INFINITESIMAL STRUCTURES RELATED TO HERMITIAN SYMMETRIC SPACES

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UDC 517.43

0. Let M be a compact, Hermitian, symmetric space (c. H. s. s.) of rank rk M > 1. In each tangent space $T_xM\ \forall x \in M$, we shall construct in a canonic way a cone K_x, such that the cones K_x corresponding to distinct points x are linearly equivalent. Given any domain on M, the local diffeomorphisms which preserve this family of cones can be extended to holomorphic automorphisms of M. On the n-dimensional complex manifold $\mathcal{M}$, we give a cone $\mathcal{K}_y \subset T_y\mathcal{M}$, $y \in \mathcal{M}$, which depends holomorphically on y and is linearly equivalent to the cones K_x. At the same time, M is distinguished as a manifold with a flat structure.

If $M = \mathbf{C}G_2^2$, then the cones K_x are the cones of the null directions of a conformal class of complex metrics. The study of this infinitesimal structure, both in the flat and curved cases, plays an essential role in R. Penrose's theory of twistors. Part of the results of the present work may be interpreted as a generalization to higher dimensions of Penrose's results.

We remark that our work is related to integral geometry (see [1, 4, 5]).

The problem of characterizing the infinitesimal structures of c. H. s. s. was formulated by S. G. Gindikin. The author is sincerely grateful to S. G. Gindikin for his great interest in this work, and for the useful discussions and suggestions.

1. Let us introduce some notation: L is a semisimple Lie group over **C**; P is a parabolic subgroup of L with Levi decomposition P = GN and such that M = L/P is a c. H. s. s., which holds if and only if the radical N is Abelian; G_0 is the semisimple part of G; $\mathcal{L}, \mathcal{P}, \mathcal{G}, \mathcal{G}_0$ are the corresponding Lie algebras; $P_x = G_xN_x$ is the stability subgroup at the point $x \in M$.

Definition. Let K_x be the cone of highest weight vectors in the G_x-module T_xM (i.e., each vector of K_x is the highest relative to some Borelian subgroup of G_x).

Proposition 1. The group of all linear automorphisms of the cone K_x equals G_x. The number of nonzero G_x-orbits in T_xM is rk M. K_x is minimal among these orbits, and $l \in L$ takes K_x into K_{lx}.

Therefore, we have associated to M, in a correct way, a cone $K(M) \subset V \simeq \mathbf{C}^n$, where n = dim M. From now on, we shall assume that L is simple (the general case reduces to this one) and that rk M > 1. Below we give the concrete realizations of K(M).

1. $M = \mathbf{C}G_n^m$. $T_xM \simeq \mathbf{C}^m \otimes \mathbf{C}^n$, $K(M) \simeq \{v \otimes w \in \mathbf{C}^m \otimes \mathbf{C}^n\}$.
2. $M = SO(n+2)/SO(n) \times SO(2)$ is a quadric in $\mathbf{C}P^{n+1}$, K(M) is a nondegenerate quadratic cone.
3. M = SO(2n)/U(n) is the connected component of the manifold of maximal isotropic subspaces relative to a nondegenerate complex metric on $\mathbf{C}^{2n}$ $K(M) \simeq \{v \wedge w \in \Lambda^2\mathbf{C}^n\}$.
4. M = Sp(n)/U(n) (the Lagrangian Grassmanian). $K(M) \simeq \{v \cdot v \in S^2\mathbf{C}^n\}$.
5. $M = E_6/\mathbf{C}O(10)$ (the complexification of the projective plane over the octonions). K(M) is isomorphic to the cone of simple half-spinors in the half-spinorial representation.
6. $M = E_7/E_6 \cdot U(1)$. K(M) is isomorphic to the cone over the irreducible idempotent elements in the Jordan algebra constructed from the complexified Hermitian matrices of order 3 over the octonions).

Let γ be the vertex of the Dynkin diagram of $\mathcal{L}$ such that the deletion of γ results in the diagram of $\mathcal{G}$. In the fundamental representation corresponding to γ, consider the cone K_γ of highest weight vectors. For $M = \mathbf{C}G_m^n$, K_γ is the cone of decomposable m-vectors, while for M = SO(2n)/U(n), K_γ is the cone of simple half-spinors in the sense of E. Cartan's definition, in the half-spinorial representation (see [2]).

Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 15, No. 3, pp. 83-84, July-September, 1981. Original article submitted March 19, 1981.

0016-2663/81/1503-0221$07.50

Proposition 2. K_γ is a cone over M. The intersection of K_γ with the subspace tangent to K_γ along the generatrix $C^*\cdot x$, where $x \in M$, is a cone over $K_x \subset T_xM$.

Using this proposition and induction, we are able to describe the connected components of the manifold of isotropic subspaces (we call them families).

Proposition 3. There is a bijective correspondence between the families in K(M) and those subgraphs of the Dynkin diagram of $\mathscr{L}$ which contain γ and are isomorphic to the diagram A_m (chains). The dimension of an isotropic subspace equals the number of vertices in the corresponding chain. The group G acts transitively on each family.

If $M = \mathbf{C}G^n_m$, we say that a given isotropic subspace has type α (respectively, β) if it is contained in a maximal isotropic subspace of the form $\mathbf{C}^m \otimes w$ (respectively, $(v \otimes \mathbf{C}^n)$).

Now let us give, on the n-dimensional complex manifold $\mathscr{M}$, a cone $\mathscr{K}_y \subset T_y\mathscr{M}$, $y \in \mathscr{M}$ such that $\mathscr{K}_y$ depends holomorphically on y and there exists a $\mathbf{C}$-linear isomorphism $A_y\colon V \to T_y\mathscr{M}$ with $A_y(K(M)) = \mathscr{K}_y$. Composing the transformations from G with the isomorphisms A_y, we obtain a G-structure on $\mathscr{M}$. Set $\mathscr{G}^{(-1)} = V$, $\mathscr{G}^{(0)} = \mathscr{G} \subset \mathscr{GL}(V)$, $\mathscr{G}^{(k)} = S^2V^* \otimes \mathscr{G}^{(k-2)} \cap V^* \otimes \mathscr{G}^{(k-1)}$. A Lie algebra structure can be introduced on $\oplus \mathscr{G}^{(k)}$ (see [3]).

Proposition 4. If rk M > 1, then $\mathscr{G}^{(1)} \simeq V^*$, $\mathscr{G}^{(2)} = 0$, $\mathscr{L} \simeq \mathscr{G}^{(-1)} \oplus \mathscr{G}^{(0)} \oplus \mathscr{G}^{(1)}$, $\mathscr{P} \simeq \mathscr{G}^{(0)} \oplus \mathscr{G}^{(1)}$, $\mathscr{G}^{(1)}_0 = 0$.

Consider a domain in $\mathbf{C}^n$ upon which there is given a flat (in the sense of G-structure) family of cones. Then it follows from Proposition 4 that the Lie algebra of the local Lie group of those diffeomorphism which preserve the given family of cones is isomorphic to $\mathscr{L}$. If $G = \mathbf{C}O(n)$, $n \geq 3$, this is Liouville's theorem. The algebra $\mathscr{P}$ is recovered as the subalgebra of $\mathscr{L}$ preserving a point.

Proposition 5. A flat G-structure is induced on M.

This is a consequence of the fact that N is Abelian.

2. Set

$$C^{k,l} = \operatorname{Hom}(\Lambda^l V, \mathscr{G}^{(k-1)}),$$
$$\partial_l f(v_1, \ldots, v_{l+1}) = \sum_{i=0}^{l} (-1)^i f(v_1, \ldots \widehat{v_{l+1-i}} \ldots, v_{l+1})(v_{l+1-i}),$$

where $f \in C^{k,l}$. Then $\partial_l f \in C^{k-1,\, l+1}$, and $\partial_l \circ \partial_{l-1} = 0$. Let $H^{k,l} = \operatorname{Ker} \partial_l \cap C^{k,l} / \operatorname{Im} \partial_{l-1} \cap C^{k,l}$. The obstruction to identifying the (k + 1)-th infinitesimal neighborhood of a point in a manifold with G-structure, with the (k + 1)-th infinitesimal neighborhood of a point in a G-flat manifold, is given by the k-th-order structure functions (s. f.) (see [3, 7]) under the assumption of annihilation of the lower order s. f. This obstruction takes values in the G-module $H^{k-1,2}$. For the conformal structure ($G = \mathbf{C}O(n)$), the first-order s. f., i.e., torsion, is equal to zero because $H^{0,2} = 0$ for $\mathbf{C}O(n)$, while the second-order s. f. is the Weyl tensor.

THEOREM 1. a) $M \neq \mathbf{C}G^n_m \Rightarrow$ the $\mathscr{G}$-module $\bigoplus_k H^{k,2}$ is irreducible. $H^{0,2} \neq 0 \Leftrightarrow \mathscr{G} \neq co(n)$. $H^{1,2} \neq 0 \Leftrightarrow \mathscr{G} = co(n)$ $n > 4$. $H^{2,2} \neq 0 \Leftrightarrow \mathscr{G} = co(3)$.

b) $M = \mathbf{C}G^n_m \Rightarrow \bigoplus_k H^{k,2} = H_+ \oplus H_-$ – the self-dual and anti-self-dual parts, which corresponds to the decomposition

$$\Lambda^2(\mathbf{C}^m \otimes \mathbf{C}^n) = \Lambda^2_+ \oplus \Lambda^2_- = S^2\mathbf{C}^m \otimes \Lambda^2\mathbf{C}^n \oplus \Lambda^2\mathbf{C}^m \otimes S^2\mathbf{C}^n,$$

H_+ and H_- are irreducible. $H^{0,2}_+ (H^{0,2}_-) = 0 \Leftrightarrow m = 2$ $(n = 2)$. $H^{0,2}_\pm = 0 \Leftrightarrow H^{1,2}_\pm \neq 0$.

Definition. A submanifold $\mathscr{N} \subset \mathscr{M}$ such that $T_x\mathscr{N} \subset \mathscr{K}_x$, $\forall x \in \mathscr{N}$, is called an integral manifold in $\mathscr{M}$

THEOREM 2. Let M be irreducible, rk M > 1, and M ≠ Sp (n) / U (n).

a) If $M \neq \mathbf{C}G^n_m$, then the G-structure on $\mathscr{M}$ is flat ⇔ given any family of isotropic subspaces of dimension >1 and any subspace in the family, there exists an integral manifold tangent to this subspace.

b) If $M = \mathbf{C}G^n_m$, then the (anti) self-dual part of the s. f. vanishes ⇔ given any family of isotropic subspaces of dimension >1 and type α (respectively, β) and any subspace in the family, there is an integral manifold tangent to this subspace. If both parts of the s. f. vanish, then the G-structure on $\mathscr{M}$ is flat.

3. Let us fix a decomposition $\operatorname{Hom}(V \wedge V, V) = C \oplus \partial_1(\operatorname{Hom}(V, \mathscr{G}))$. Since $\mathscr{G}$ is reductive, there is a canonical choice for C. For the G_0-structure, there is a canonical G_0-connection on $\mathscr{M}$. If $G_0 = SO(n)$, then the latter is

the Levi–Civita connection. While there is no canonical G-connection on $\mathcal{M}$, one may define a Cartan connection (about Cartan connections see [7]). One may also define O-geodesics on $\mathcal{M}$.

Consider the bundle $\pi: \mathcal{M}_0 \to \mathcal{M}$ whose fiber $\pi^{-1}(x)$ is the projectivization of the cone $\mathcal{K}_x$.

THEOREM 3. The choice of the complement C determines a field of directions $l(y)$, $y \in \mathcal{M}_0$, and $l(y)$ projects isomorphically into a generatrix of the cone $\mathcal{K}_{\pi(y)}$.

The O-geodesics are precisely the projections of the integral curves of this field of directions.

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A REMARK ON EXTENSION OF MEASURES

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Let $(\Omega, \mathfrak{A})$ be a measurable space, $\mathfrak{B}$ be a σ-subalgebra of the σ-algebra $\mathfrak{A}$, and μ_0 be a σ-additive measure defined on $\mathfrak{B}$. Suppose that there exists a σ-homomorphism h of the σ-algebra $\mathfrak{A}$ onto the algebra $\mathfrak{B}/\mu_0$ of the μ_0-equivalence classes that is an extension of the canonical σ-homomorphism g_0 of the σ-algebra $\mathfrak{B}$ onto $\mathfrak{B}/\mu_0$. Then a σ-additive measure μ that is defined on $\mathfrak{A}$ and extends μ_0 can be put in correspondence with the measure μ_0. Indeed, for this it is sufficient to set $\mu(A) = \mu_0(A')$ for $A \in \mathfrak{A}$, where A' is an arbitrary representative of the class $g_0^{-1}(h(A))$.

It is clear that not every extension μ of the measure μ_0, if it exists, has the above form. Nevertheless, the existence of at least one extension of the measure μ_0 to a σ-additive measure defined on $\mathfrak{A}$ implies the existence of a σ-homomorphism $h: \mathfrak{A} \to \mathfrak{B}/\mu_0$ that extends the σ-homomorphism g_0. The present communication is devoted to a proof of this statement.

1. A well-known theorem of Sikorski (see [1, Theorem 33.1]) asserts that complete Boolean algebras are injective objects in the category of Boolean algebras with homomorphisms as morphisms of the category. This theorem becomes invalid when the category of σ-algebras with σ-homomorphisms (see [2, Example]) or the category of complete algebras with complete homomorphisms (see [3, Theorem 4]) is considered. However, Sikorski's theorem remains valid for the category of the σ-algebras that satisfy the σ-chain condition (and, by the same token, for the category of the complete Boolean algebras) with σ-homomorphisms. More precisely, each object of this category is injective.

In order to show this, we introduce the following notation. Let X and Y be the Stone spaces of Boolean algebras $\mathfrak{C}$ and $\mathfrak{D}$, respectively, and h_0 and h_0' be isomorphisms of the algebra $\mathfrak{C}$ onto the algebra $\Phi(X)$ of the clopen subsets of X and of the algebra $\mathfrak{D}$ onto the algebra $\Phi(Y)$ of the clopen subsets of Y, respectively. Further, let h be a homomorphism of the algebra $\mathfrak{C}$ into the algebra $\mathfrak{D}$. It defines a homomorphism h' of the algebra $\Phi(X)$ into the algebra $\Phi(Y)$ in a natural manner; namely, $h'(F) = h_0' h h_0^{-1}(F)$ for each $F \in \Phi(X)$. By virtue of Theorem 11.1 and a remark in [1], this homomorphism is induced by a continuous mapping ψ of the space Y into X, and, by definition, $\psi^{-1}(F) = h'(F)$ for each $F \in \Phi(X)$.

Voroshilovgrad Machine-Building Institute. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 15, No. 3, pp. 85-86, July-September, 1981. Original article submitted August 14, 1979; revision submitted December 2, 1980.