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Methods of Integral Geometry and Recovering a Function with Compact Support from Its Projections in Unknown Directions

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Abstract. A method for recovering a function with compact support on a plane from its Radon transform is suggested. In particular, from the given projections of a function in unknown directions, these directions may be recovered (up to a rotation or a reflection), if there are ≥ 7 of them. This result may be applied in tomography.

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1. Introduction

1.1. FORMULATION OF RESULTS

Let $\rho(x_1, x_2)$ be a function with compact support describing the distribution density of the matter of a planar particle. As is customary, its projection in the direction of a vector ω is defined by the set of integrals of $\rho(x)$ over all the lines parallel to ω, such that $|\omega| = 1$.

For instance, the projection in the direction (0, 1) (or the projection onto the line $x_2 = 0$) is the function

$$p(x_1) = \int \rho(x_1, x_2)\, \mathrm{d}x_2 .$$

The term projection originates from tomography because a Röntgen photo (or a microphoto obtained with the help of an electron microscope) is the projection of the distribution function of the investigated object.)

To define projections of $\rho(x)$ in all the directions, is to define its Radon transform

$$\check{\rho}(\omega, p) = \int p(x)\, \delta(\langle \omega, x\rangle - p)\, \mathrm{d}^2 x .$$

As was shown by Radon in 1917 [1, 2], a function $\rho(x)$ (not necessarily with compact support) can be recovered from $\check{p}(\omega, p)$ by the formula

$$\rho(x_1^0, x_2^0) = \iint_{-\infty}^{\infty} \check{f}(\omega, p)/(p - \langle \omega, x^0 \rangle)^2 \, dp^0$$

During the past 20 years, Radon's work has found ever wider application in Röntgen (X-ray), nuclear magnetic resonance and ultrasonic tomography of the human body and other objects, electron microscopy of biomolecules, radioastronomy and other fields.

We show that if ρ is a function with compact support, then it is possible to recover it up to a motion of the plane from its projections in unknown directions.

From the point of practical applications, the following result is of the most interest. From the given projections of an asymmetric function ρ in unknown directions $\omega_1, \ldots, \omega_n$, for $n \geq 7$, we may recover a system of vectors $\omega_1, \ldots, \omega_n$ up to a rotation of the whole system by some angle or a reflection with respect to a line. After that, we get the old problem: to recover the structure from the projections in the known directions.

All the considerations are based on the properties of moments of projections found in 1961 by Gelfand and Graev (Paley–Wiener's theorem for the Radon transformation). It is rather obvious that not only Paley-Wiener's theorem, but also the other ideas and methods of integral geometry [4, 5] will be applied to practical problems.

1.2. POSSIBLE PRACTICAL APPLICATIONS

A typical setting which immediately gives rise to the problem of recovering the structure of a planar particle from its projections in unknown directions, is that electron microscopy: the study of identically stretched in one direction biological particles. The particles are precipitated on a layer so that the direction in which they are stretched is parallel to the layer's plane. Around this direction they are allowed to rotate arbitrarily.

Since the direction in which a particle is stretched is clear from the projection, we assume that the particle is rotated by unknown angles $\varphi_1, \ldots, \varphi_n$ around a fixed axis and then is mapped onto the plane.

Note that the projection of the centre of mass of a particle can be found from the projection of the particle, since its center of mass is projected onto that of the projection. Consider the planar section through the center of mass perpendicular to the axis around which the particle is rotated. Let x_1, x_2 be coordinates in the plane of a section so that the center of mass of the planar section of the particle is at the origin $(0, 0)$. Denote by $\rho(x_1, x_2)$ the distribution function of the planar section of the particle.

We get the following problem: a planar particle is rotated by unknown angles

$\varphi_1, \ldots, \varphi_n$ around its center of mass and projected onto the line $x_2 = 0$. Find the relative angles $\varphi_k - \varphi_1$, $k = 2, \ldots, n$. Clearly, we may assume $\varphi_1 = 0$.

2. Properties of the Moments of Projections

2.1. The distribution function of a particle rotated by an angle φ counterclockwise is determined by the formula

$$\rho_\varphi(x_1, x_2) = \rho(x_1 \cos\varphi + x_2 \sin\varphi, -x_1 \sin\varphi + x_2 \cos\varphi).$$

For the projection

$$p_\varphi(x_1) = \int \rho_\varphi(x_1, x_2)\, \mathrm{d}x_2$$

of $\rho_\varphi(x_1, x_2)$ onto the line $x_2 = 0$, determine its kth moment $M_k(\varphi)$ setting

$$M_k(\varphi) = \int p_\varphi(x_1) x_1^k\, \mathrm{d}x_1. \tag{1}$$

All the arguments are based on the fact that $M_k(\varphi)$ is a homogeneous trigonometric polynomial of degree k, i.e.,

$$M_k(\varphi) = \lambda_{k0} \cos^k\varphi + \lambda_{k-1,1} \cos^{k-1}\varphi \sin\varphi + \cdots + \lambda_{0k} \sin^k\varphi.$$

In fact, performing the change of variables

$$y_1 = x_1 \cos\varphi + x_2 \sin\varphi, \qquad y_2 = -x_1 \sin\varphi + x_2 \cos\varphi$$

(consequently $x_1 = y_1 \cos\varphi - y_2 \sin\varphi$) in (1) we get

$$\begin{aligned} M_k(\varphi) &= \int \rho(y_1, y_2)(y_1 \cos\varphi - y_2 \sin\varphi)^k\, \mathrm{d}y_1\, \mathrm{d}y_2 \\ &= \left(\int \rho(y_1, y_2) y_1^k\, \mathrm{d}y_1\, \mathrm{d}y_2\right) \cos^k\varphi + \\ &\quad + \left(-C_K^1 \int \rho(y_1, y_2) y_1^{k-1} y_2\, \mathrm{d}y_1\, \mathrm{d}y_2\right) \cos^{k-1}\varphi \sin\varphi + \\ &\quad + \cdots + \left((-1)^k \int \rho(y_1, y_2) y_2^k\, \mathrm{d}y_1\, \mathrm{d}y_2\right) \sin^k\varphi. \end{aligned} \tag{2}$$

I. M. Gelfand suggested calling the conditions on $\mu_k(\varphi)$ *Kavalieri's conditions* since, for $k = 0$, the condition $M_0(\varphi) = \text{const}$ mirrors the known Kavalieri principle: the area of a body may be recovered from the lengths of its sections by a bunch of parallel lines.

REMARK. Until now, in problems of three-dimensional reconstruction, only the Radon transformation itself (on the plane or in the space) and the Radon inversion formula were applied.

2.2. A homogeneous trigonometric polynomial of degree k is uniquely determined by its values at any $k+1$ points $\varphi_1, \ldots, \varphi_{k+1}$ $(0_i \leq \varphi_i \leq 2\pi)$ such that $\varphi_i - \varphi_j \neq \pm\pi$. (The last restriction is necessary since $M_k(\varphi \pm \pi) = (-1)^k M_k(\varphi)$.) In fact, we know the values of the function

$$M_k(\varphi)/\cos^k \varphi = \lambda_{0k} \tan^k \varphi + \cdots + \lambda_{k0}$$

at $\varphi_1, \ldots, \varphi_{k+1}$ such that $\tan \varphi_i \neq \tan \varphi_j$ and, as we know, a kth degree polynomial is uniquely determined by its values at $k+1$ different points.

Simultaneously considering several moments, we get a system of equations for their coefficients and angles φ_i. As k grows, this system becomes highly overdefined which enables us to find all the angles φ_i.

3. Recovering the Angles $\varphi_2, \ldots, \varphi_n$ for an Asymmetric Particle and for $n \geq 7$

Since the center of mass of the projection $p_\varphi(x_1)$ is $x_1 = 0$, then $M_1(\varphi) = \int p_\varphi(x_1) x_1 \, dx_1 = 0$.

For asymmetric particles, the higher moments do not, in general, identically vanish. In what follows to avoid misunderstanding, we set

$$m_k(i) = \int \rho_{\varphi_i}(x_1) x_1^k \, dx_1 \equiv M_k(\varphi_i).$$

Consider the equations

$$\begin{aligned} &\lambda_{20} \cos^2 \varphi_i + \lambda_{11} \cos \varphi_i \sin \varphi_i + \lambda_{02} \sin^2 \varphi_i = m_2(i), \\ &\lambda_{30} \cos^3 \varphi_i + \lambda_{21} \cos^2 \varphi_i \sin \varphi_i + \lambda_{12} \cos \varphi_i \sin^2 \varphi_i + \lambda_{03} \sin^3 \varphi_i = m_3(i). \end{aligned} \tag{3}$$

Since $\varphi_i = 0$, then $\lambda_{20} = m_2(1)$, $\lambda_{30} = m_3(1)$ are known. We have obtained a system of $2(n-1)$ equations for $5+(n-1) = n+4$ variables λ_{11}, $\lambda_{02}, \lambda_{21}, \lambda_{03}, \varphi_2, \ldots, \varphi_n$. Therefore, for $n \geq 7$ we get an overdefinite system of equations, solving which we find $\varphi_2, \ldots, \varphi_n$.

Actually we can seek a minimum of nonnegative function

$$\begin{aligned} &\sum_{2 \leq i \leq n} (\lambda_{20} \cos^2 \varphi_i + \lambda_{11} \cos \varphi_i \sin \varphi_i + \lambda_{02} \sin^2 \varphi_i - m_2(i))^2 + \\ &+ \sum_{2 \leq i \leq n} (\lambda_{30} \cos^3 \varphi_i + \lambda_{21} \cos^2 \varphi_i \sin \varphi_i + \lambda_{12} \cos \varphi_i \sin^2 \varphi_i + \\ &+ \lambda_{03} \sin^3 \varphi_i - m_3(i))^2. \end{aligned}$$

It suffices to take $n = 7$. Then this is the function of 11 variables. For a justification of the assumption of overdefiniteness of the system (3) for large n, see the Appendix.

4. Finding the Angles φ_i when the Values of Two Angles are Known

4.1. Suppose we know the values of two more angles φ_2 and φ_3. In practice, to find these two angles we may (e.g. if our electron microscope is endowed with a gonyometer) turn the film with particles twice by angles φ_2 and φ_3 and then take their photos. Then, solving the system of linear equations

$$\lambda_{20} \cos^2 \varphi_i + \lambda_{11} \cos \varphi_i \sin \varphi_i + \lambda_{02} \sin^2 \varphi_i = m_2(i),$$

where $i = 1, 2, 3$ ($\varphi_1 = 0$) we find $\lambda_{20}, \lambda_{11}, \lambda_{02}$, i.e. $M_2(\varphi)$.

It is essential that, for this, we may take sufficiently small values of φ_2 and φ_3.

Clearly, $M_2(\varphi) = a_1 \cos 2(\varphi + \psi) + a_2$, where a_1, a_2, ψ are some parameters easily recovered from $\lambda_{20}, \lambda_{11}, \lambda_{02}$.

Let $\check{\varphi}_i^{(1)}$, $\check{\varphi}_i^{(2)}$, $\check{\varphi}_i^{(1)} + \pi$, $\check{\varphi}_i^{(2)} + \pi$ be the solutions of the equation

$$a_1 \cos 2(\varphi_i + \psi) + a_2 = m_2(i), \quad \mu \leqslant i \leqslant n. \tag{4}$$

We are interested in those solutions that satisfy the equations $M_k(\varphi_i) = m_i(i)$. Actually, it suffices to take only one equation, $M_3(\varphi_i) = m_3(i)$.

Note that since $M_3(\varphi + \pi) = -M_3(\varphi)$, then two of the four roots of Equation (4) may be immediately discarded. Denote the remaining two roots of $\varphi_i^{(1)}$ and $\varphi_i^{(2)}$. Let $M_3^{(1)}(\varphi)$ and $M_3^{(2)}(\varphi)$ be homogeneous trigonometric polynomials of degree 3 uniquely determined by the conditions

$$M_3^{(1)}(\varphi_i) = m_3(i) \quad \text{for } i = 1, 2, 3; \qquad M_3^{(1)}(\varphi_4^{(1)} = m_3(4)$$

and

$$M_3^{(2)}(\varphi_i) = m_3(i) \quad \text{for } i = 1, 2, 3; \qquad M_3^{(2)}(\varphi_4^{(2)}) = m_3(4),$$

respectively. Then we may find the values

$$M_3^{(1)}(\varphi_j^{(1)}), \quad M_3^{(2)}(\varphi_j^{(2)}), \quad M_3^{(2)}(\varphi_j^{(1)}), \quad M_3^{(2)}(\varphi_j^{(2)}), \quad 4 < j \leqslant n,$$

making use of Lagrange's interpolation formula for $M_3(\varphi)/\cos^2 \varphi$:

$$\begin{aligned} M_3(\varphi) = \cos^3 \varphi \Bigg[& \frac{(\tan \varphi - \tan \varphi_2)(\tan \varphi - \tan \varphi_3)(\tan \varphi - \tan \varphi_4)}{(\tan \varphi_1 - \tan \varphi_2)(\tan \varphi_1 - \tan \varphi_3)(\tan \varphi - \tan \varphi_4)} m_3(1) + \\ & + \frac{(\tan \varphi - \tan \varphi_1)(\tan \varphi - \tan \varphi_3)(\tan \varphi - \tan \varphi_4)}{(\tan \varphi_2 - \tan \varphi_1)(\tan \varphi_2 - \tan \varphi_3)(\tan \varphi_2 - \tan \varphi_4)} m_3(2) + \cdots \\ & + \frac{(\tan \varphi - \tan \varphi_1)(\tan \varphi - \tan \varphi_2)(\tan \varphi - \tan \varphi_3)}{(\tan \varphi_4 - \tan \varphi_1)(\tan \varphi_4 - \tan \varphi_2)(\tan \varphi_4 - \tan \varphi_3)} m_3(4), \end{aligned}$$

one of which should be equal (in practice, almost equal) to $m_3(j)$.

If, e.g., $M_3^{(2)}$ $(\varphi^{(1)}) = m_3(j)$, then $\varphi_4^{(2)}$ and $\varphi_j^{(1)}$ are 'genuine' angles, i.e. are exactly found. If necessary, we may similarly make use of the higher moments. For this we first construct homogeneous trigonometric 4th degree polynomials $M_4^{1,1}(\varphi)(M_4^{(1,2)}(\varphi), M_4^{(2,1)}(\varphi), M_4^{2,2}(\varphi))$ determined by the condition that it takes values $m_4(4)$, $m_4(5)$ at $\varphi = \varphi_4^{(1)}$, $\varphi_5^{(1)}$ (for $\varphi = \varphi_4^{(1)}$, $\varphi_4^{(2)}$; $\varphi = \varphi_4^{(2)}$, $\varphi_4^{(1)}$; $\varphi = \varphi_4^{(2)}$, $\varphi_4^{(2)}$) etc.

4.2. The procedure described in Section 4.1 for finding the angles $\varphi_4, \dots, \varphi_n$, admits the following geometric interpretation.

To a distribution function $\rho(x_1, x_2)$ assign the quadratic form

$$\begin{aligned} Q_\rho(x) &= \lambda_{20}x_1^2 + \lambda_{11}x_1x_2 + \lambda_{02}x_2^2 \\ &= \int \rho(y_1, y_2)(y_1x_1 + y_2x_2)^2\, \mathrm{d}y_1\, \mathrm{d}y_2 \geqslant 0 \end{aligned}$$

and consider the ellipse $Q_\rho(x) = 1$.

It is easy to verify that the ellipse coorresponding to $\rho_\varphi(x_1, x_2)$ is obtained from the ellipse corresponding to $\rho(x_1, x_2)$ under the rotation by φ counterclockwise. Note that from the projection $p_\varphi(x_1)$, the length of the section of the ellipse $Q_{\rho_\varphi}(x) = 1$ is recovered by the line $x_2 = 0$. The form of the ellipse is recovered from the angles φ_2 and φ_3 (the lengths of the ellipse's principal axes and their directions). For an asymmetric particle, the lengths of the principal axes of this ellipse are different, generally. Therefore, the ellipse $Q_{\rho_\varphi}(x) = 1$ is determined after a reflection with respect to the line $x_2 = 0$.

5. Direct Recovering of a Function $\rho(x)$ with Compact Support from the Moments of its Projections

5.1. Recall that the (s, l)th moment of a function $\rho(x_1, x_2)$ is

$$M_{s,l}(\rho) = \int \rho(x_1, x_2)x_1^s x_2^l\, \mathrm{d}x_1\, \mathrm{d}x_2 \tag{5}$$

By formula (1), the kth moments of the projections, $M_k(\varphi)$, define and, in turn, are defined by the numbers $M_{s,l}(\rho)$, where $s + l = k$. In particular, since $M_k(\varphi)$ is a trigonometric polynomial of degree k and, therefore, is determined by its values at $k + 1$ points, then we may recover all the moments $M_{s,l}(\rho)$ with $s + l \leqslant k$ from the $k + 1$ projections.

Let us show how knowing all the moments $M_{s,l}(\rho)$ find a function $\rho(x_1, x_2)$ with compact support. For simplicity, assume that the ρ's support belongs to the interior of the square $-1 \leqslant x_1, x_2 \leqslant 1$. The general case is reduced to this one by the change of a scale.

5.2. RECAPITULATION

Legendre polynomials (see [8, Ch. II, Section 8]).

The nth (normed) Legendre polynomial is for $n \geqslant 1$

$$\begin{aligned}\varphi_n(x) &= \sqrt{\frac{2n+1}{2}}\frac{1}{2^n n!}\frac{d^n(x^2-1)^n}{dx^n} \\ &= \sqrt{\frac{2n+1}{2}}\frac{1}{2^n n!}\sum_{\substack{0\leq k\leq n\\ 2k-n\geq 0}}(-1)^{n-k}\binom{n}{k}\frac{(2k)!}{(2k-n)!}x^{2k-n}\end{aligned} \tag{6}$$

and $\varphi_0(x)=\sqrt{1/2}$.

The Legendre polynomials are obtained after orthonormalizing the sequence $1, x, x^2, \ldots$ on the segment $[-1, 1]$. Their characteristic feature is the following:

$$\int_{-1}^{1}\varphi_{k_1}(x)\varphi_{k_2}(x)\,dx = 1 \quad \text{for } k_1=k_2,\ 0 \text{ otherwise.}$$

Any partially-smooth function $\rho(x_1, x_2)$ on the square $-1\leq x_1, x_2\leq 1$ expands into the series

$$\rho(x_1, x_2) = \sum_{s,l\geq 0} a_{s,l}\varphi_s(x_1)\varphi_l(x_2)$$

converging to this function, where $a_{s,l}$ are given by the formula

$$a_{s,l} = \int_{-1}^{1}\int_{-1}^{1}\rho(x_1, x_2)\varphi_s(x_1)\varphi_l(x_2)\,dx_1\,dx_2\,. \tag{7}$$

Now, note that knowing the moments $M_{s,l}(\rho)$ for $s+l\leq k$, we may find $a_{s,l}$ for $s+l=k$ by formulas (5)–(7). In particular, knowing all the moments $M_{s,l}(\rho)$ we may determine $\rho(x_1, x_2)$.

5.3. As a corollary, note that we have obtained a new proof of the known Solmon's theorem (see [9, Proposition 7.5]): an object is determined by any infinite number of Röntgenogrammes, i.e., a function with compact support is uniquely recovered from any infinite number of the function's projections.

It is essential that, unlike any of the earlier proofs based on the analytic continuation and, therefore, absolutely inapplicable in practice, our proof is strictly constructive. Knowing a finite number of moments $M_{s,l}(\rho)$, $s+l\leq k$, we may construct a polynomial

$$\sum_{s+l\leq k} a_{s,l}\varphi_s(x_1)\varphi_l(x_2)$$

approximating $\rho(x_1, x_2)$.

This method is worth applying to problems with a small number of known projections and where high accuracy in recovering the object is not needed, e.g., in problems of the three-dimensional reconstruction in electron microscopy of biomolecules.

The main advantage of this method is that we do not put any constraints on the directions of projecting. For instance, the angles φ determined by these projections with the abscissa axis vary in the interval $\varphi_0 \leqslant \varphi \leqslant \varphi_0 + \epsilon$, where ϵ is small. (This is exactly the case often encountered in practice.)

6. Appendix. Multi-Dimensional Generalization and a Sketch of Justification of the Method. Recovering a Mutual Arrangement of *k*-Dimensional Subspaces of an *N*-dimensional Space from the Projections of an (Unknown) Function onto these Subspaces

6.1. In [2, 3] the relations among the moments of projections are found for the Radon transformations in the N-dimensional space (Paley–Wiener's theorem). Namely, if

$$\check{f}(\omega, p) = \int f(x)\delta(\langle x, \omega\rangle - p)\, \mathrm{d}^N x,$$

where $\omega_1^2 + \cdots + \omega_N^2 = 1$, is the projection of a function $f(x_1, \ldots, x_N)$ onto the line with the directing vector ω, then the kth moment of the projection

$$M_k(\omega) = \int \check{f}(\omega, p) p^k \, \mathrm{d}p$$

is a kth degree polynomial in coordinates $\omega_1, \ldots, \omega_N$ of ω.

Paley–Wiener's theorem enables one to investigate the following problem adjacent to the problem of recovering mutual orientation of particles from their projections.

Given projections of an unknown function $f(x_1, \ldots, x_N)$ in the N-dimensional space onto k-dimensional subspaces $\pi_1, \ldots, \pi_n$, find mutual arrangements of these subspaces, i.e., find the set $\pi_1, \ldots, \pi_n$ up to a rotation of the N-dimensional space.

The most nontrivial case is $k = 1$. Let us consider it in detail. Given n functions $\check{f}(\omega^{(1)}, p), \ldots, \check{f}(\omega^{(n)}, p)$ of one variable p, find the configuration of vectors $\omega^{(1)}, \ldots, \omega^{(n)}$.

For this consider, the system of equations for coordinates of $\omega^{(1)}, \ldots, \omega^{(n)}$ and coefficients of polynomials $M_k(\omega)$, where $2 \leqslant k \leqslant k(N)$ ($k(N)$ is determined from N):

$$M_k(\omega^{(i)}) = \int \check{f}(\omega^{(i)}, p) p^k \, \mathrm{d}p. \tag{8}$$

The configuration of vectors $\omega^{(1)}, \ldots, \omega^{(n)}$ is defined up to a rotation of the N-dimensional space. To get rid of this arbitrariness, assume that

$$\omega_t^{(s)} = 0 \quad \text{for } t > s \quad \text{and} \quad \omega_s^{(s)} \geqslant 0, \quad \text{where } s = 1, \ldots, N.$$

Then the system (8) which, for a sufficiently large n becomes over-defined, has one solution which can be actually found by the least-squares method. The justification of this statement will be carried out in Subsection 6.2 in the case most important from the practical point of view: $N = 2$. In this case, system (8) turns into system (3) of Section 3. Note that though in this system the number of equations exceeds the number of unknowns, this is not sufficient to prove the uniqueness of the solution. In fact, considering φ_1 as an unknown quantity (i.e. without assuming $\varphi_i = 0$), the number of equations in system (3) becomes, with the growth of n, greater than the number of unknowns but the solution is clearly not unique: it is defined up to a change of φ_i by $\varphi_i + \psi$, where ψ is an arbitrary angle.

6.2. Consider the curve S on the plane (y_1, y_2) determined by the formulas

$$y_1 = M_2(\sin\varphi, \cos\varphi), \qquad y_2 = M_3(\sin\varphi, \cos\varphi), \tag{9}$$

where $M_2(\xi_1, \xi_2)$, $M_3(\xi_1, \xi_2)$ are homogeneous polynomials of degrees 2 and 3, respectively. As follows from the basics of algebraic geometry the degree of S is 6, and if S_1 and S_2 are two curves of the form (9), then (by Bezout's theorem) either S_1 coincides with S_2 or their intersection contains no more than deg $S_1 \cdot$ deg $S_2 = 36$ points.

Making use of the peculiarity of the problem, we may show that we may replace 36 by a considerably lesser number, 7 (cf. Section 3). However, a qualitative deduction is more interesting: if S_1 and S_2 have sufficiently many common points, they coincide. This means that knowing sufficiently many moments

$$\int \check{f}(\omega^{(2)}, p)p^2\,\mathrm{d}p, \qquad \int \check{f}(\omega^{(i)}, p)p^3\,\mathrm{d}p,$$

we may uniquely draw the curve (9) on the plane (y_1, y_2). Knowing the curve, we may find the graph of the function $y_1 = M_2(\sin\varphi, \cos\varphi)$ up to a shift $\varphi \mapsto \varphi + \psi$.

In fact, let $y_{\max}$ and $y_{\min}$ be the maximal and minimal values of the coordinate y_1 of the points of S.

Since

$$y_1 = M_2(\sin\varphi, \cos\varphi) = a_1 \cos 2(\varphi + \psi) + a_2,$$

then

$$a_1 = (y_{\max} - y_{\min})/2, \qquad a_2 = (y_{\max} + y_{\min})/2,$$

i.e., the graph of the function $y_1 = M_2(\sin\varphi_1 \cos\varphi)$ is found up to a shift.

The condition $\varphi = 0$ allows us to determine four possible values of ψ. Introducing the third moment, $M_3(\varphi)$, we find the desired value of ψ (this is justified by the same arguments as in Section 4).

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