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INTEGRAL GEOMETRY AND MANIFOLDS OF MINIMAL DEGREE IN $\mathbf{CP}^n$

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1. INTRODUCTION

1. An n-parameter family of submanifolds $B_\xi \subset B$, $\dim B = n$, is said to be admissible if the value of any smooth function f at each point x can be reconstructed, knowing only the integrals of f over the submanifolds of the family passing through an infinitesimal neighborhood of the point x. (A rigorous definition will be given in Sec. 2.)

The classical example is the family of all hyperplanes in $\mathbf{R}^{2n+1}$ or $\mathbf{C}^n$. Its admissibility follows from the locality of the inversion formula for the Radon transformation (cf. Sec. 2).

The goal of this paper is to construct a large class of admissible families of hypersurfaces. In Sec. 7 we prove that in this way one gets all admissible families of curves on algebraic surfaces up to birational isomorphism. Explicit local inversion formulas are obtained.

2. We recall that if X is a submanifold in $\mathbf{CP}^n$, which does not lie in a hyperplane (nondegenerate submanifold), then

$$\deg X \geqslant \operatorname{codim} X + 1, \tag{1}$$

where $\deg X$ is the number of points of intersection of X with a generic plane of complementary dimension. Indeed neither the degree nor the codimension changes under passage to a hyperplane section so that arguing by induction one can assume that $\dim X = 0$. In this case X is a collection of points not lying in any hyperplane.

In 1885 geometer Federigo Enriques discovered that all nondegenerate irreducible submanifolds for which equality holds in (1) can be simply and beautifully described ([12], cf. also Sec. 3).

Example 1.1. a) Let X_d be an irreducible nondegenerate curve of degree d in $\mathbf{CP}^d$. Then it is projectively equivalent to the Veronese curve $(x_0 : x_1) \mapsto (x_0^d : x_0^{d-1}x_1 : \ldots : x_1^d)$ (it is also called a rational normal curve [11, p. 196]).

b) Del Pezzo proved [11, p. 561] that any irreducible nondegenerate surface of degree $n-1$ in $\mathbf{CP}^n$ is either a Veronese surface

$$(x_0 : x_1 : x_2) \mapsto (x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2) \tag{2}$$

in $\mathbf{CP}^5$, or a surface S_k constructed as follows:

We take two Veronese curves lying in crossing planes of dimensions k and $n-k-1$ in $\mathbf{CP}^n$ and we establish an isomorphism between them. The surface S_k consists of lines joining

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corresponding points of these curves (for $k = 0$, $n = 2$ and $k = 1$, $n = 3$ one gets CP^2 and a quadric in CP^n).

From now on, taking some liberties with language, we shall call nondegenerate, irreducible manifolds for which equality holds in (1), manifolds of minimal degree in CP^n.

THEOREM A. A family consisting of hyperplane sections of an n-dimensional manifold of minimal degree $X^n \subset CP^n$, tangent to the algebraic submanifolds $M_1, \ldots, M_{N-n}$ is admissible.

Remark 1.2. The only restriction on the submanifolds M_i is that the family defined by them should depend on n parameters. This is always so for generic M_i because tangency with a submanifold is a condition of codimension 1 for hypersurfaces.

We note that until now only separate examples of admissible families of hypersurfaces of dimension greater than 1 were known, namely the Radon transform, the horospherical transformation in Lobachevskii space [3], and all admissible complexes of quadrics in CP^3 [9].

Example 1.3. a) $X^n = CP^n$ $(N = n)$. Then "hyperplane sections" are simply hyperplanes in CP^n.

b) X^n is a quadric $Q_n \subset CP^{n+1}$. In this case the family of all hyperplane sections is birationally isomorphic to the family of all spheres $\sum_{i=1}^{n} (z_i - a_i)^2 = r^2$ in C^n.

In order to see this we consider the stereographic projection $\pi_x: Q_n \to CP^n$ from the point $x \in Q_n$. We recall that from the projective point of view the sphere is a quadric in CP^n, containing the quadric $\sum_{i=1}^{n} z_i^2 = 0$ in the hyperplane at infinity $(z_1 : \ldots : z_n)$. Hence if H_x is a hyperplane in CP^{n+1}, tangent to Q_n at the point x, then π_x is regular outside the cone $H_x \cap Q_n$, and the complement of its vertex projects to a hyperplane quadric in CP^n. Consequently, the projection of the hyperplane section Q_n contains it.

c) The map (2) transforms conics into hyperplane sections of a Veronese surface.

Following the classics, we shall call n-parameter families of submanifolds of an n-dimensional manifold complexes.

In Sec. 5 we prove that the complexes described in Theorem A are precisely all admissible complexes in general position consisting of hyperplane sections of manifolds of minimal degree in CP^n. Moreover, we construct all others.

Up to now we have defined admissible families of submanifolds $B_\xi \subset B$, where ξ belongs to a manifold of parameters Γ only in the case when $\dim B = \dim \Gamma$. Although the same definition makes sense for $\dim B < \dim \Gamma$ admissibility in this case should be defined differently, imposing a considerably more stringent condition: the existence of a universal local inversion formula (cf. [1] and point 1 of Sec. 5).

THEOREM B. a) The family of hyperplane sections of a surface of minimal degree in dim × B < dim Γ tangent to the algebraic curves CP^n, tangent to the algebraic curves $M_1, \ldots, M_k$ on it with multiplicities $c_1, \ldots, c_k$, where $c_1 + \ldots + c_k \leqslant n - 2$ is admissible.

b) Any admissible family of irreducible curves in the category of algebraic manifolds is birationally isomorphic to a family from point a).

c) Families from point a) are birationally isomorphic if and only if they are isomorphic.

The study of admissible families of reducible curves reduces to the case of irreducible curves (cf. Lemma 7.1).

Thus, any admissible family of curves on an algebraic surface has a canonical realization by hyperplane sections of either a surface S_k in CP^n, or a Veronese surface in CP^5 uniquely determined up to isomorphism.

In connection with Theorem B one should stress that admissibility is a purely local concept. Hence it is natural to classify admissible families at least up to local isomorphism. In the category of algebraic manifolds this is classification up to birational isomorphism.

In Sec. 6 we give the following integral-geometric characterization of manifolds of minimal degree in CP^n.

THEOREM C. Let us assume that the family consisting of hyperplane sections of an irreducible manifold $X \subset \mathbb{C}P^n$, tangent to codim X generic submanifolds is admissible. Then X is a manifold of minimal degree.

3. Differential-Geometric Applications. Let $\{B_\xi\}$ be a 2-parameter family of curves on the surface B, $\xi \in \Gamma$. Then a generic point $x \in B$ defines a curve Γ_x on the surface of parameters Γ whose points parametrize the curves of the family $\{B_\xi\}$ passing through x, i.e., $\Gamma_x := \{\xi \in \Gamma \mid x \in B_\xi\}$. We have obtained a 2-parameter family $\{\Gamma_x\}$ of curves in Γ called dual to the original one.

In [5], Gel'fand, Gindikin, and Shapiro, using a weaker condition for admissibility (which we shall call formal admissibility), which in return makes sense for families of real curves too, proved the following remarkable theorem.

THEOREM 1.4 [5]. A complex of curves on a surface is formally admissible if and only if it is the dual complex to the geodesics of an affine connection.

We call the family of geodesics of an affine connection algebraic if it is a family of curves in the category of algebraic manifolds.

THEOREM 1.5. a) For any of the complexes described in point a) of Theorem B the dual family consists of the geodesics of a connection.

b) This construction gives all algebraic families of geodesics up to birational isomorphism.

A conformal manifold is called a Weyl–Einstein manifold if there exists an affine connection of zero curvature on it which preserves the conformal structure, whose Ricci tensor $R_{(ij)}$ is proportional to the metric: $R_{(ij)} = \Lambda(x)g_{ij}$.

Using the idea of [1, 4] one can prove the following generalization of the theorem of Gel'fand–Gindikin–Shapiro.

THEOREM 1.6. a) Three-dimensional manifolds parametrizing formally admissibly families of curves on surfaces are canonically endowed with a Weyl–Einstein manifold structure.

b) All Weyl–Einstein manifolds are obtained in this way.

See [4] about the Weyl–Einstein equations and their twistor interpretation.

We call a solution of the Weyl–Einstein equations algebraic if the family of curves corresponding to it according to Theorem 1.6 is algebraic. If we assume in point a) of Theorem B that $c_1 + \ldots + c_k = n - 3$ then the three-dimensional manifolds parametrizing this family are all algebraic solutions of the Weyl–Einstein equations.

4. Content of the Paper. Theorem A is proved in Secs. 3-5. First of all we need an explicit description of all manifolds of minimal degree in $\mathbb{C}P^n$ (Sec. 3). In Sec. 4 we introduce the concept of composition of double bundles, prove that the composition of admissible double bundles is admissible, and decompose the double bundle corresponding to the family of hyperplane sections of a generic manifold of minimal degree into a composition of two admissible ones. In Sec. 5 we give explicit local inversion formulas.

Theorem C is proved in Sec. 6. In Sec. 7 we prove Theorem B. I thank S. G. Gindikin for his interst in the work and S. L. Tregub for helpful discussions.

2. DEFINITION OF ADMISSIBILITY

1. It is natural to talk about families of submanifolds in the language of double bundles. We recall that by a double bundle is meant a diagram of manifolds

$$\begin{tikzcd} & A \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ B & & \Gamma \end{tikzcd}, \qquad (3)$$

for which $\pi_1 \times \pi_2 \colon A \to B \times \Gamma$ is an imbedding. For $x \in B$ and $\xi \in \Gamma$ we set $B_\xi := \pi_1 \circ \pi_2^{-1}(\xi)$, $\Gamma_x := \pi_2 \circ \pi_1^{-1}(x)$.

Thus, a double bundle defines a family $\{B_\xi\}$ of submanifolds of B and also the family $\{\Gamma_x\}$ of submanifolds of Γ dual to it. Conversely, a family of submanifolds $\{B_\xi\}$ of B defines the incidence submanifold $A := \{(x, \xi) \subset B \times \Gamma \mid x \in B_\xi\}$. Its projection to the factors defines a double bundle.

2. We choose a density μ_ξ on B_ξ and define an integration operator

$$I\colon C_0^\infty(B) \to C^\infty(\Gamma);\quad I\colon f(x) \mapsto \int_{B_\xi} f(x)\,\mu_\xi.$$

In other words, the Schwartz kernel of the operator I is a distribution of the form $\mu(x, \xi)\delta(A)db$ on $B \times \Gamma$, where $\delta(A)$ is the δ-function of the submanifold $A \subset B \times \Gamma$, db is the volume element on B, $\mu(x, \xi)$ is a function on A.

By local inversion formula for the integral transform I we shall mean the inverse operator J: $C^\infty(\Gamma) \to C^\infty(B)$ whose Schwartz kernel has the form $L\delta(A)d\gamma$, where L is a differential operator in a neighborhood of A.

One can rewrite the local operator J in the form $J\varphi(x) = \int_{\Gamma_x} (L_x\varphi)\,\nu_x$, where L_x is a differential operator in a neighborhood of Γ_x and ν_x is a density on Γ_x.

If $\dim B_\xi$ is odd, then there are never local inversion formulas. Hence we shall as a rule work in the category of complex-analytic manifolds although as before we shall integrate smooth functions. For example, the Radon transform in $\mathbf{C}^n$ looks as follows:

$$\varphi(a) := \int f(z)\,\delta\Big(z_n - \sum_{i=1}^{n-1} a_i z_i - a_0\Big)\, d^n z\, d^n \bar{z}$$

and has a local inversion formula for any n (c_n are constants):

$$f(z) = c_n \cdot \int \varphi(a)\,\delta^{(n-1,\,n-1)}\Big(z_n - \sum_{i=1}^{n-1} a_i z_i - a_0\Big)\, d^n a\, d^n \bar{a}.$$

If the integral transform I is invertible, then clearly $B \leqslant \dim \Gamma$.

Definition 2.1. Let $\dim B = \dim \Gamma$. A family of submanifolds $\{B_\xi\}$ is called admissible if there exist (k, 0)-forms μ_ξ on B_ξ such that the integral transform

$$I\colon C_0^\infty(B) \to C^\infty(\Gamma),\quad I\colon f(x) \mapsto \int_{B_\xi} f(x)\,\mu_\xi \bar{\mu}_\xi,$$

admits a local inversion formula with Schwartz kernel of the form $L\bar{L}\delta(A)d\gamma d\bar{\gamma}$, where L is a holomorphic differential operator in a neighborhood of A and $d\gamma$ is an (n, 0)-form on Γ.

3. MANIFOLDS OF MINIMAL DEGREE IN $\mathbf{C}P^n$

1. Let $d_1, \ldots, d_r$ be nonnegative integers, $d = d_1 + \ldots + d_r + r - 1$. We take r planes $H_1, \ldots, H_r$ in $\mathbf{C}P^d$ in general position, $\dim H_i = d_i$. Let $X_{d_i} \subset H_i$ be a Veronese curve in H_i. We fix isomorphisms between X_{d_r} and the remaining curves: $f_i\colon X_{d_r} \simeq X_{d_i}$.

For $x \in X_{d_r}$ we consider the (r − 1)-dimensional plane spanned by the points x, $f_1(x), \ldots, f_{r-1}(x)$. When x runs through the whole curve X_{d_r} these planes sweep out an r-dimensional submanifold in $\mathbf{C}P^d$ (possibly singular) which we denote by $X_{(d_1, \ldots, d_r)}$, or for short, $X_{\vec{d}}$ (cf. Fig. 1).

LEMMA 3.1.

Proof. A generic hyperplane h containing $H_1, \ldots, H_{r-1}$ intersects X_{d_r} at d_r points $x_1, \ldots, x_{d_r}$. Hence $h \cap X_{\vec{d}}$ is the union of d_r (r − 1)-dimensional planes of the form $\overline{x_i, f_1(x_i), \ldots, f_{r-1}(x_i)}$ and the submanifolds $X_{(d_1, \ldots, d_{r-1})} \subset \overline{H_1 \ldots H_{r-1}}$. Hence

$$\deg(h \cap X_{(d_1, \ldots, d_r)}) = d_r + \deg(h \cap X_{(d_1, \ldots, d_{r-1})}). \blacksquare$$

The manifold $X_{(d_1, \ldots, d_r)}$ is nonsingular if and only if $d_i > 0$ for all i. If $d_1 \geqslant \ldots \geqslant d_r$, $d_i \neq 0$, $d_{i+1} = 0$ then $X_{\vec{d}}$ is a cone over $X_{(d_1, \ldots, d_i)}$ with "vertex" in the (r − i)-plane.

Further, the degree of a cone in P^{5+r} over a Voronese surface whose "vertex" is an r-dimensional plane is equal to 4.

THEOREM (Enriques [12, 13]). Let X be an irreducible submanifold of $\mathbf{C}P^d$, not lying in a hyperplane, and $\deg X = \operatorname{codim} X + 1$. Then X is one of the submanifolds listed below:

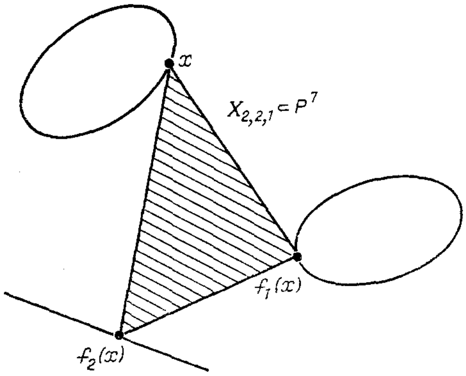

Fig. 1

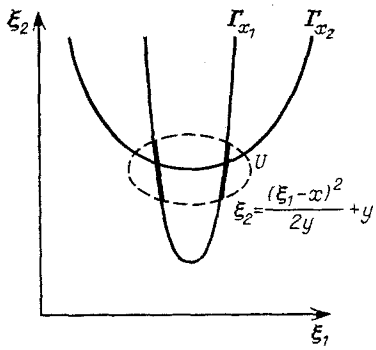

Fig. 2

1) $\mathbf{C}P^d$;

2) a quadric in $\mathbf{C}P^d$;

2) a Veronese surface in P^5 or a cone over it in P^{5+r};

4) a manifold $X_{(d_1,\ldots,d_r)}$.

2. We need another construction of the manifolds $X_{\vec{d}}$. Let $E = \bigoplus_{i=1}^{r} \mathcal{O}(d_i)$ be a vector bundle over P^1, $\hat{P}(E)$ be the manifold of hyperplanes in its fibers, and $\pi\colon \hat{P}(E) \to P^1$ be the canonical projection. We denote by $\mathcal{O}_\pi(1)$ the line bundle over $\hat{P}(E)$ whose fiber over the point x is the quotient of the fiber of E over x by the hyperplane corresponding to the point x. Then $\pi_*\mathcal{O}_\pi(1) = E$. The line bundle $\mathcal{O}_\pi(1)$ defines a map

$$\varphi\colon \hat{P}(E) \to \hat{P}(H^0(\hat{P}(E), \mathcal{O}_\pi(1))) \simeq \hat{P}(H^0(P^1, E)).$$

Namely, $\varphi(x)$ is the hyperplane in $H^0(\hat{P}(E), \mathcal{O}_\pi(1))$, consisting of sections which vanish at x.

The subbundle $E_i := \mathcal{O}(d_1) \oplus \ldots \oplus \widehat{\mathcal{O}(d_i)} \oplus \ldots \oplus \mathcal{O}(d_r)$ defines a curve $X_i \subset \hat{P}(E)$. The restriction of $\mathcal{O}_\pi(1)$ to it is isomorphic to $\mathcal{O}(d_i)$, so that $\varphi(X_i)$ is a Veronese curve in a d_i-plane $H_i \subset \hat{P}(H^0(\hat{P}(E), \mathcal{O}_\pi(1))$. After the identification

$$\hat{P}(H^0(\hat{P}(E), \mathcal{O}_\pi(1))) = (H^0(P^1, E)^* \setminus 0)/\mathbf{C}^*$$

this plane coincides with $(H^0(P^1, E_i)^\perp \setminus 0)/\mathbf{C}^*$.

The map φ carries the fiber of $\hat{P}(E)$ over the point $x \in P^1$ into an $(r-1)$-plane intersecting each of the curves $\varphi(X_i)$ in a point. Hence $\varphi(\hat{P}(E))$ can be identified canonically with $X_{(d_1,\ldots,d_r)}$.

4. COMPOSITION OF DOUBLE BUNDLES

1. Let the manifold Γ parametrize the submanifolds C_ξ in C (i.e., $\xi \in \Gamma$) and C in its own right parametrize the submanifolds $\tilde{B}_c$ in B, while $\dim C_\xi + \dim \tilde{B}_c < \dim B$.

In this case we can consider the family of submanifolds

$$B_\xi := \bigcup_{c \in C_\xi} \tilde{B}_c. \tag{4}$$

It will be convenient for us to say the same thing in the language of double bundles. Namely, let

$$\begin{array}{ccc} & A_1 & \\ {}^{\alpha_1}\swarrow & & \searrow^{\alpha_2} \\ B & & C \end{array} \quad \text{and} \quad \begin{array}{ccc} & A_2 & \\ {}^{\beta_1}\swarrow & & \searrow^{\beta_2} \\ C & & \Gamma \end{array} \tag{5}$$

be double bundles corresponding to the families $\{\tilde{B}_c\}$ and $\{C_\xi\}$. We set $A = \{(a_1, a_2) \in A_1 \times A_2 \mid \alpha_2(a_1) = \beta_1(a_2)\}$. We get the following commutative diagram:

$$\begin{tikzcd}
& & A \arrow[dl, "\gamma_1"'] \arrow[dr, "\gamma_2"] & & \\
& A_1 \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & & A_2 \arrow[dl, "\beta_1"'] \arrow[dr, "\beta_2"] & \\
B & & C & & \Gamma
\end{tikzcd} \tag{6}$$

(γ_1 and γ_2 are the projections of $A \subset A_1 \times A_2$ onto the factors). It is easy to verify that the double bundle $B \xleftarrow{\alpha_1 \circ \gamma_1} A \xrightarrow{\beta_2 \circ \gamma_2} \Gamma$ corresponds to the family (4) of submanifolds B_ξ in B defined above. We shall say that it is the composition of the double bundles (5).

<u>LEMMA 4.1.</u> The composition of admissible double bundles is admissible.

<u>Proof.</u> We choose admissible densities χ_c and ψ_ξ on $\tilde{B}_c$ and C_ξ, respectively, and we define an integral transform I: $C_0^\infty(B) \to C^\infty(\Gamma)$ by the formula

$$I\colon f(x) \mapsto \int_{C_\xi} \Big(\int_{\tilde{B}_c} f(x)\, \chi_c \Big) \psi_\xi .$$

Then $I = I_\psi \circ I_\chi$, where

$$C_0^\infty(B) \xrightarrow{I_\chi} C^\infty(C) \xrightarrow{I_\psi} C^\infty(\Gamma),$$

$$I_\chi\colon f(x) \mapsto \int_{\tilde{B}_c} f(x)\, \chi_c; \quad I_\psi\colon \varphi(c) \mapsto \int_{C_\xi} \varphi(c)\, \psi_\xi .$$

We denote by J_ψ and J_χ local inverse operators for I_ψ and I_χ, respectively. Then $J = J_\chi \circ J_\psi$ is the inverse operator for I. We prove that it is local. We set

$$\tilde{\Gamma}_c = \beta_2 \circ \beta_1^{-1}(c); \quad C_x = \alpha_2 \circ \alpha_1^{-1}(x).$$

Then if φ is a function on Γ then $(J_\varphi)(x)$ is the integral of $L\varphi$ (where L is a differential operator) over the manifold $\bigcup_{c \in C_x} \Gamma_c$. It remains to note that this manifold coincides with

$$\Gamma_x = (\alpha_2 \circ \gamma_2) \circ (\alpha_1 \circ \gamma_1)^{-1}(x). \blacksquare$$

2. Let

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] & \\
\varphi(\hat{P}(E)) & & P(H^0(P^1, E))
\end{tikzcd}$$

be the double bundle corresponding to the family of hyperplane sections of the Veronese manifold $\varphi(\hat{P}(E)) \subset \hat{P}(H^0(P^1, E))$. We consider the following commutative diagram:

$$\begin{tikzcd}
& & & A \arrow[dl] \arrow[dr] & & \\
& & A_1 \arrow[dl] \arrow[dr] & & A_2 \arrow[dl] \arrow[dr] & \\
(7l) & \varphi(\hat{P}(E)) & & P(E) & & P(H^0(P^1, E)
\end{tikzcd} \tag{7}$$

Here the points of P(E) parametrize hyperplanes in the fibers of $\hat{P}(E)$ over P^1 so that the double bundle in the left corner is admissible: the corresponding integral transform is the Radon bundle transform.

The double bundle in the right corner of the diagram describes the family of all sections of the bundle π: $P(E) \to P^1$. (All such sections are projectivizations of sections of the bundle E over P^1.)

We recall that a family of compact smooth submanifolds $B_\xi \subset B$, $\xi \in \Gamma$ is called <u>complete</u> if the canonical map $T_\xi\Gamma \to \Gamma(B_\xi, N_{B_\xi}B)$ is an isomorphism.

For example, the family of all sections of the bundle $P(E) \to P^1$ is complete.

THEOREM 4.2 [2, 6]. Let Γ' be a complete family of compact nonsingular rational curves covering the entire manifold C. Then the complex of curves tangent to $\dim\Gamma' - \dim C$ algebraic hypersurfaces M_i is admissible. ∎

In Sec. 5 we explicitly write local inversion formulas for these complexes in the case of interest to us when Γ' is the family of all sections of the bundle π: $P(E) \to P^1$ and thus we get a proof of this theorem which is independent of [2, 6].

The remaining admissible subcomplexes are obtained by degenerations of these. I. N. Bernshtein and S. G. Gindikin described them completely. In order to explain precisely how, we consider a tower of σ-processes

$$\mathcal{A}\colon C^q \xrightarrow{\sigma^q} C^{q-1} \xrightarrow{\sigma^{q-1}} \ldots \xrightarrow{\sigma^1} C^0 \equiv C,$$

where $\sigma^i\colon C^i \to C^{i-1}$ is a σ-process with center in an irreducible algebraic submanifold $Y_{i-1} \subset C_{i-1}$. Let $Z_1,\ldots,Z_m$ be algebraic hypersurfces in C^q and $\ell_1,\ldots,\ell_m$ be integers. We denote by $\Gamma(C; \mathcal{A}; Z_1, \ldots, Z_m; l_1, \ldots, l_m)$ the family of all (smooth rational) curves C_ξ whose lift to C^q intersects the preimages of $Y_0,\ldots,Y_{q-1}$ and is tangent to $Z_1,\ldots,Z_m$ with multiplicities $\ell_1,\ldots,\ell_m$, respectively. Let $d_j = \operatorname{codim} Y_j$.

It is easy to show that the family constructed must depend on $\dim\Gamma' - \sum_{i=1}^{m} l_i - \sum_{j=0}^{q}(d_j - 1)$ parameters (cf. Remark 1.1). Let this number be at least $\dim D$.

THEOREM 4.3 [2, 6]. a) The family $\Gamma(C; \mathcal{A}; Z_1, \ldots, Z_m; l_1, \ldots, l_m)$ is admissible.

b) This construction gives all admissible subfamilies of a complete family of rational curves C_ξ covering C. ∎

Inversion formulas for degenerate complexes are obtained by passing to the limit from the formulas for complexes in general position.

Applying Lemma 4.1, we get that the composition of the double bundle of (7ℓ) standing in the left corner of (7) with any admissible complex of rational curves on P(E) is admissible.

To prove Theorem A for the submanifolds $\varphi(\widehat{P}(E))$ it remains to show that here the admissible complexes from Theorem 4.2 give precisely those complexes which are indicated in Theorem A. For this we need to translate the definition of composition of double bundles into the language of symplectic geometry which will be done below.

Explicit inversion formulas for quadrics and cones over a Veronese surface will be given elsewhere.

3. Lagrangian Manifolds and Envelopes. Let X and Y be symplectic manifolds, Λ (respectively, L_Y) be a Lagrangian submanifold of X × Y (respectively, Y). We set

$$\Lambda \circ L_Y := \rho_X(X \times L_Y \cap \Lambda),$$

where ρ_X is the projection of X × Y to Y.

LEMMA 4.4 [7, Sec. 4, Chap. IV]. If $X \times L_Y$ is transverse to Λ then $\Lambda \circ L_Y$ is an immersed Lagrangian submanifold of X. ∎

We denote by $T^*_A B$ the conormal bundle to the submanifold A and B. It is easy to verify that this is a Lagrangian submanifold of T^*B which is homogeneous (with respect to the action of $\mathbf{C}^*$ on the fibers).

LEMMA 4.5. Each homogeneous Lagrangian irreducible algebraic submanifold of T^*B has the form $T^*_{A_0}B$, where A_0 is the nonsingular part of the irreducible algebraic submanifold $A \subset B$. ∎

We consider an arbitrary double bundle (3). Let $Y \subset \Gamma$.

Definition 4.6. The manifold E_Y of critical values of the map $\pi_1 : \pi_2^{-1}(Y) \to B$ is called the envelope of the family $\{B_y\}$, where $y \in Y$.

Let $\pi_B\colon T^*B \to B$ be the canonical projection.

Proposition 4.7. $\pi_B(T^*_\Lambda(B \times \Gamma) \circ T^*_Y\Gamma) = E_Y$.

Proof. The vector $v \in T_y\Gamma$ defines a section $\gamma_v(b)$ of the normal bundle $N_{B_\xi}B$. Hence the formula $[b = \pi_B(\lambda)]$

$$\nu_y \colon \lambda \mapsto -\langle \lambda, \gamma_v(b) \rangle \tag{8}$$

defines a map $\nu_y \colon T^*_{By} B \to T^*_y \Gamma$.

If $x \in E_Y$, then by definition there exists a $\xi \in T^*_x B$, such that for $a = (x, y) \in A$

$$\xi|_{d_a \pi_1 (T_a(\pi_2^{-1}(Y))} \equiv 0. \tag{9}$$

Let $T^*_{y,Y}\Gamma$ be the fiber of the conormal bundle at the point $y \in Y$.

LEMMA 4.8. a) $(x, \xi; y, \nu_y(\xi)) \in T^*_{a,A}(B \times \Gamma)$.

b) $(y, \nu_y(\xi)) \in T^*_{y,Y}\Gamma$.

Proof. a) It is necessary to verify that for any $(v_1, v_2) \in T_a A$

$$\langle (\xi, \nu_y(\xi)), (v_1, v_2) \rangle = \langle \xi, v_1 \rangle + \langle \nu_y(\xi), v_2 \rangle = 0,$$

but this is also the definition (8) for ν_y.

b) This follows from (8) and (9). ∎

It follows from Lemma 4.8 that $E_Y \subset \pi_B(T^*_A(B \times \Gamma) \circ T^*_Y \Gamma)$. To prove the opposite inclusion we note that if $(x, \xi) \in T^*_A(B \times \Gamma) \circ T^*_Y\Gamma$, then by definition there exists a $(y, \eta) \in T^*\Gamma$ for which $(x, \xi; y, \eta) \in T^*_A(B \times \Gamma)$. For any $v_1 \in T_x B_y$ the vector $(v_1, 0) \in T_a A$, so that $\langle \xi, v_1 \rangle = 0$ and (9) holds. Consequently, x is a critical value for the map $\pi_1 \colon \pi_2^{-1}(Y) \to B$.

In the double bundle (7) as an abbreviation we shall denote $\varphi(\hat{P}(E))$ by B. Then the projection

$$T^*_{A_1}(B \times P(E)) \subseteq T^*(B \times P(E)) \equiv T^*B \times T^*P(E)$$

to the second factor is a bijection at a generic point. It is easy to verify this directly or to derive it from the admissibility of the double bundle using Theorem 4.9 (cf. below).

Hence, if $(x, \xi; y, \eta) \in T^*_{A_1}(B \times P(E))$, then the formula $(x, \xi) \to (y, -\eta)$ defines a homogeneous symplectomorphism $\Phi \colon T^*P(E) \to T^*B$. Consequently, for any Lagrangian submanifold $L \subset T^*P(E)$

$$\Phi(L) = T^*_{A_1}(B \times P(E)) \circ L. \tag{10}$$

Let $\xi \in P(H^0(P^1, E))$. We denote by C_ξ (respectively, B_ξ) a curve in P(E) (respectively, a hypersurface in B). Then according to (10) and Proposition 4.7

$$T^*_{B_\xi} B = \Phi(T^*_{C_\xi} P(E)).$$

Let $\Phi \colon T^*B_1 \to T^*B_2$ be a homogeneous symplectomorphism. Then by Lemma 4.5 there exists a submanifold of B_2 which we denote by $\Phi(X)$ such that $\Phi(T^*_X B_1) = T^*_{\Phi(X)} B_2$.

The tangency-intersection conditions with the submanifolds of Theorem 4.2 mean precisely that $T^*_{C_\xi} P(E) \setminus 0$ intersects $T^*_{M_k} P(E) \setminus 0$. The condition of tangency of the hypersurfaces B_ξ with $\Phi(N_k)$ have the same interpretation. Thus, in the composition of the double bundle (7ℓ) with the complex of curves C_ξ tangency-intersection conditions are singled out which are obtained from exactly one of the complexes described in Theorem A. Theorem A is completely proved for the manifolds $\varphi(\hat{P}(E))$. ∎

THEOREM 4.8. Any admissible complex of hyperplane sections of the manifold $\varphi(\hat{P}(E))$ is obtained by composition of the double bundle (7) and an admissible complex of curves C_ξ on P(E).

Proof. We consider the following diagram:

$$\begin{tikzcd} & T^*_A(B \times \Gamma) \arrow[dl, "\rho_B"'] \arrow[dr, "\rho_\Gamma"] & \\ T^*B & & T^*\Gamma \end{tikzcd}, \tag{11}$$

where ρ_B and ρ_Γ are the projections of the submanifold $T^*_A(B \times \Gamma) \subset T^*B \times T^*\Gamma$ onto the factors. If $\dim B = \dim \Gamma$ then all three manifolds in (11) have the same dimension. The degree of the map ρ_Γ is called the codegree of the double bundle.

THEOREM 4.9 [8]. If (3) is an admissible double bundle and $\dim B = \dim \Gamma$ then its codegree is equal to 1. ∎

LEMMA 4.10. In composition of double bundles the codegrees multiply.

Proof. In the diagram (6)

$$T_A^*(B \times \Gamma) = T_{A_1}^*(B \times C) \circ T_{A_2}^*(C \times \Gamma).$$

Here $\circ$ is composition, respectively, in $T^*B \times T^*C$ and $T^*C \times T^*\Gamma$. ∎

If in Theorem 4.8 one omits the word "admissible," then the corresponding result is obtained in point 2.

It is shown in Sec. 1 of [8] that the condition "codegree equal to 1" for complexes of curves is equivalent to the main condition of admissibility of [1]. Hence it follows from Theorem 4.9, Lemma 4.10, and the admissibility of the double bundle (71) that any complex of hyperplane sections $\varphi(\hat{P}(E))$, having codegree 1 can be obtained from an admissible complex of curves and consequently is admissible by Lemma 4.1. ∎

5. EXPLICIT LOCAL INVERSION FORMULAS

1. Universal Local Inversion Formula [5]. We consider a family $\{B_\xi\}$ of curves in B where $\xi \in \Gamma$ and $\dim B < \dim \Gamma$. Let $\varkappa_x$: $C^\infty(\Gamma) \to \Omega^{1,0}(\Gamma_x)$ be a holomorphic differential operator of the first order while $d\varkappa_x(If) = 0$ for $f \in C_0^\infty(B)$.

If γ is a two-dimensional cycle in Γ_x and $\dim_{\mathbf{C}} B \geqslant 3$, then

$$\int_\gamma \varkappa_x \wedge \bar{\varkappa}_x(If) = c(\gamma) f(x), \tag{12}$$

where $c(\gamma)$ is independent of f. Indeed the integral (12) is unchanged under deformation of the cycle γ. Hence it defines a generalized function on B with support at the point x. In fact for any point $y \neq x$ one can deform the cycle γ so that the (conical) surface $\bigcup_{\xi \in \gamma} B_\xi$ does not touch y. It follows from homogeneity conditions that this generalized function is proportional to $\delta(x)$.

The same inference is true if $\dim_{\mathbf{C}} B = 2$, but B can be imbedded in a three-dimensional manifold B' so that the family $\{B_\xi\}$ becomes part of a large family of curves in B' for which $\varkappa_x$ exists. This is precisely the case we encounter below.

2. The manifold P(E) is obtained from $(\mathbf{C}^2 \setminus 0) \times (\mathbf{C}^n \setminus 0)$ with coordinates $(t_0, t_1; x_1, \ldots, x_n)$ by the identification

$$(t_0, t_1; x_1, \ldots, x_n) \sim (\lambda t_0, \lambda t_1; \lambda^{k_1} x_1, \ldots, \lambda^{k_r} x_r) \sim (t_0, t_1; \beta x_1, \ldots, \beta x_2).$$

In the affine part of P(E) with coordinates $(t; x_1, \ldots, x_{r-1}) = (1, t; x_1, \ldots, x_{r-1}, 1)$ the sections of P(E) have the form $x_i = P_i(t)/P_r(t)$, $1 \leqslant i \leqslant r-1$, where $P_i(t) = a^i_{k_r} t^r + \ldots + a^i_1 t + a^i_0$.

We define the integral transform I by the formula

$$I\colon f(t, x) \mapsto \int f\left(t, \frac{P_1(t)}{P_r(t)}, \ldots, \frac{P_{r-1}(t)}{P_r(t)}\right) dt\, d\bar{t} =: If(a^i_j).$$

Let $a_0^r = 1$. We set

$$\varkappa = \sum_i \sum_j \frac{\partial}{\partial a^i_j} da^i_{j+1} - D_{a_0} da^r_1,$$

where we sum over $0 \leqslant i \leqslant r$, $0 \leqslant j \leqslant k_i - 1$ $(i, j) \neq (r, 0)$ and

$$D_{a_0} := \sum_{1 \leqslant i \leqslant r-1} a^i_j \frac{\partial}{\partial a^i_j} + \sum_{1 \leqslant j \leqslant k_r} a^r_j \frac{\partial}{\partial a^r_j}.$$

Let Γ_0 be the manifold of curves of the family passing through the point $(t_0, x_0) \in P(E)$.

Proposition 5.1. The restriction of $\varkappa(If)$ to Γ_0 is a closed 1-form.

Proof. The function of $a^i_1, \ldots, a^i_{k_i}$, where $1 \leqslant i \leqslant r$, form a system of coordinates on Γ_0. We calculate $d\varkappa(If)|_{\Gamma_0}$ in the simplest case $r = 2$, $k_1 = k_2 = 1$ (in general the same mechanism works):

$$I\colon f \mapsto \int f\left(t, \frac{a_1 t + a_0}{b_1 t + b_0}\right) dt\, d\bar{t} =: If(a_0, a_1, b_0, b_1).$$

If $b_0 = 1$ then

$$\varkappa(If) = \frac{\partial(If)}{\partial a_0} da_1 - \left(a_0 \frac{\partial}{\partial a_0} + a_1 \frac{\partial}{\partial a_1} + b_1 \frac{\partial}{\partial b_1}\right)(If)\, db_1,$$

$$d\varkappa(If)|_{\Gamma_0} = \left(\frac{\partial^2(If)}{\partial a_0 \partial b_1} + \frac{\partial}{\partial a_1}\left(a_0 \frac{\partial}{\partial a_0} + a_1 \frac{\partial}{\partial a_1} + b_1 \frac{\partial}{\partial b_1}\right)(If)\right) db_1 \wedge da_1.$$

In order to verify that this expression is equal to 0, it is convenient first not to impose the condition $b_0 = 1$. Then obviously the function $If(a_0, a_1, b_0, b_1)$ satisfies the differential equations

$$\left(\frac{\partial^2}{\partial a_0 \partial b_1} - \frac{\partial^2}{\partial a_1 \partial b_0}\right)(If) = 0, \quad \left(a_0 \frac{\partial}{\partial a_0} + a_1 \frac{\partial}{\partial a_1} + b_0 \frac{\partial}{\partial b_0} + b_1 \frac{\partial}{\partial b_1}\right) If = 0.$$

If we now impose the condition $b_0 = 1$ then in the first of the equations it is necessary to substitute for $\frac{\partial}{\partial b_0}$ the operator $\left(a_0 \frac{\partial}{\partial a_0} + a_1 \frac{\partial}{\partial a_1} + b_1 \frac{\partial}{\partial b_1}\right)$.

We note that $\varkappa\colon C^\infty(\Gamma) \to \Omega^{1,0}(\Gamma_0)$ is a first-order differential operator. Hence, for any point $\xi \in \Gamma_0$ there exists a linear operator $M\colon T_\xi \Gamma_0 \to T_\xi \Gamma$ such that for $v \in T_\xi \Gamma_0$

$$\varkappa(If)(v) = d(If)(Mv). \tag{13}$$

We denote by Γ_N the manifold of all curves of the family tangent to an analytic hypersurface N.

<u>LEMMA 5.2.</u> The restriction of $\varkappa(If)$ to $\Gamma_0 \cap \Gamma_N$ depends only on $If|_{\Gamma_N}$.

<u>Proof.</u> Without loss of generality one can assume that $t_0 = 0$. Let $\xi \in \Gamma_0 \cap \Gamma_N$; C_ξ be the corresponding curve in P(E), tangent to N at the point (t, x). It is easy to verify that the tangent space at the point ξ to the manifold of all curves C_ξ passing through (t, x) lies in $T_\xi \Gamma_N$. We have

$$-t\varkappa(If) + d(If) = \sum_{i=1}^{r-1} \sum_{j=1}^{k_i} \left(\frac{\partial}{\partial a_j^i} - t \frac{\partial}{\partial a_{j-1}^i}\right)(If)\, da_j^i,$$

where in place of $\partial/\partial a_0$ it is necessary to substitute D_{a_0}. The manifold of curves passing through (t, x) is defined by the equations $P_i(t) = xP_r(t)$. Hence the vector field $\partial/\partial a_j^i - t\cdot\partial/\partial a_{j-1}^i$ is tangent to it and $t\cdot\varkappa(If) + d(If)$ and consequently, $\varkappa(If)$, too can be calculated from the restriction of If to Γ_N.

Let $N_1, \ldots, N_{d-k}$ be analytic hypersurfaces in P(E), $k \geqslant r$, and $\Gamma(N) := \bigcap \Gamma_{N_j}$ $(1 \leqslant j \leqslant d-k)$.

<u>LEMMA 5.3.</u> The restriction of $\varkappa(If)$ to $\Gamma_0 \cap \Gamma(N)$ depends only on the restriction of If to $\Gamma(N)$.

The proof follows quickly from Lemma 5.2 and (13). ■

The local inversion formula for the family of curves $\Gamma(N)$ is obtained by integration of the 2-form $\varkappa \wedge \bar{\varkappa}(If)$ over $\Gamma_0 \cap \Gamma(N)$. Here we must require that this analytic subset of is a cycle. In particular, it must be closed and consequently, by Chow's theorem, algebraic (since $\Gamma_{(t,x)}$ is a projective algebraic manifold). If the submanifolds N_j are algebraic, then this is always so.

One can show that if $\Gamma_{(t,x)} \cap \Gamma(N)$ is algebraic for any $(t, x) \in P(E)$, then N_j is an algebraic submanifold.

6. PROOF OF THEOREM C

Above we defined for a double bundle (3) a map $T^*_{B_\xi} B \to T^*_\xi \Gamma$ [cf. (9)].

<u>LEMMA 6.1 [8, Lemma 4.2].</u> The diagram

$$\begin{tikzcd} & \bigcup_\xi T^*_{B_\xi} B \arrow[dl] \arrow[dr, "\bigcup_\xi \nu_\xi"] & \\ T^*B & & T^*\Gamma \ (=\bigcup_\xi T_B^*\Gamma) \end{tikzcd}$$

can be canonically identified with the diagram (12). ■

COROLLARY 6.2. The codegree of the double bundle (3) is equal to the degree of the map ν_ξ. ■

Let Γ' be a submanifold of Γ, $\nu'_\xi\colon T^*_{B_\xi}B \to T^*_\xi\Gamma'$ be the corresponding map $i^*_\xi\colon T^*_\xi\Gamma \to T^*_\xi\Gamma'$ be the canonical projection corresponding to the imbedding $T_\xi\Gamma' \subset T_\xi\Gamma$. Then it follows from (9) that one has $\nu'_\xi = i^*_\xi \circ \nu_\xi$.

LEMMA 6.3. $\nu'_\xi = i^*_\xi \circ \nu_\xi$. ■

Let $\Gamma' \subset \widehat{CP}^n \equiv \Gamma$ be a complex singled out by conditions of tangency with codim X hypersurfaces in $X^m \subset CP^n$, h_ξ be the hypersurface in CP^n, corresponding to the point $\xi \in \Gamma$ and $B_\xi := X \cap h_\xi$.

Proposition 6.4. The projectivization of the submanifold $\nu^*_\xi(T^*_{B_\xi}, X) \subset T^*_\xi\widehat{CP}^n$ is canonically isomorphic to the submanifold B_ξ of the hyperplane h_ξ.

Proof. $PT^*_\xi\widehat{CP}^n$ can be canonically identified with the hyperplane h_ξ. Further, for an arbitrary family $\{B_\xi\}$ of hypersurfaces the projectivization of the map (9)

$$P\nu_\xi : B_\xi \to PT^*_\xi\Gamma = \hat{P}T_\xi\Gamma$$

can be defined as follows: for a generic point $x \in B_\xi$ $P\nu_\xi(x)$ is the hyperplane $PT_\xi\Gamma_x$ in $PT_\xi\Gamma$. Applying this recipe in our situation, we get the proof of Proposition 6.4. ■

Let x_i be a point of tangency of the generic hypersurface B_ξ with the submanifold M_i $(1 \leqslant i \leqslant \operatorname{codim} X)$. By hypothesis one can assume that this is a generic point on B_ξ. Then according to Lemma 6.3 and Proposition 6.4, $P\nu'_\xi\colon B_\xi \to CP^{m-1}$ is a projection of the submanifold $B_\xi \subset h_\xi$ with center in the plane spanned by the points $x_1, \ldots, x_{\operatorname{codim} X}$. Hence the degree of B_ξ and h_ξ is equal to $\deg P\nu'_\xi + \operatorname{codim} X$. From this, according to Corollary 6.3, $\deg X = \deg B_\xi = 1 + \operatorname{codim} B_\xi$. ■

7. PROOF OF THEOREM B

1. The family of hyperplane sections of the surface $S_k \subset CP^n$ (cf. Example 1.2, b) is isomorphic to the family of all sections of the bundle $P(E) \to CP^1$, where $E = \mathcal{O}(k) \oplus \mathcal{O}(n-k)$. Hence all its admissible subfamilies and the inversion formulas for them are described in Sec. 5.

LEMMA 7.1. If the curves of an admissible complex are reducible, then one of their component also forms an admissible complex.

Proof. Let $B_\xi = B^{(1)}_\xi \cup B^{(2)}_\xi$. Then the curves of the dual family are also reducible: $\Gamma_x = \Gamma^{(1)}_x \cup \Gamma^{(2)}_x$ (the family $\{\Gamma^{(1)}_x\}$ is dual to $\{B^{(1)}_\xi\}$). If each of the components forms a 2-parameter family, and the admissible density is different from 0 on the whole curve B_ξ then the complex $\{B_\xi\}$ cannot be admissible because its codegree is not less than 2 (cf. Theorem 4.9 and its proof in [8]). Hence an admissible density is different from 0 only on one of the components which also gives an admissible complex.

Now let the family $\{\Gamma^{(2)}_x\}$ depend on one parameter. By Theorem 1.4 the curves Γ_x are the geodesics of a connection on Γ. But then $\{\Gamma^{(1)}_x\}$ is also a family of geodesics. Hence again by Theorem 1.4, the complex $\{B^{(1)}_\xi\}$ is admissible.

Conversely, if one of the components of a family of reducible curves forms an admissible complex, then the whole family is also admissible. (An admissible density should be extended by zero to the whole curve.)

We start the proof of Theorem B. First let $\dim \Gamma = 2$.

Proposition 7.2. The family $\{B_\xi\}$ is birationally isomorphic to a family of smooth compact rational curves.

LEMMA 7.3. Let $\{B_\xi\}$ be an admissible family of curves on a surface, $\xi \in \Gamma$. Then there exists a Zariski open set $\Gamma' \subset \Gamma$ such that the curves parametrizable by it have singularities only at a finite set of points $S \subset B$.

Derivation of Proposition 7.2 from Lemma 7.3. Let $\sigma\colon B_1 \to B$ be a blow up with center in the set $S \subset B$ from Lemma 7.3. Applying Lemma 7.3 again to the family of proper preimage of curves B_ξ in B_1, where $\xi \in \Gamma'$, and iterating this procedure, in a finite number of steps we get a family of smooth compact rational curves on the surface B_k. ■

Proof of Lemma 7.3. According to Theorem 1.4 for generic points $x \in B$ the curves Γ_x are geodesics of an affine connection on Γ. If we consider all of its geodesics, then we get a family in which through any point in each direction exactly one curve leaves. Hence their lift to the projectivized tangent bundle $PT\Gamma$ is a foliation. There exists a small domain $U \subset \Gamma$ for which the foliation in PTU has a factor, the surface $\tilde{B}_U$.

The fibers of the bundle π: $PTU \to U$ are isomorphic to $\mathbf{C}P^1$ and transverse to the curves of the foliation. Hence, their images in $\tilde{B}_U$ are a 2-parameter family of smooth compact rational curves $\{\tilde{B}_\xi\}$ with normal bundle $\mathcal{O}(1)$. (The index of intersection of these curves is equal to 1 because two points in a small domain U are joined by exactly one geodesic lying entirely in U.)

Let γ be a geodesic in U and let there exist a family of curves $\{\Gamma_z\}$ such that $\Gamma_z \cap U = \gamma$. Then all of them are reducible: $\Gamma_z = \Gamma_z^{(1)} \cup \Gamma_z^{(2)}$, where $\Gamma_z^{(1)}$ is independent of z (fixed component): $\gamma \subset \Gamma_z^{(1)} \cap U$.

In all in Γ the number of such fixed components is finite (because otherwise the curves B_ξ are reducible). One can assume that they lie outside U.

Let us now assume that Lemma 7.3 is false. Then the singularities of the curves B_ξ fill a curve $C \subset B$. Let ξ be a generic point of U and B_ξ have a singularity at the generic point $c \in C$.

We take a neighborhood $\mathscr{V}$ of the point c for any point x of which $\dim \Gamma_x = 1$. The connected components of the curve $\Gamma_x \cap U$ for $x \in \mathscr{V}$ are geodesics in U because by Theorem 1.4 this is so for the generic point x and the limit of geodesics is a geodesic. We denote the domain in $\tilde{B}_U$ parametrizing all these geodesics by $\tilde{B}_U'$. Then there is a canonical map f: $\tilde{B}_U' \to \tilde{B}_U$. Namely, if $y \in \tilde{B}_U'$ and γ_y is the corresponding geodesic in U, then there exists a unique point $x \in \mathscr{V}$ such that $\gamma_y \subset \Gamma_x \cap U$. Let $f(y) := x$. We note that $\#f^{-1}(x) = \pi_0(\Gamma_x \cap U)$ and as a rule is greater than 1, and the images $f(\tilde{B}_\xi) = B_\xi$ are singular curves.

Since all the curves $\tilde{B}_\xi$ in $\tilde{B}_U'$ are smooth, c is a critical value of the map f. Since c is a generic point of the curve C, for $y \in f^{-1}(c)$ the kernel of $d_y f$ is one-dimensional. Hence the number of curves $\tilde{B}_\xi$ passing through y and tangent to $\operatorname{Ker} d_y f$ is finite. This contradicts the existence of a 1-parameter family of curves $\tilde{B}_\xi$ with singularity at c.

Example 7.4. $\{B_\xi\}$ are horocycles $(x - \xi_1)^2 + (y - \xi_2)^2 = \xi_2^2$ on the Lobachevskii plane. The dual family consists of the parabolas $\xi_2 = (\xi_1 - x)^2/(2y) + y$ (Fig. 2). Through close points with different abscissas in the (ξ_1, ξ_2) plane there pass exactly two such parabolas. One of them, Γ_{x_1} lies in a small neighborhood U of these points and the other, Γ_{x_2} gets out of it strongly. Hence, $\Gamma_{x_2} \cap U$ consists of two components and $\#f^{-1}(x_2) = 2$, i.e., $f: \tilde{B}_U \to B$ is a 2-sheeted covering. Here the curves $\tilde{B}_\xi$ on $\tilde{B}_U$ have normal bundle $\mathcal{O}(1)$, at the same time that the curves B_ξ have normal bundle $\mathcal{O}(2)$.

LEMMA 7.5. If on the algebraic surface B there is a 2-parameter family of rational curves, then it is rational.

Proof. Let $\{B_S\}$ be a family of rational curves on B parametrized by the curve S. Let $X_S = \{(x, s) \mid x \in B_S\}$. Then X_S is fibered over S by rational curves. Hence $\mathbf{C}(X_S)$, the field of rational functions on X_S, is isomorphic to the field $\mathbf{C}(S)$ (rational function over $\mathbf{C}(S)$. Since X_S is mapped onto B, $\mathbf{C}(B)$ is a subfield of $\mathbf{C}(S)(t)$. By Lüroth's theorem it is isomorphic to the field of rational functions over $\mathbf{C}(S')$, where $\mathbf{C}(S') \subset \mathbf{C}(S)$. This means that B is fibered over S' by rational curves. Let B_ξ be a rational curve on B different from the fibers of this projection. Then it projects to the whole curve S' and again by Lüroth's theorem S' is rational. Consequently [11] the surface B is also rational. ∎

Since on a rational surface algebraic equivalence for divisors coincides with rational equivalence, the family of all curves obtained by deformations of a curve B_{ξ_0} coincides with the complete linear system $|L|$ of all curves (or, better said, divisors) on B linearly equivalent to B_{ξ_0}.

According to Proposition 7.2, we can realize the family $\{B_\xi\}$ by smooth compact rational curves on the surface B' and in particular, assume that $B_{\xi_0} \cong \mathbf{C}P^1$. Then all curves of the linear system $|L|$ are rational, because the arithmetic genus of B_{ξ_0} is equal to 0 and is unchanged under deformation. (Example: the linear system containing rational singular curves $(x - a)^2 = (y - b)^3$ consists of all curves of degree 3 in $\mathbf{C}P^2$.)

Let l: $B' \to P^N$ be the map defined by the linear system $|L|$. Then $\ell(B_{\xi_0})$ is the hyperplane section $\ell(B')$ so that the degree of $\ell(B')$ in P^N is equal to the index of self-intersection of the curve B_{ξ_0} on B'. On the other hand, by Kodaira's theorem $N = \dim H^0(B_{\xi_0}, N_{B_{\xi_0}}B')$ from which it follows that $N_{B_{\xi_0}}B'$ is isomorphic to the bundle $\mathcal{O}(N-1)$ on $\mathbf{C}P^1$. Hence the index of self-intersection of B_{ξ_0} is equal to N − 1 and deg ℓ(B') = N − 1.

Let $\bar{L}$ be a minimal linear system on ℓ(B') containing $\{l(B_\xi)\}$. We write it in the form $|L - C - \sum_{i=1}^{k} m_i x_i|$, where C is a divisor and x_i are points on ℓ(B') (possibly infinitely close, cf. [15, Chap. 5, Sec. 4]). It follows from the irreducibility of $\ell(B_\xi)$ that C = 0. Further, m_i = 1 because the arithmetic genus of $\ell(B_\xi)$ is equal to 0. Hence $\bar{L}$ defines the map ℓ(B') on the complete surface $\bar{B} \subset \mathbf{C}P^n$ of degree n − 1 (where n = N − k). Here the complex $\{\ell(B_\xi)\}$ becomes a complex $\{\bar{\ell}(B_\xi)\}$ of hyperplane sections of $\bar{B}$. We note that through any point of the surface $\bar{B}$ there passes exactly a 1-parameter family of curves $\bar{\ell}(B_\xi)$ because otherwise the linear system $\bar{L}$ will not be minimal. Hence in describing the admissible complex $\{\bar{\ell}(B_\xi)\}$ in terms of tangency-intersections conditions (cf. Theorem 4.3) only tangency conditions figure. This is also the canonical realization of the admissible complex of curves. Theorem B is completely proved for the case dim Γ = 2. (We note that one can easily avoid references to Theorem 4.3.)

Now let $\dim \Gamma \geqslant 3$. Then the condition of tangency with dim Γ − 2 generic curves on B singles out an admissible complex of curves (cf. Theorem 4.3). Applying Lemma 7.3 to it, we get that the singularities of generic curves B_ξ (where $\xi \in \Gamma$)) are concentrated in a finite set $S \subset B$ (independent of ξ). The rest of the proof is preserved word for word. ∎

2. Thus, in each class of birationally isomorphic admissible families of curves on a surface there is a canonical model. (As is known there is no such model for the surface itself.) The surface $\bar{B}$ on which it is realized can be defined as the manifold of moduli of the dual family $\{\Gamma_x\}$ of subschemes of codimension 1 in Γ. Here the curves M_i figuring in the formulation of Theorem B are singled out by the fact that they parametrize irreducible subschemes ("multiple" hypersurfaces). Their structural sheaf has the form $\mathcal{O}_\Gamma/I_x^{c_i+1}$, where I_x is the defining ideal corresponding to the reduced subscheme and c_i is the multiplicity with which B_ξ is tangent to the curve M_i. [Thus, in Example 7.3 the point (x, 0) in the (x, y) plane parametrizes the subscheme $(\xi_1 - x)^2 = 0$ in the (ξ_1, ξ_2) plane.]

On the other hand, for a (germ of a) neighborhood of a curve of any formally admissible family of curves on an analytic surface there is a canonical analytic model, the surface $\tilde{B}_U$ constructed in the course of proving Theorem B. It is characterized by the fact that the curves $\tilde{B}_\xi$ on it form a complete family of smooth compact rational curves [with normal bundle $\mathcal{O}(\dim\Gamma - 1)$].

Thus, for a neighborhood of a curve of an admissible family on an algebraic surface there are two canonical models: an algebraic and an analytic. It is curious that as a rule they do not coincide. More precisely, there is a canonical map $f\colon \tilde{B}_U \to \bar{B}$. Namely, if γ_y is a geodesic in U corresponding to the point $y \in \tilde{B}_U$, then (as is clear from theorem B) there exists a unique point $f(y) \in \bar{B}$, such that $\gamma_y = \Gamma_{f(y)} \cap U$. The map f has ramification of degree c_i at the points of the curve M_i. Hence the analytic model coincides with the algebraic only for the family of all hyperplane sections of a surface of minimal degree in $\mathbf{C}P^2$. (In particular, for dim Γ = 2 one has coincidence only for the complex of all lines in $\mathbf{C}P^2$.)

3. In the course of proving Theorem B, we incidentally proved the following theorem.

THEOREM 7.5. A complex of curves on an algebraic surface is admissible if and only if it has codegree 1 and is birationally equivalent to a family of smooth compact curves. ∎

Analogous arguments show that the same thing is also true for complexes of curves on algebraic manifolds of any dimension.

4. The analog of Theorem 7.5 for complexes of surfaces is false.

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