$

The columns of the matrix (a_j^i) form n–tuples of vectors in the coordinate space F^n. Another basis in S leads us to a $GL_n(F)$–equivalent n–tuple of vectors. So the configuration is defined correctly.

Conversely, let $(\ell_1, \dots, \ell_n)$ be a generic configuration of vectors in V_m. Then

$$\ell_{m+i} = \sum_{j=1}^{n-m} b_i^j \ell_j$$

and the subspace in W , generated by the vectors

$$e_1 + \sum_{j=1}^{n-m} b_1^j e_{m+j} \, ; \, \dots \, ; \, e_m + \sum_{j=1}^{n-m} b_m^j e_{m+j} \, , \qquad (7.1)$$

corresponds to the configuration $(\ell_1, \dots, \ell_n)$. In this case the matrix (a_j^i) has the form

$$\begin{bmatrix} 1 & 0 \dots\dots 0 & b_1^1 \dots\dots b_1^{n-m} \\ 0 & 1 \quad \vdots & \vdots \qquad\quad \vdots \\ \vdots & \ddots \quad \vdots & \vdots \qquad\quad \vdots \\ 0 & \dots\dots\dots 1 & b_m^1 \dots\dots b_m^{n-m} \end{bmatrix} \equiv (I_m, B) \ . \qquad (7.2)$$

Both constructions give the same configuration of vectors because the restriction of the functionals

$$f^{m+j} - \sum_{i=1}^{m} b_i^j f^i \ , \ j = 1, \dots ,m$$

to the subspace S is 0 .

The correspondence

$$S \in \hat{G}_m(n) \longmapsto S^{\perp} := \{ f \in W^* \text{ such that } f|_S \equiv 0 \}$$

defines the duality $\hat{G}_m(n) \xrightarrow{\sim} \hat{G}_{n-m}(n)$.

The projections $\tilde{e}_i$ of the vectors e_i in W/S form a configuration, corresponding to the subspace $S^{\perp}$ (because $(S^{\perp})^* = W/S$).

Note, that $\tilde{e}_i = -\sum_{j=1}^{n} b_i^j \tilde{e}_{m+j}$. So the columns of the matrix

$$\begin{bmatrix} -b_1^1 \ldots\ldots -b_m^1 & 1 & 0\ldots\ldots 0 \\ \vdots \qquad\qquad \vdots & 0 & 1 \qquad 0 \\ \vdots \qquad\qquad \vdots & \vdots & \qquad\quad \vdots \\ -b_1^{n-m} \ldots -b_m^{n-m} & 0 & 0\ldots\ldots 1 \end{bmatrix} \equiv (-B^t, I_m)$$

give a configuration of vectors in F^{n-m}, dual to the initial one in F^m (formed by the columns of the matrix (7.2)).

Later on we will be mainly interested in the case $n = 2m$. A generic configuration of $2m$ vectors in F^m may be represented by the $m \times m$ matrix B with non–zero minors. Namely, such a matrix represents a configuration, defined by the columns of an $m \times 2m$ matrix (I,B). In this case the dual configuration is given by the matrix $-(B^{-1})^t$.

Let $T^n \subset GL(W)$ be the maximal torus preserving all 1–dimensional coordinate subspaces $\{\lambda e_i\}$. It acts freely on $\hat{G}_m(n)$. The quotient $\hat{G}_m(n)/T^n$ can be canonically identified with the configurations of n points in generic position in P_F^{m-1}. So we get a duality

$$\left\{\begin{array}{l}\text{configurations of } n\\ \text{points in generic}\\ \text{position in } P_F^{m-1}\end{array}\right\} \overset{*}{\longleftrightarrow} \left\{\begin{array}{l}\text{configurations of } n\\ \text{points in generic}\\ \text{position in } P_F^{n-m-1}\end{array}\right\}.$$

2. Geometrical definition of the duality of configurations. We start with the configurations of points in P_F^m (projective configurations).

Note that a configuration of hyperplanes in P_F^{m-1} gives a configuration of points in $\hat{P}_F^{m-1}$. Let us choose a (projective) isomorphism $g : P_F^{m-1} \xrightarrow{\sim} \hat{P}_F^{m-1}$. Then we get a configuration of points in P_F^{m-1}. This configuration does not depend on the choice of g, because every two such isomorphisms differ by an element of $PGL(n)$ and hence give the same configuration. So from now on we will identify configurations of points and hyperplanes in P_F^{m-1}.

Let $(\ell_1, \dots, \ell_{2m})$ be a configuration of points in P_F^{m-1} in generic position. Let us denote by L_I (respectively L_{II}) the $(m-1)$–simplex with vertices $\ell_1, \dots, \ell_m$ (respectively $\ell_{m+1}, \dots, \ell_{2m}$). Then the codimension 1 faces of these simplices form a configuration of $2m$ hyperplanes in P^{m-1} and hence a configuration of $2m$ points in P^{m-1}.

More precisely, let L_i (respectively L_{m+i}) be the codimension 1 face of L_I (L_{II}) that does not contain $\ell_i(\ell_{m+i})$.

Proposition 7.1. The configuration of hyperplanes $(L_1, \dots, L_{2m})$ is dual to the configuration $(\ell_1, \dots, \ell_{2m})$.

Proof. Let $(\ell_1, \dots, \ell_{2m})$ be a configuration of vectors in V that projects to a configuration $(\ell_1, \dots, \ell_{2m})$ in $P(V)$, $\dim V = m$. Let us denote by $f_1, \dots, f_m$ (respectively $f_{m+1}, \dots, f_{2m}$) the basis in V^* dual to the basis $\ell_1, \dots, \ell_m$ (respectively $\ell_{m+1}, \dots, \ell_{2m}$) in V.

Lemma 7.2. The configuration of vectors $(f_1, \dots, f_m, -f_{m+1}, \dots, -f_{2m})$ is dual to the configuration $(\ell_1, \dots, \ell_{2m})$.

Proof. If $\ell_{m+i} = \sum_{j=1}^{m} b_i^j \ell_j$ then $f_{m+i} = \sum_{j=1}^{m} c_i^j f_j$, where we have the relation $C = (B^{-1})^t$ between the matrices $C = (c_i^j)$ and $B = (b_i^j)$. Indeed, $< f_{m+i}, \ell_{m+i'} > = \sum_{j=1}^{n} c_i^j \cdot b_{i'}^j = \delta_{ii'}$. But we have already proved above that the dual configuration of vectors is described by the matrix $-(B^{-1})^t$. ■

Proposition 7.1 follows from Lemma 7.2. ■

There is a rather surprising geometrical corollary of Proposition 7.1.

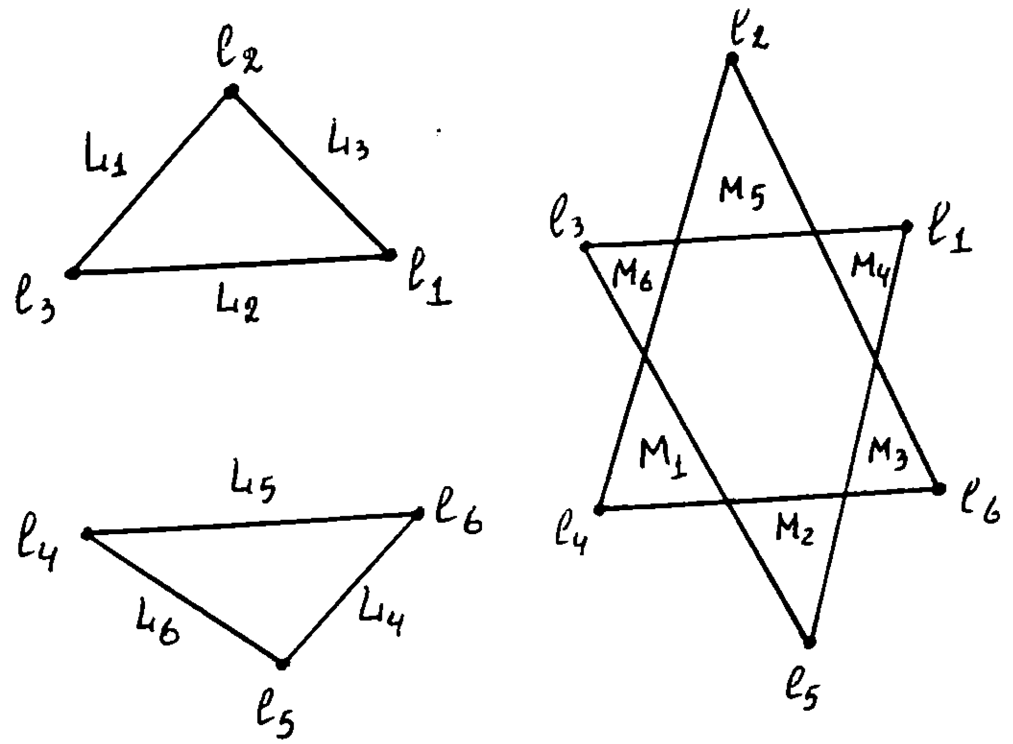
Corollary 7.3. Let $\ell_1, \dots, \ell_{2m}$ be a $(2m)$–tuple of points in generic position in P_F^{m-1} and

$$\{1, \dots, 2m\} = \{i_1, \dots, i_m\} \cup \{j_1, \dots, j_m\}.$$

Let us denote by M_{i_k} (respectively M_{j_k}) the hyperplane generated by the points $\ell_{i_1}, \dots, \hat{\ell}_{i_k}, \dots, \ell_{i_m}$ (respectively $\ell_{j_1}, \dots, \hat{\ell}_{j_k}, \dots, \ell_{j_m}$).

Then there exists a projective transformation $g \in PGL(m)$ such that $g \cdot M_i = L_i$, $1 \le i \le 2m$.

In other words, the configurations $(L_1, \dots, L_{2m})$ and $(M_1, \dots, M_{2m})$ coincide, (see fig. 7.1). ■



$$(L_1, \dots, L_6) = (M_1, \dots, M_6) = *(\ell_1, \dots, \ell_6)$$

fig. 7.1

Proposition 7.1 permits to define the duality geometrically for any $n \geq m + 2$. Namely, let $(\ell_1, \dots, \ell_{m+k})$ be a $(m + k)$ – tuple of points in generic position in P^{m-1} and $2 \leq k \leq m$. Let us denote by H the $(k - 1)$ – dimensional plane generated by the points $\ell_{m+1}, \dots, \ell_{m+k}$. Set (see fig. 7.2)

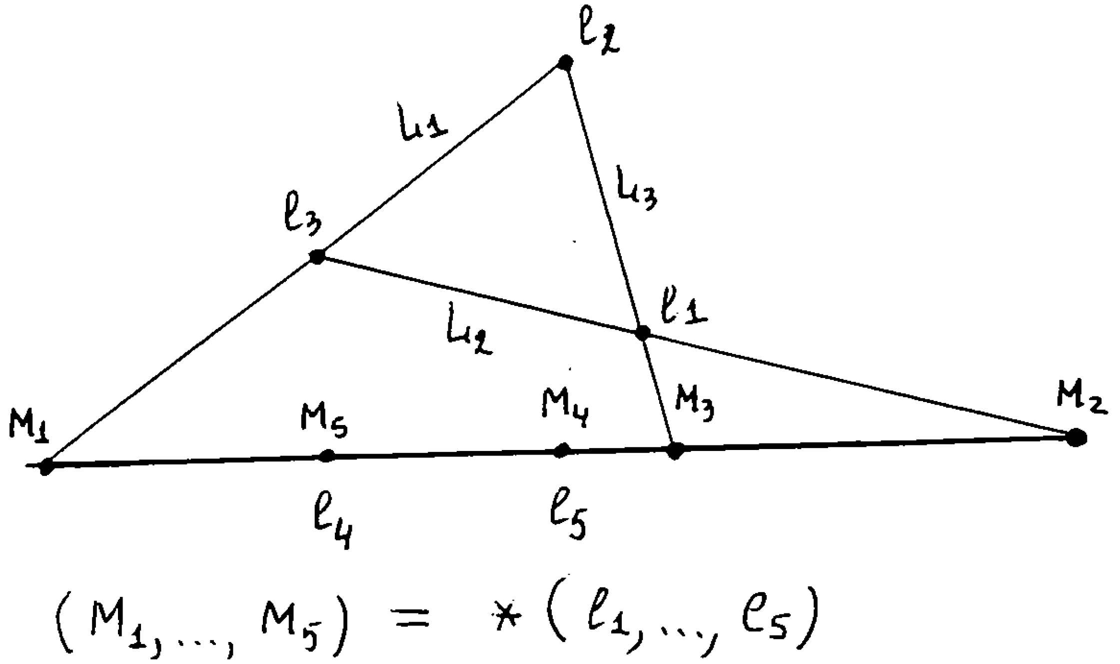


fig. 7.2

$$L_i = < \ell_1, \dots, \hat{\ell}_i, \dots, \ell_m > ,$$

$$M_{m+j} = < \ell_{m+1}, \dots, \hat{\ell}_{m+j}, \dots, \ell_{m+k} > ,$$

$$M_i = L_i \cap H \quad (1 \leq i \leq m) .$$

<u>Proposition</u> 7.4. The configuration $(M_1, \dots, M_{m+k})$ of hyperplanes in H is dual to the configuration $(\ell_1, \dots, \ell_{m+k})$ of points in P^{m-1} .

In order to prove this proposition we need the following

Lemma 7.5. Let $(\ell_1, \dots, \ell_n)$ be a configuration of points in generic position in P^{m-1}, $(Y_1, \dots, Y_n)$ the dual configuration of hyperplanes in P^{n-m-1}.

Then the configuration $(\ell_1, \dots, \hat{\ell}_i, \dots, \ell_n)$ in P^{m-1} is dual to the configuration $(Y_1 \cap Y_i, \dots, Y_n \cap Y_i)$ in Y_i.

Proof. Choose an $s \in \hat{G}_{n-m}(n)$ such that $s \cdot T_n$ corresponds to a configuration $(e_1, \dots, e_n)$ in P^{m-1}. Then by definition, the projections of the basis vectors $\ell_1, \dots, \ell_n$ onto $P(W/s)$ form a configuration that coincides with $(\ell_1, \dots, \ell_n)$. The configuration of hyperplanes $P(Y_i \cap s)$ in Ps is dual to it. Lemma 7.5 follows immediately from these considerations. ■

Corollary 7.6. Let us suppose that $(y_1, \dots, y_n) = *(x_1, \dots, x_n)$ ($*$ is the operation of duality on configurations). Then $(y_1, \dots, \hat{y}_i, \dots, y_n) = *(x_i \mid x_1, \dots, \hat{x}_i, \dots, x_n)$. ■

Proposition 7.4 follows from Lemma 7.5 and Proposition 7.1 by induction. Namely, let $(\ell_1, \dots, \ell_{2m})$ be a configuration of points in generic position containing $(\ell_1, \dots, \ell_{m+k})$. The dual configuration can be represented by codimension 1 faces $L_1, \dots, L_m$ and $L_{m+1}, \dots, L_{2m}$ of simplices $(\ell_1, \dots, \ell_m)$ and $(\ell_{m+1}, \dots, \ell_{2m})$. The configuration $(\ell_1, \dots, \ell_{2m-1})$ is dual to the one $(L_1 \cap L_{2m}, \dots, L_{2m-1} \cap L_{2m})$. The last $(m-1)$ planes are just codimension 1 faces of the simplex $(\ell_{m+1}, \dots, \ell_{2m-1})$ in L_{2m} and soon the geometrical description of the duality in the case $n > 2m$ can be obtained by inversion of this construction. Namely, let $(L_1, \dots, L_n)$ be a configuration of hyperplanes in P^{m-1}. Let us realize it as a configuration of hyperplanes in an $(m-1)$-plane $H \subset P^{n-m-1}$ $(n-m-1 > m-1)$. Let $M_1, \dots, M_{n-m}$ be hyperplanes in P^{n-m-1} such that $M_i \cap H = L_i$. They determine a simplex in P^{n-m-1}

with vertices $m_i = \bigcap_{j \neq i} M_j$, $1 \leq i \leq n - m$. Let $m_{n-m+i} = \bigcap_{j \neq i} L_{n-m+j}$ be the vertices of the simplex $(L_{n-m+1}, \dots, L_n)$ in H . Then the configuration $(m_1, \dots, m_n)$ is dual to the one $(L_1, \dots, L_n)$.

The following description of the duality between the configurations of $n + 3$ points in P^n and P^1 may be useful.

Recall that an irreducible curve in P^n_F that does not lie in a hyperplane has degree $\geq n$. Such curves of minimal possible degree n are called rational normal curves. If a rational normal curve has a point over the field F , then it is projectively equivalent to the following one

$$(x_0 : x_1) \longmapsto (x_0^n : x_0^{n-1} x_1 : \dots : x_0 x_1^{n-1} : x_1^n) .$$

For example in the case $n = 2$ such a curve is a conic.

It is known that through every $n + 3$ points in generic position in P^n passes exactly one rational normal curve.

Let $\ell_1, \dots, \ell_{n+3}$ be $n + 3$ points in generic position in P^n_F and C be the rational normal curve passing through these points. Let us identify C with P^1_F . Then we get a configuration $(y_1, \dots, y_{n+3})$ of $n + 3$ points in P^1_F .

Lemma 7.7. It is dual to the initial one.

Proof. Let $(\tilde{y}_1, \dots, \tilde{y}_{n+3})$ be a configuration of $n+3$ points in P^1_F dual to the one $(\ell_1, \dots, \ell_{n+3})$. Then according to Corollary 7.5

$$(\tilde{y}_1, \dots, \hat{\tilde{y}}_i, \dots, \tilde{y}_{n+3}) = *(x_i \mid x_1, \dots, \hat{x}_i, \dots, x_{n+3}) .$$

By induction we get

$$(\tilde{y}_1, \dots, \hat{\tilde{y}}_i, \dots, \tilde{y}_{n+3}) = (y_1, \dots, \hat{y}_i, \dots, y_{n+3})$$

So we have

$$(\tilde{y}_1, \dots, \tilde{y}_{n+3}) = (y_1, \dots, y_{n+3}) . \quad \blacksquare$$

§ 8. Projective duality and the group $\mathcal{B}_3(F)$.

Let $(\ell_0, \dots, \ell_5)$ be a configuration of 6 points such that there are no 4 points lying on a line. Let us denote by $*(\ell_0, \dots, \ell_5)$ the dual configuration.

Theorem 8.1. In the group $\mathcal{B}_3(F)$

$$*(\ell_0, \dots, \ell_5) + (\ell_0, \dots, \ell_5) = 0 .$$

Proof. Recall that if $a_1, \dots, a_4$ are 4 distinct points on a line, then (see s. 2 of § 4)

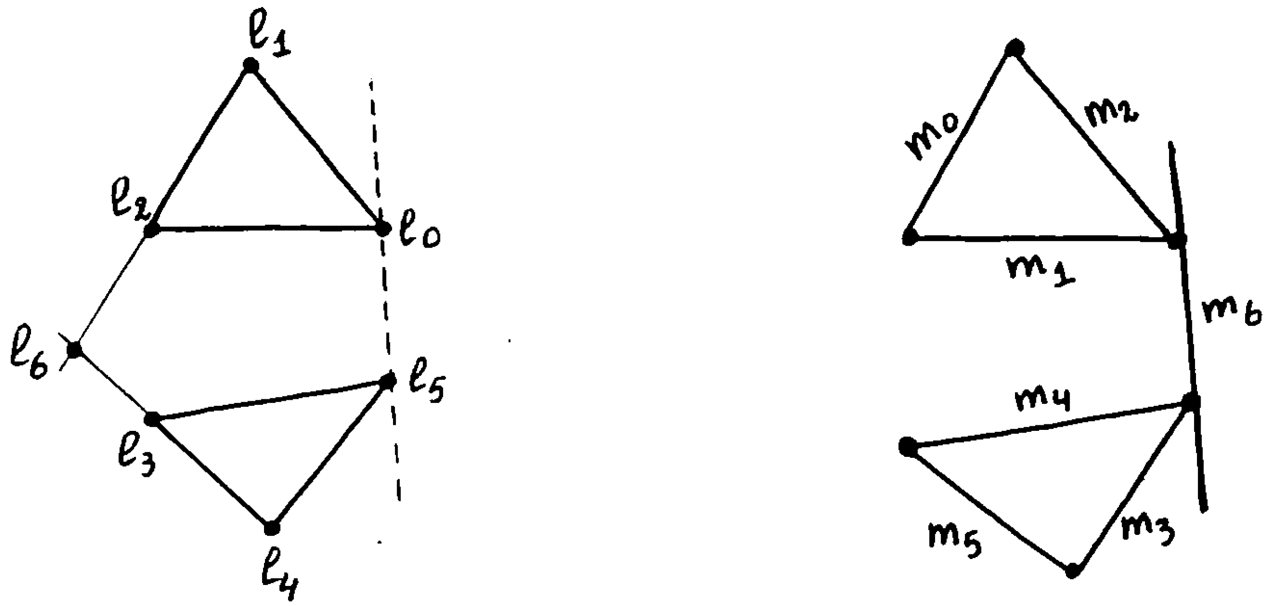
$$L_3'\{r(a_1, \dots, a_4)\} = -L_3\{r(a_1,a_2,a_3,a_4)\} - 2L_3\{r(a_1,a_3,a_2,a_4)\} + \eta_3$$

and

$$L_3'\{x\} + L_3'\{x^{-1}\} = 0 . \tag{8.1}$$

We will abbreviate $L_3'\{r(\ell_j | \ell_{i_1}, \dots, \ell_{i_4})\}$ by writing $(j | i_1 \dots i_4)$.

Set $\ell_6 := \overline{\ell_1\ell_2} \cap \overline{\ell_3\ell_4}$ (see fig. 8.1 a)). Using the 7–term relation for a configuration $(\ell_0, \dots, \ell_6)$, Relation R3 and Proposition 4.12 we have



a) b)

fig. 8.1

$$(\ell_0, \dots, \ell_5) = (5|2634) - (5|1634) + (5|1234) - (5|1264) + (5|1263)$$

$$- (0|2534) + (0|2536) - (0|2546) + (2|0534) - (2|0536) + (2|0546)$$

$$- (5|0234) + (5|0236) - (5|0246) + (0|1563) - (0|1564) + (0|1534)$$

$$- (1|0563) + (1|0564) - (1|0534) + (5|0163) - (5|0164) + (5|0134)$$

(8.2)

$$- (0|4512) + (0|4516) - (0|4526) + 4|0512) - (4|0516) + (4|0526)$$

$$-(5|0412)+(5|0416)-(5|0426)+(0|3512)-(0|3516)+(0|3526)-(3|0512)$$

$$+(3|0516)-(3|0526)+(5|0312)-(5|0316)+(5|0326)-(0|2634)+(0|1634)$$

$$-(0|1234)+(0|1264)-(0|1263)\,.$$

Now let us denote by m_0, m_1, m_2, m_3, m_4, m_5 the lines $\overline{\ell_1\ell_2}$, $\overline{\ell_0\ell_2}$, $\overline{\ell_0\ell_1}$, $\overline{\ell_4\ell_5}$, $\overline{\ell_3\ell_5}$, $\overline{\ell_3\ell_4}$, – see fig. 8.1 b). We will consider them as points in $\hat{P}^2_F$. Then according to Corollary 7.6 $(m_0, \dots, m_5) = *(\ell_0, \dots, \ell_5)$.

Let $m_6 := \overline{m_1m_2} \cap \overline{m_3m_4}$. The corresponding line in P^2_F is $\overline{\ell_0\ell_5}$.

We can express $(m_0, \dots, m_5)$ as a sum of $2 \cdot 5 + 4 \cdot 9 = 46$ terms of type $L_3'\{r(m_j | m_{i_1}, \dots, m_{i_4})\}$ just in the same way as $(\ell_0, \dots, \ell_5)$ – see formula (8.2). Let us prove that the sum of this formulae is equal to zero.

For every term of type $L_3'\{r(\ell_5 | \ell_{i_1}, \dots, \ell_{i_4})\}$ occurring in formula (8.2) there exists a unique term of type $L_3'\{r(m_0 | m_{j_1}, \dots, m_{j_4})\}$ such that either

$$(\ell_5 | \ell_{i_1}, \dots, \ell_{i_4}) \equiv (m_0 | m_{j_1}, \dots, m_{j_4})$$

and the corresponding terms have opposite sign in our sum $(\ell_0, \dots, \ell_5) + (m_0, \dots, m_5)$ and so cancel out, or

$$(\ell_5 \,|\, \ell_{i_1}, \ell_{i_2}, \ell_{i_3}, \ell_{i_4}) \equiv (m_0 \,|\, m_{j_2}, m_{j_1}, m_{j_3}, m_{j_4})$$

and in this case the corresponding terms have the same sign, so according to (8.1) their sum is again zero.

For example, we have

$$L_3'\{r(\ell_5 \,|\, \ell_1, \ell_2, \ell_3, \ell_4)\} - L_3'\{r(m_0 \,|\, m_1, m_2, m_3, m_4)\} = 0$$

because (see fig. 8.2)

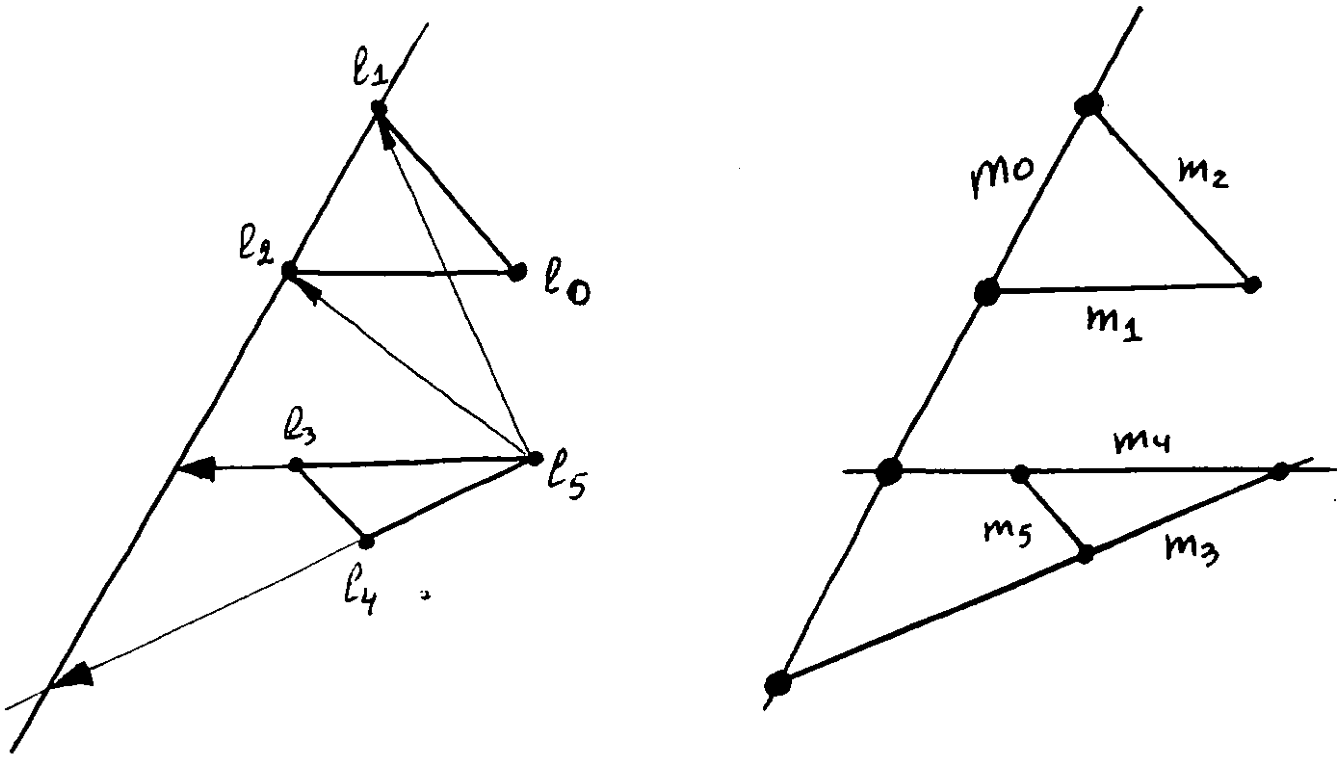


fig. 8.2

$$(\ell_5 \,|\, \ell_1, \ell_2, \ell_3, \ell_4) \equiv (m_0 \,|\, m_2, m_1, m_4, m_3) \ . \tag{8.3}$$

Indeed, by definition the right configuration of 4 points coincides with the one $(m_0 \cap m_2 \ , m_0 \cap m_1 \ , m_0 \cap m_4 \ , m_0 \cap m_3)$ on a line m_0 (in formula (8.3) the lines m_i are considered as points of the dual projective plane), and (8.3) is clear from fig. 8.2.

Another example:

$$- L_3'\{r(\ell_5 \,|\, \ell_1, \ell_6, \ell_3, \ell_4)\} - L_3'\{r(m_0 \,|\, m_5, m_2, m_3, m_4)\} = 0$$

because (see fig. 8.3)

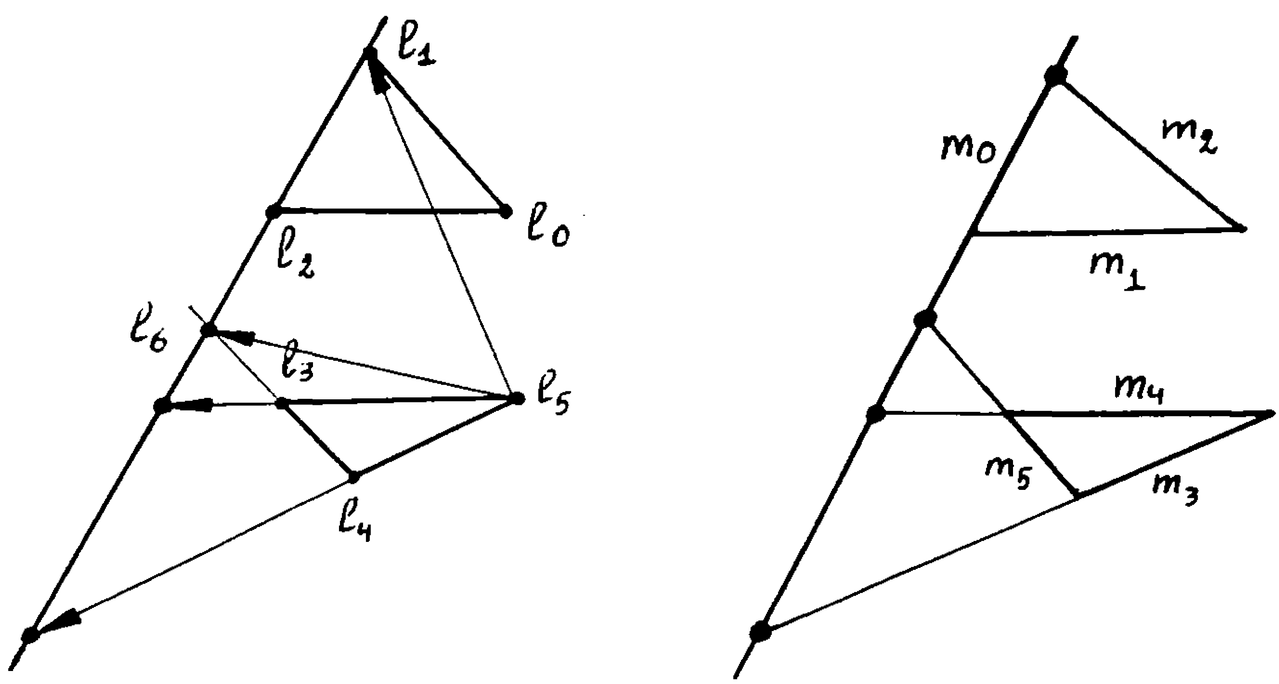


fig. 8.3

$$(\ell_5 \mid \ell_1, \ell_6, \ell_3, \ell_4) \equiv (m_0 \mid m_2, m_5, m_4, m_3) .$$

Similarly (by duality) the same assertion is true for $L_3'\{r(\ell_0 \mid \ell_{j_1}, \dots, \ell_{j_4})\}$ and $L_3'\{r(m_5 \mid m_{i_1}, \dots, m_{i_4})\}$.

Lemma 8.2.

a) $L_3'[\{r(\ell_2 \mid \ell_0, \ell_5, \ell_3, \ell_4)\} - \{r(m_1 \mid m_0, m_5, m_3, m_4)\}] = 0$.

b) $L_3'[-\{r(\ell_2 \mid \ell_0, \ell_5, \ell_3, \ell_6)\} - \{r(m_4 \mid m_0, m_5, m_1, m_6)\}] = 0$.

c) $L_3'[\{r(\ell_2 \mid \ell_0, \ell_5, \ell_4, \ell_6)\} - \{r(m_3 \mid m_0, m_5, m_1, m_6)\}] = 0$.

Proof. a) Let $x = \overline{\ell_0 \ell_2} \cap \overline{\ell_3 \ell_4}$ (see fig. 8.4).

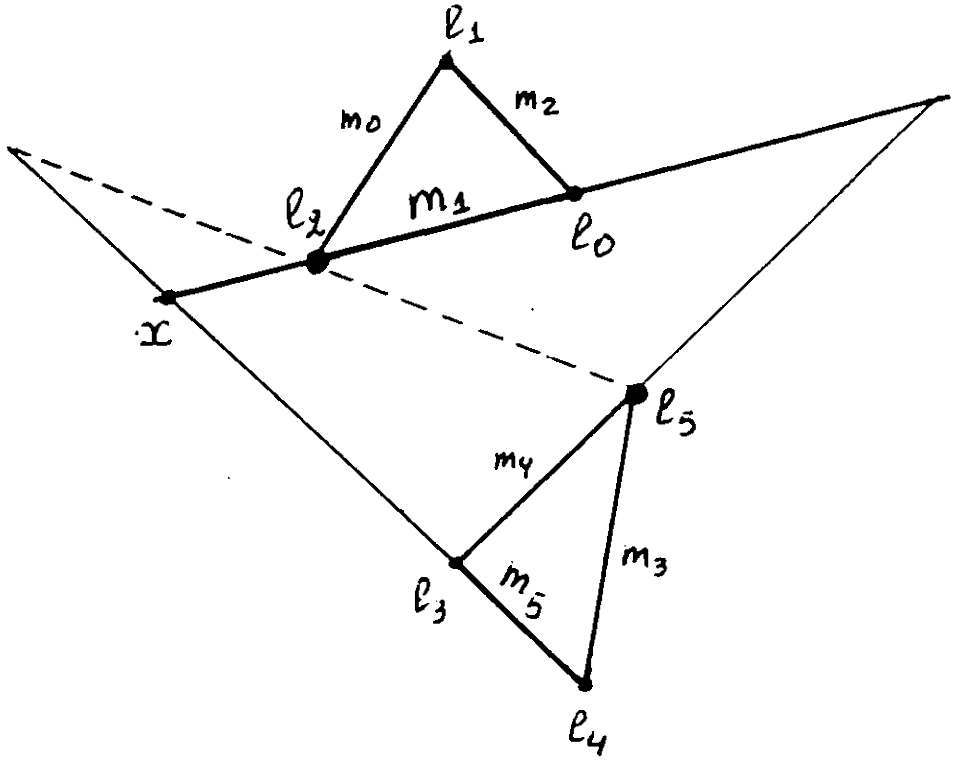


fig. 8.4

Projection onto the line m_5 shows that

$$(\ell_2 \mid \ell_0, \ell_5, \ell_3, \ell_4) \equiv (\ell_5 \mid x, \ell_2, \ell_3, \ell_4)\,.$$

Projection onto the line m_1 gives

$$(\ell_5 \mid x, \ell_2, \ell_3, \ell_4) \equiv (m_1 \mid m_5, m_0, m_4, m_3)\,.$$

So we get a).

b) Projection onto the line m_4 gives (see fig. 8.5)

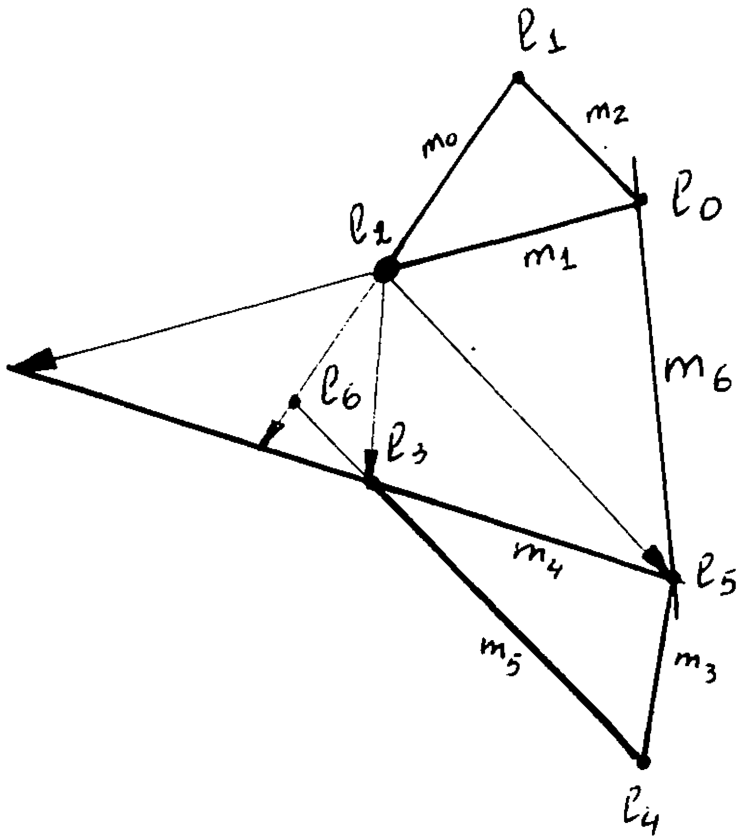


fig. 8.5

It remains to use (8.1).

c) It is proved in complete analogy with b). ∎

Similar lemmas are valid for the projections with center at the points ℓ_1, ℓ_3 and ℓ_4, occurring in formula 8.2. Theorem (8.1) is proved.

§ 9. Theorems 1.9 and 1.10

1. There is the following complex $\mathrm{Meas}(C_\bullet(\mathbb{C}P^n))$ of measurable functions on configurations of points in $\mathbb{C}P^2$ (see 1.18 a)):

$$\longrightarrow \mathrm{Meas}\ C_5(\mathbb{C}P^2) \longrightarrow \mathrm{Meas}\ C_6(\mathbb{C}P^2) \longrightarrow \mathrm{Meas}\ C_7(\mathbb{C}P^2) \longrightarrow \dots$$

Theorem 9.1. $\dim H^6(\mathrm{Meas}\ C_\bullet(\mathbb{C}P^2)) = 2$.

Proof. (Compare with proof of Theorem 7.4.5 in [Bl 1]). There is a complex of $PGL_3(\mathbb{C})$–modules $(C^\bullet, \partial)$, where $C^i := \mathrm{Meas}\,((\mathbb{C}P^2)^{i+1}, R)$ is the space measurable functions on $(\mathbb{C}P^2)^{i+1}$. It is well–known ([Bl 1]) that $C^\bullet$ is a resolution of R by topological $PGL_3(\mathbb{C})$ – modules. Indeed, if $f \in C^i$ is a cocycle, $i \geq 1$, then $\sum_{j=0}^{i+1} (-1)^j f(x_0, \dots, \hat{x}_j, \dots, x_{i+1}) = 0$ (a.e.). Choose $y \in \mathbb{C}P^2$ such that

$$f(x_0, \dots, x_i) = \sum_{j=0}^{i} (-1)^j f(y, x_0, \dots, \hat{x}_j, \dots, x_i)$$

for almost all $(x_0, \dots, x_i)$. Taking $g(x_0, \dots, x_{i-1}) = f(y, x_0, \dots, x_{i-1})$ we get $\partial g = f$. If $f \in C^0$, $\partial f = 0$ then $f \equiv \mathrm{const}$.

So we have $H^*_{cont}(PGL_3(\mathbb{C}), R) = H^*_{cont}(PGL_3(\mathbb{C}), C^\bullet)$. Let us compute the spectral sequence

$$E_1^{p,q} = H^q_{cont}(PGL_3(\mathbb{C}), C^p) \longrightarrow H^{p+q}_{cont}(PGL_3(\mathbb{C}), R) .$$

Let P_i be the stabilisator of i generic points of $\mathbb{C}P^2$. Then $P_5 = P_4 = \{e\}$; $P_3 = (\mathbb{C}^*)^2$ and P_2 (respectively P_1) is a semidirect product of $(\mathbb{C}^*)^2$ (resp. $\mathbb{C}^* \times PGL_2(\mathbb{C})$) and the abelian group $\mathbb{C}^2$. A measurable version of Shapiro's lemma ([Gu]) shows that

$$E_1^{p,q} = H^q(P_{p+1},R) .$$

The standard trick (see [S1], § 1 for a discrete version) shows that

$$H^*_{cts}(P_2,R) = H^*_{cts}((\mathbb{C}^*)^2,R)$$

$$H^*_{cts}(P_1,R) = H^*_{cts}(\mathbb{C}^* \times PGL_2(\mathbb{C}),R) .$$

So (see fig. 9.1)

$$E_1^{4,1} = 0 \ , \ E_1^{3,1} = E_1^{3,2} = 0 \ ; \ E_1^{2,3} = E_1^{1,3} = 0$$

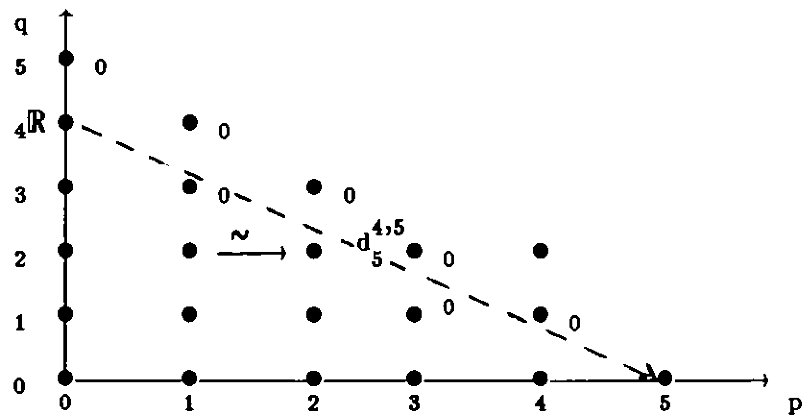


fig. 9.1

Lemma 9.2. $d_1^{1,2} : E_1^{1,2} \longrightarrow E_1^{2,2}$ is an isomorphism.

Note that $E_1^{4,0} = H^4_{cts}(\mathbb{C}^* \times PGL_2(\mathbb{C}),R) = R$. But $H^4_{cts}(PGL_3(\mathbb{C}),R) = 0$. So we have a nontrivial differential

$$d_5^{4,0} : E_5^{4,0} \cong E_1^{4,0} \longrightarrow E_5^{0,5} \cong E_1^{0,5} .$$

Therefore

$$\dim E_1^{0,5} = \dim H^5_{cts}(PGL_3(\mathbb{C}),R) + \dim H_{cts}(\mathbb{C}^* \times PGL_2(\mathbb{C}),R) = 2 .$$

Theorem 9.1 is proved. ∎

Corollary 9.3 $\dim H^5(\text{Meas } C_\bullet(\mathbb{C}P^2)) = 0$.

Proof. Follows immediately from the proof of Theorem 9.1. ∎

Analoguous but more complicated arguments prove a continuous version of Theorem 1.10 (see Theorem 1.9).

The complex involution $z \longrightarrow \bar{z}$ acts on the 2–dimensional vector space $E_1^{0,5} = H^6(\text{Meas } C_\bullet(\mathbb{C}P^2))$ with eigenvalues $+1$ and -1 . The corresponding eigenvectors are $\mathcal{M}_3(\ell_0, \dots ,\ell_5)$ and $d^* f_1^{(3)}(\ell_0, \dots ,\ell_5)$ – see s. 5 and s. 7 of § 1. Their restriction to a degenerate configuration presented in fig. 1.12 is $\mathcal{L}_3(z)$ and $\mathcal{D}_2(z) \cdot \log|z|$, where $z = r(\ell_5 | \ell_0,\ell_2,\ell_1,\ell_2)$ (see fig. 1.12). This proves Theorem 1.9.

Theorem 1.10 follows immediately from Theorem 1.9. Indeed, we have the isomorphism $M_3 : \mathcal{J}_3(F)_{\mathbb{Q}} \xrightarrow{\sim} B_3(F)_{\mathbb{Q}}$, so any continuous function f satisfying the functional equation $f(R_3(a,b,c)) = 0$ defines a continuous skew–symmetric function $M_3 \circ f$ on stable configurations of 6 points in $\mathbb{C}P^2$ satisfying $d_6^*(M_3 \circ f) = 0$. Therefore we have

$$M_3 \circ f = d_5^*\varphi + \lambda_1 \cdot \mathcal{M}_3 + \lambda_2 \cdot (d^* f_1^{(3)}) .$$

($\mathcal{M}_3 \equiv \mathcal{M}_3(\ell_0, \dots ,\ell_5)$ and $d^* f_1^{(3)}(\ell_0, \dots ,\ell_5)$ are functions constructed in s. and s. 7 of § 1).

But the restriction of $d_s^*\varphi$ to a degenerate configuration represented in fig. 1.12 is 0 (see proof of Proposition 1.11), and the restriction of the other terms is $\lambda_1 \mathcal{L}_3(z) + \lambda_2 \mathcal{D}_2(z) \cdot \log|z|$ $(z := r(\ell_5 \,|\, \ell_0,\ell_2,\ell_1,\ell_3))$.

2. Let $C_n(P^2)$ be the abelian group generated by all n–tuples of points in P^2 and $C_n^{alt}(P^2)_{\mathbb{Q}}$ the subspace of skew–invariants in $C_n(P^2)_{\mathbb{Q}}$ with respect to the action of the permutation group S_n. Then $C_\bullet^{alt}(P^2)_{\mathbb{Q}}$ is a resolution of the trivial $PGL_3(F)$–module $\mathbb{Q}$ and

$$H_*(PGL_3(F),\mathbb{Q}) = H_*(PGL_3(F),C_\bullet^{alt}(P_F^2)) .$$

So we have a spectral sequence associated with the stupid filtration on $C_\bullet^{alt}(P_F^2)_{\mathbb{Q}}$. It is easy to prove that $E^2_{0,5}$ is generated by classes of (degenerate) configurations in fig. 1.12 (in fact, we already used the necessary arguments in s. 6, 7 of § 1). (Unpleasant) computation of higher differentials in this spectral sequence shows that $\ker(\mathbb{Q}[P_F^1] \longrightarrow B_2(F)_{\mathbb{Q}} \otimes F_{\mathbb{Q}}^*)$ maps to $H_5(PGL_3(F),\mathbb{Q})$. In particular a configuration in fig. 1.7 (that corresponds to $\lambda\{1\} \in \mathbb{Q}[P^1_F]$) gives a class in $H_5(PGL_3(F),\mathbb{Q})$ for an arbitrary field F .

J. Dupont told me that he studied this spectral sequence several years ago and got similar results (unpublished, private communication). In fact his arguments are more clear and elegant. I hope we will have the pleasure to read his paper in the near future.

Recall that in s. 5 of § 1 we have constructed an honest, everywhere defined but discontinuous 5–cocycle of $PGL_3(\mathbb{C})$. Its restriction to a class in $H_5(PGL_3(\mathbb{C}),\mathbb{Q})$ represented by $\{1\}$ is equal to $\mathcal{L}_3(1) = \zeta_{\mathbb{Q}}(3)$ (because the restriction of the function $\mathcal{M}_3(\ell_0, \dots ,\ell_5)$ to the configuration in fig. 1.7 is just $\mathcal{L}_3(1)$ – see § 1). On the other hand by the Borel theorem for $F = \mathbb{Q}$ the restriction of the Borel class to a class in $H_5(PGL_3(\mathbb{Q}),\mathbb{Q})$ is a rational multiple of $\zeta_{\mathbb{Q}}(3)$. So the class constructed above is a rational multiple of the Borel class in $H^5_{cts}(PGL_3(\mathbb{C}),R)$. Another proof of the non–triviality of the constructed class in $H^5_{cts}(PGL_3(\mathbb{C}),R)$ follows from Theorem 1.9 – see s. 7 of § 1. The non–triviality of this class follows also from explicit formulae of the next § combined with recent results of J. Yang [J2] about the relation of the Hain–MacPherson trilogarithm and $H^5_{cts}(GL_3(\mathbb{C}),R)$.

§ 10. Explicit formula for the Grassmannian trilogarithm.

Recall that $\hat{G}_m(n)$ is the manifold of all m–dimensional subspaces in the n–dimensional coordinate vector space W transversal to coordinate hyperplanes. R. MacPherson considered the truncated simplicial Grassmannian $\hat{G}^{(3)}$:

$$\overset{s_0}{\underset{s_6}{\underset{\longrightarrow}{\overset{\longrightarrow}{\cdots}}}} \hat{G}_3(6) \overset{s_0}{\underset{s_5}{\underset{\longrightarrow}{\overset{\longrightarrow}{\cdots}}}} \hat{G}_2(5) \overset{s_0}{\underset{s_4}{\underset{\longrightarrow}{\overset{\longrightarrow}{\cdots}}}} \hat{G}_1(4)\,. \tag{10.1}$$

Here s_i denotes the intersection with the i–th coordinate hyperplane. We have the following homomorphism of abelian groups:

$$m : \mathbb{Z}[\hat{G}_m(n)] \longrightarrow C_n(n-m) \tag{10.2}$$

where m(S) is the image of the coordinate vectors in W/S . Applying it to (10.1) we get a truncated simplicial abelian group

$$\overset{s_0}{\underset{s_6}{\underset{\longrightarrow}{\overset{\longrightarrow}{\cdots}}}} C_6(3) \overset{s_0}{\underset{s_5}{\underset{\longrightarrow}{\overset{\longrightarrow}{\cdots}}}} C_5(3) \overset{s_0}{\underset{s_4}{\underset{\longrightarrow}{\overset{\longrightarrow}{\cdots}}}} C_4(3)\,.$$

The corresponding complex of abelian groups is just the Grassmannian complex $C_\bullet(3)$.

In § 3 – 4 we have constructed a canonical homomorphism of complexes $f : C_\bullet(3) \longrightarrow B_F(3)$:

$$\begin{array}{ccccc} \xrightarrow{d} C_6(3) & \xrightarrow{d} & C_5(3) & \xrightarrow{d} & C_4(3) \\ \downarrow f_2^{(3)} & & \downarrow f_1^{(3)} & & \downarrow f_0^{(3)} \\ B_3(F) & \xrightarrow{\delta} & B_2(F)\otimes F^* & \xrightarrow{\delta} & \Lambda^3 F^* \end{array}$$

Now let $F = \mathbb{C}(X)$ be the field of functions on an (open) manifold $X/\mathbb{C}$. Let us construct explicitly a homomorphism of complexes

$$\begin{array}{ccccc} B_3(\mathbb{C}(X)) & \xrightarrow{\delta} & B_2(\mathbb{C}(X)) \otimes \mathbb{C}(X)^* & \xrightarrow{\delta} & \Lambda^3 \mathbb{C}(X)^* \\ \downarrow r_0^{(3)} & & \downarrow r_1^{(3)} & & \downarrow r_2^{(3)} \\ \Omega_X^0 & \longrightarrow & \Omega_X^1 & \longrightarrow & \Omega_X^2 \end{array}$$

(Ω_X, d) is the C^∞–de Rham complex on X. Then the composition $r_i^{(3)} \circ f_{3-i}^{(3)} \circ m$ defines an i–form ω_i on $\hat{G}_{3-i}(6-i)$. The collection of these forms will represent a cochain ω in the complex computing the Deligne cohomology $H^6(\hat{G}^{(3)}, R(3)_{\mathscr{D}})$ of the truncated simplicial Grassmannian such that $D\omega = \mathrm{Re}(\mathrm{vol}_3)$, where D is the total differential in this complex and vol_3 is the canonical holomorphic 3–form with logarithmic singularities on $\hat{G}_1(4) = (\mathbb{C}^*)^3$ $\left[\mathrm{vol}_3 = \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3} \right]$. More precisely this means that we will construct a collection of forms ω_i such that

$$d\omega_i = \sum_j (-1)^j s_j^* \omega_{i+1} \quad (i = 0,1), \quad d\omega_2 = \mathrm{Re}(\mathrm{vol}_3).$$

Set

$$r_0^{(3)}\{f(z)\}_3 := \mathscr{L}_3(f(z))$$

$$r_1^{(3)}\{f(z)\}_2 \otimes g(z) := -\mathcal{L}_2(f(z))\text{darg}\, g(z) +$$

$$\frac{1}{3}\log|g|(\log|1-f|\ \text{dlog}\ |f| - \log|f|\ \text{dlog}\ |1-f|)$$

$$r_2^{(3)} f_1(z)\wedge f_2(z)\wedge f_3(z) :=$$
$$\frac{1}{6}\sum_{\sigma\in S_3}(-1)^{|\sigma|}\sigma\cdot(3\log|f_1|\,\text{darg}\, f_2\wedge\text{darg}\, f_3 - \log|f_1|\,\text{dlog}|f_2|\wedge\text{dlog}|f_3|).$$

Lemma 10.1.

a) $dr_2^{(3)}(f_1 \wedge f_2 \wedge f_3) = -\text{Re}\left[\frac{df_1}{f_1}\wedge\frac{df_2}{f_2}\wedge\frac{df_3}{f_3}\right]$

b) $d \circ r_i^{(3)} = r_{i+1}^{(3)} \circ d$ for $i = 0,1$.

Proof. a) – clear;

b) $$d\mathcal{L}_3(z) = -\mathcal{L}_2(z)\ \text{darg}\ z +$$
$$+\frac{1}{3}\log|z|(\log|1-z|\ \text{dlog}\ |z| - \log|z|\ \text{dlog}\ |1-z|),$$
$$d(-\mathcal{L}_2(z)\ \text{darg}\ w + \frac{1}{3}\log|w|(\log|1-z|\ \text{dlog}\ |z| - \log|z|\ \text{dlog}\ |1-z|)) =$$
$$= (\log|1-z|\ \text{darg}\ z - \log|z|\ \text{darg}\ (1-z))\wedge\text{darg}\ w -$$
$$-\frac{1}{3}(\log|1-z|\ \text{dlog}\ |z| - \log|z|\ \text{dlog}\ |1-z|)\wedge\text{dlog}|w|.\quad\blacksquare$$

In the following paper we will see how these formulae enable one to compute explicitly the (3–d)–th Chern class in the Deligne cohomology of an n–dimensional vector bundle over X .

Appendix

The duality of configurations of points in the plane and a "resolution" for $K_2(F)$.

The Bloch–Suslin complex $B_F(2)$ can be considered as a "resolution" for $K_2(F)$. More precisely, $\mathbb{Z}[P^1_F\backslash 0,1,\infty]$ is the free abelian group generated by all Steinberg relations in $\Lambda^2 F^*$ and $R_2(F) \subset \mathbb{Z}[P^1_F\backslash 0,1,\infty]$ is a subgroup of the kernel of the homomorphism

$$\mathbb{Z}[P^1_F\backslash 0,1,\infty] \xrightarrow{\ \delta\ } \Lambda^2 F^*$$

$$\delta : [x] \longmapsto (1-x) \wedge x$$

which is defined universally for all fields F . So (by Suslin's theorem) $K_3^{ind}(F)_{\mathbb{Q}}$ is the quotient of $\mathrm{Ker}\,\delta$ by the "universal" kernel of δ (modulo torsion).

Now let us try to continue the process of constructing of the "resolution" for $K_2(F)$. For this let us consider the homomorphism

$$C_5(P^1_F) \xrightarrow{\ \delta\ } C_4(P^1_F)/\{(x_0,x_1,x_2,x_3) + (x_1,x_0,x_2,x_3)\} \equiv \qquad \text{(A 1)}$$

$$\mathbb{Z}[P^1_F\backslash 0,1,\infty]/\{[x] + [x^{-1}]\}$$

$$(x_0, \dots ,x_4) \longmapsto \sum_{i=0}^{4} (-1)^i (x_0, \dots ,\hat{x}_i, \dots ,x_4) .$$

(We factorize by the skew–symmetry relations $[x] + [x^{-1}]$ only for convenience).

Then by definition $R_2(F) = \delta(C_5(P^1))$.

It is obvious that $\delta(C_6(P^1_F)) \subset C_5(P^1_F)$ lies in the kernel of the homomorphism (A 1). Let us construct elements in this kernel that do not lie in $\delta(C_6(P^1_F))$.

Let us define an involution $s : C_5(P^1_F) \longrightarrow C_5(P^1_F)$ as follows. For a configuration $(x_0, \dots, x_4) \in C_5(P^1_F)$ consider the configuration of 6 points in P^2_F as in fig. A 1.1 such that

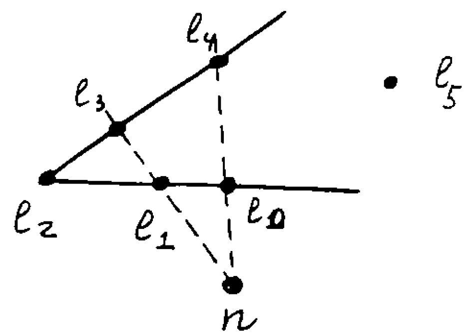


fig. A 1

$$(\ell_5 \,|\, \ell_0, \dots, \ell_4) = (x_0, \dots, x_4) .$$

Put $n = \overline{\ell_0\ell_4} \cap \overline{\ell_1\ell_3}$. Set

$$s(x_0, \dots, x_4) := (\ell_5 \,|\, \ell_0, \ell_1, n, \ell_3, \ell_4) .$$

We will prove that $s^2 = \mathrm{id}$ a little bit later.

Lemma A 1

$$(\ell_5 | \ell_0, n, \ell_3, \ell_4) = (\ell_5 | \ell_0, \ell_1, \ell_3, \ell_2)$$
$$(\ell_5 | \ell_1, n, \ell_3, \ell_4) = (\ell_5 | \ell_1, \ell_0, \ell_2, \ell_4)$$
$$(\ell_5 | \ell_0, \ell_1, n, \ell_3) = (\ell_5 | \ell_0, \ell_2, \ell_4, \ell_3)$$
$$(\ell_5 | \ell_0, \ell_1, n, \ell_4) = (\ell_5 | \ell_2, \ell_1, \ell_3, \ell_4) .$$

Proof. Consider the projection onto the line $\overline{n\ell_0\ell_4}$ – see fig. A 1 – we have

$$(\ell_5 | \ell_0, n, \ell_3, \ell_4) = (\ell_3 | \ell_0, n, \ell_5, \ell_4) = (\ell_3 | \ell_0, \ell_1, \ell_5, \ell_2) .$$

The projection onto the line $\overline{\ell_0\ell_1\ell_2}$ gives

$$(\ell_3 | \ell_0, \ell_1, \ell_5, \ell_2) = (\ell_5 | \ell_0, \ell_1, \ell_3, \ell_2). \qquad \blacksquare$$

It follows immediately from this lemma that $\delta((x_0, \dots ,x_4) - s \cdot (x_0, \dots ,x_4)) = 0$ in $\mathbb{Z}[P^1_F \backslash 0,1,\infty] / \{ [x] + [x^{-1}] \}$. Now let $S_2(F)$ be the subgroup of $C_5(P^1_F)$ generated by $\delta(C_6(P^1_F))$ and the elements $(x_0, \dots ,x_4) - s(x_0, \dots ,x_4)$. Then we have the following complex $\tilde{B}_2(F)$

$$C_5(P^1_F)/S_2(F) \xrightarrow{\delta} \mathbb{Z}[P^1_F \backslash 0,1,\infty] / \{ [x] + [x^{-1}] \} \xrightarrow{\delta} \Lambda^2 F^*$$

(the left group is placed in degree 0 .)

It was A.A. Suslin who first considered the subgroup $S_2(F)$ (unpublished). In fact, he defines in coordinates the elements $(x_0, \dots, x_4) - s(x_0, \dots, x_4)$. (Our contribution is an invariant geometrical definition.) A.A. Suslin proved that $H^0(\tilde{B}_2(F)_{\mathbb{Q}}) = K_4^{[2]}(F)_{\mathbb{Q}}$. According to the rank conjecture 1.22 and the Beilinson–Soulé conjecture for $K_4(F)$ the last group should be zero. So $S_2(F)$ should give all relations between 5–term relations for the dilogarithm.

It would be interesting to find all relations between the functional equations $R_3(a,b,c)$ for the trilogarithm.

Now set

$$(m_0, \dots, m_5) = *(\ell_0, \dots, \ell_5) .$$

Lemma A 2. If $(\ell_0, \dots, \ell_5)$ is as in fig. A 1 then the dual configuration is as in fig. A 2 and

$$(m_2 \,|\, m_0, m_1, m_5, m_3, m_4) = (\ell_5 \,|\, \ell_4, \ell_3, n, \ell_1, \ell_0) . \qquad \text{(A 2)}$$

Proof. Let us use the geometrical definition of the duality of configurations. Consider a pair of triangles (ℓ_1, ℓ_2, ℓ_3) and (ℓ_0, ℓ_4, ℓ_5). Then we have 3 sides $m_0 := \overline{\ell_4 \ell_5}$, $m_1 := \overline{\ell_2 \ell_3}$, $m_5 := \overline{\ell_0 \ell_4}$ that contain ℓ_4, and 3 sides $m_3 := \overline{\ell_0 \ell_1}$, $m_4 := \overline{\ell_0 \ell_5}$, $m_5 := \overline{\ell_0 \ell_4}$ containing ℓ_0. So the dual configuration $(m_0, \dots, m_5)$ is as on fig. A 2. Further, the intersection points of the lines m_0, m_1, m_5, m_3, m_4 with the line m_2 are obtained by projection of the points $\ell_4, \ell_3, n, \ell_1, \ell_0$ with the center at ℓ_5. So we have (A 2). ■

It follows from Lemma A 2 that $s^2 = \mathrm{id}$.

Note also, that Lemmas A 1 and A 2 prove Theorem 8.1 for configurations $(\ell_0, \dots, \ell_5)$ as in fig. A 1.

<u>Proposition</u> A 3. For a generic configuration $(\ell_0, \dots, \ell_5)$ of 6 points in P^2_F

$$\delta\Big(\sum_{i=0}^{5} (-1)^i (\ell_i \,|\, \ell_0, \dots, \hat{\ell}_i, \dots, \ell_5) - \sum_{j=0}^{5} (-1)^j (m_j \,|\, m_0, \dots, \hat{m}_j, \dots, m_5)\Big) = 0 . \blacksquare$$

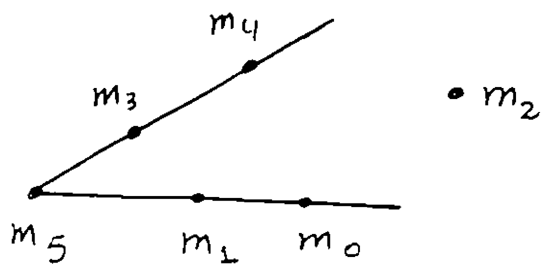


fig. A 2

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