

THE CLASSICAL TRILOGARITHM, ALGEBRAIC K -THEORY OF FIELDS, AND DEDEKIND ZETA FUNCTIONS

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ABSTRACT. In this paper we show how to express the values of $\zeta_F(3)$ for arbitrary number field F in terms of the trilogarithms (D. Zagier's conjecture) and how to relate this result to algebraic K -theory.

1. THE CLASSICAL POLYLOGARITHM FUNCTION

The classical polylogarithm function

$$(1.1) \quad \text{Li}_p(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^p} \quad (z \in \mathbb{C}, |z| \leq 1, p \in \mathbb{N})$$

during the last 200 years was the subject of much research—see [L]. Using the inductive formula $\text{Li}_p(z) = \int_0^z \text{Li}_{p-1}(t)t^{-1} dt$, $\text{Li}_1(z) = -\log(1-z)$, the p -logarithm can be analytically continued to a multivalued function on $\mathbb{C} \setminus \{0, 1\}$. However, D. Wigner and S. Bloch introduced [B1] the single-valued cousin of the dilogarithm, namely

$$(1.2) \quad D_2(z) := \text{Im}(\text{Li}_2(z)) + \arg(1-z) \cdot \log|z|.$$

Of course, for Li_1 such function is $-\log|z|$. Analogous functions $D_p(z)$ for $p \geq 3$ were introduced in [R] and computed explicitly in [Z]. Let us consider the slightly modified function

$$(1.3) \quad \mathcal{L}_3(z) := \text{Re} \left[\text{Li}_3(z) - \log|z| \cdot \text{Li}_2(z) + \frac{1}{3} \log^2|z| \cdot \text{Li}_1(z) \right].$$

Such modified functions were considered also for all p by D. Zagier, A. A. Beilinson and P. Deligne [Z3, Be1]. $\mathcal{L}_3(z)$ is real-analytic on $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$ and continuous on $\mathbb{C}P^1$.

Let F be a field. Let P_F^1 be the projective line over F , and let $\mathbb{Z}[P_F^1 \setminus \{0, 1, \infty\}]$ be the free abelian group generated by symbols $\{x\}$, where $x \in P_F^1 \setminus \{0, 1, \infty\}$.

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We may consider \mathcal{L}_3 as defining a homomorphism

$$(1.4) \quad \mathcal{L}_3: \mathbf{Z}[P_{\mathbf{C}}^1 \setminus \{0, 1, \infty\}] \rightarrow \mathbf{R}, \quad \mathcal{L}_3: \Sigma n_i \{x_i\} \mapsto \Sigma n_i \mathcal{L}_3(x_i).$$

We can do the same for any other real-valued function on $P_{\mathbf{C}}^1 \setminus \{0, 1, \infty\}$, in particular for D_2 .

2. FORMULA FOR $\zeta(3)$

Now let F be an arbitrary algebraic number field, d_F the discriminant of F , r_1 and r_2 the number of real and complex places, σ_j all possible embeddings $F \hookrightarrow \mathbf{C}$, $1 \leq j \leq r_1 + 2r_2$, and $\overline{\sigma_{r_1+k}} = \sigma_{r_1+r_2+k}$. Set $A_{\mathbf{Q}} := A \otimes \mathbf{Q}$. Let us consider the homomorphism

$$(1.5) \quad \begin{aligned} \Delta: \mathbf{Q}[P_F^1 \setminus \{0, 1, \infty\}] &\rightarrow (\Lambda^2 F^* \otimes F^*)_{\mathbf{Q}}, \\ \Delta: \{x\} &\mapsto (1-x) \wedge x \otimes x. \end{aligned}$$

Theorem 1. *Let $\zeta_F(s)$ be the Dedekind zeta function of F . Then there exist $y_1, \dots, y_{r_1+r_2} \in \text{Ker } \Delta \subset \mathbf{Q}[P_F^1 \setminus \{0, 1, \infty\}]$ such that $\zeta_F(3)$ is equal to $\pi^{3r_2} \cdot |d_F|^{-1/2}$ times the $(r_1 + r_2)$ -determinant $\|\mathcal{L}_3(\sigma_j y_i)\| \cdot (1 \leq j \leq r_1 + r_2)$.*

For $s = 2$ a similar result was proved in [Z2]. It also follows directly from results of [Bo, B1, Su]. D. Zagier conjectured that an analogous fact should be valid for all integers $s \geq 3$ [Z3].

To prove Theorem 1 we give an explicit formula expressing the Borel regulator $r_3: K_3(\mathbf{C}) \rightarrow \mathbf{R}$ by $\mathcal{L}_3(z)$, and then use the Borel theorem [Bo]. Below we indicate some ingredients of the proof which are of independent interest.

3. GENERIC 3-VARIABLE FUNCTIONAL EQUATION FOR $\mathcal{L}_3(z)$

The dilogarithm satisfies a remarkable 2-variable functional equation, discovered in the 19th century by W. Spence, N. H. Abel and others [L]. Its version for $D_2(z)$ is as follows. Let $r(x_1, \dots, x_4)$ be the crossratio of a 4-tuple of different points on P^1 . For every five different points on P^1 set

$$(3.1) \quad \begin{aligned} R_2(x_0, \dots, x_4) &:= \\ \sum_{i=0}^4 (-1)^i [r(x_0, \dots, \hat{x}_i, \dots, x_4)] &\in \mathbf{Z}[P^1 \setminus \{0, 1, \infty\}]. \end{aligned}$$

Then $D_2(R_2(x_0, \dots, x_4)) = 0$ in the sense of formula (1.4). Note that (3.1) depends actually on two variables because of the PGL_2 -

invariance of the crossratio. It seems that any other functional equation for $D_2(z)$ can be deduced formally from this one.

It turns out that the analogous functional equation for $\mathcal{L}_3(z)$ corresponds to a special configuration of seven points in the plane. Namely, let x_1, x_2, x_3 be vertices of a triangle in P_F^2 (i.e. these points are not on a line); y_1, y_2, y_3 points on its "sides" $\overline{x_1x_2}$, $\overline{x_2x_3}$, and $\overline{x_3x_1}$, and z a point in generic position (see Figure 1). Further, denote by $(y_1|y_2, y_3, x_3, z)$ the configuration of four points on a line obtained by projection of points y_2, y_3, x_3, z with center at the point y_1 . Set

$$\begin{aligned}
 R_3(x_i, y_i, z) := & (1 + \tau + \tau^2) \\
 & \circ [\{r(y_1|y_2, y_3, x_2, z)\} - \{r(y_1|y_2, y_3, x_3, z)\} \\
 & + \{r(z|x_3, y_3, x_1, y_2)\} + \{r(z|y_3, y_1, x_1, y_2)\} \\
 & + \{r(z|y_1, x_2, x_1, y_2)\} \\
 & + \{r(z|x_2, x_3, x_1, y_2)\} - \{r(z|x_3, y_1, x_1, y_2)\}] \\
 & + \{r(y_1|y_2, y_3, x_2, x_3)\} - 3\{1\}
 \end{aligned}$$

where $\tau: x_i \rightarrow x_{i+1}, y_i \rightarrow y_{i+1}$ (indices modulo 3) (for example, $\tau^2 \circ \{r(y_1|y_2, y_3, x_2, z)\} = \{r(y_3|y_1, y_2, x_1, z)\}$) and, by definition, $\{1\} = \{x\} + \{1-x\} + \{1-x^{-1}\}$ for any $x \in F^* \setminus 1$. As we will see below the choice of x is inessential for our purposes.

Theorem 2. *In the case $F = \mathbb{C}$, $\mathcal{L}_3(R_3(x_i, y_i, z)) = 0$. Note, that $\mathcal{L}_3(\{x\} - \{x^{-1}\}) = 0$ and $\mathcal{L}_3(\{x\} + \{1-x\} + \{1-x^{-1}\}) = \zeta_{\mathbb{Q}}(3)$.*

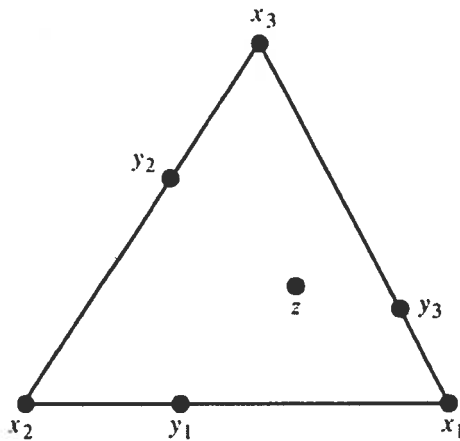


FIGURE 1

A configuration $(x_1, x_2, x_3, y_1, y_2, y_3, z)$ of seven points in P_F^2 depends on three parameters. Consider a specialization of this configuration, when z lies on the line $\overline{x_3 y_1}$. It depends on two parameters, and the corresponding functional equation coincides with the classical Spence-Kemmer functional equation for the trilogarithm, discovered by Spence in 1809 [S] and, independently, by E. Kummer in 1840 [K] (see Chapter VI in [L]).

It is also possible to deduce the Spence-Kummer equation formally from Theorem 2 (as a linear combination of relations $\mathcal{L}_3(R_3(x_i, y_i, z)) = 0$). The validity of the inverse statement is an interesting problem.

Conjecture 1. Any functional equation for $\mathcal{L}_3(z)$ can be formally deduced from Theorem 2.

4. ALGEBRAIC K-THEORY OF A FIELD

Now let F be an arbitrary field. Set $B_2(F) := \mathbb{Z}[P_F^1 \setminus 0, 1, \infty] / R_2$, where R_2 is generated by elements $R_2(x_0, \dots, x_4)$ —see (3.1). Then there is the well-known Bloch complex $B_2(F) \xrightarrow{\delta} \Lambda^2 F^*$, where $\delta[x] = (1-x) \wedge x$. (It is not hard to prove that $\delta(R_2) = 0$.) Thanks to Matsumoto, we know that $\text{Coker } \delta = K_2(F)([M])$. Using some ideas of S. Bloch [B1], A. Suslin proved that $K_3^{\text{ind}}(F) := \text{Coker}(K_3^M(F) \rightarrow K_3(F))$ coincides with $\ker \delta$ modulo torsion [Su].

Note also that $K_1(F) = F^*$ has an interpretation in the same spirit: $F^* = \mathbb{Z}[P_F^1 \setminus 0, 1, \infty] / R_1$, where R_1 is generated by expressions $[x] + [y] - [xy]$, reminiscent of the functional equation for $\ln|\cdot|$.

Let us define a complex $Q(3)_{\mathcal{A}}$ as follows:

$$(4.1) \quad \mathbb{Q}[P_F^1 \setminus 0, 1, \infty] / R_3 \xrightarrow{\delta_1} (B_2(F) \otimes F^*)_{\mathbb{Q}} \xrightarrow{\delta_2} (\Lambda^3 F^*)_{\mathbb{Q}}$$

(the left group placed in degree 1), where $\delta_2[x] \otimes y = (1-x) \wedge x \wedge y$, $\delta_1\{x\} = [x] \otimes x$, and the subgroup R_3 is generated by $\{x\} - \{x^{-1}\}$, $(\{x\} + \{1-x\} + \{1-x^{-1}\}) - (\{y\} + \{1-y\} + \{1-y^{-1}\})$ and $R_3(x_i, y_i, z)$ (see Equation 3.2).

Theorem 2'. $\delta_1(R_3) = 0$ in $B_2(F) \otimes F^*$.

Hence the complex $Q(3)_{\mathcal{A}}$ is well defined. Recall, that $K_n(F) := \pi_n(BGL(F)^+)$, where $BGL(F)^+$ is an H -space. Hence, by the Milnor-Moore theorem [MM] $K_n(F) \otimes \mathbb{Q} = \text{Prim } H_n(GL(F), \mathbb{Q})$.

A. Suslin proved [Su2] that $H_n(GL_n(F), \mathbf{Z}) = H_n(GL(F), \mathbf{Z})$. Therefore $K_n(F) \otimes \mathbf{Q} = \text{Prim } H_n(GL_n(F), \mathbf{Q})$. So $\text{Im}(H_n(GL_{n-i}) \rightarrow H_n(GL_n))$ gives a canonical filtration $K_n(F)_{\mathbf{Q}} \supset K_n^{(1)}(F)_{\mathbf{Q}} \supset \dots$. Set $K_n^{(m)}(F)_{\mathbf{Q}} := K_n^{(m)}(F)_{\mathbf{Q}} / K_n^{(m+1)}(F)_{\mathbf{Q}}$.

Theorem 3. *There are canonical maps*

$$c_1: K_3^{(2)}(F)_{\mathbf{Q}} \rightarrow H^1(\mathbf{Q}(3)_{\mathcal{A}})$$

$$c_1: K_4^{(1)}(F)_{\mathbf{Q}} \rightarrow H^2(\mathbf{Q}(3)_{\mathcal{A}}).$$

Conjecture 2. c_1 and c_2 are isomorphisms.

Note, that according to [Su2]

$$K_3^{(0)}(F)_{\mathbf{Q}} \cong H^3(\mathbf{Q}(3)_{\mathcal{A}}) \cong K_3^M(F)_{\mathbf{Q}}.$$

(A. A. Beilinson and S. Lichtenbaum conjectured that there should exist complexes $\mathbf{Q}(j)_{\mathcal{A}}$ computing all $K_n(F)$ —see [Be2, Li].)

5. THE GROUP $B_3(F)$

For a G -space X , points of $G \backslash X \times \dots \times X$ are called configurations. Let $\mathbf{Z}[C_6(P_F^2)]$ be the free abelian group generated by all possible configurations (l_0, \dots, l_5) of 6 points in P_F^2 .

Let us define a homomorphism $L_3: \mathbf{Z}[P_F^1 \setminus \{0, 1, \infty\}] \rightarrow \mathbf{Z}[C_6(P_F^2)]$ as follows: $L_3(x) = (x_1, x_2, x_3, y_1, y_2, y_3)$, where $r(y_i | x_1, x_2, y_3) = x$ (this configuration was described in §3). The (unique) configuration where y_1, y_2, y_3 are on a line will be denoted η_3 .

Definition. $B_3(F)$ is the quotient of the group $\mathbf{Z}[C_6(P_F^2)]$ by the following relations

- (R1) $(l_0, \dots, l_5) = 0$, if two of the points l_i coincide or four lie on a line.
- (R2) (The seven-term relation.) For any seven points (l_0, \dots, l_6) in P_F^2

$$\sum_{i=0}^6 (-1)^i (l_0, \dots, \widehat{l}_i, \dots, l_6) = 0.$$

(R3) Let (m_0, \dots, m_5) be a configuration of six points in P_F^2 , such that $m_2 = \overline{m_0 m_1} \cap \overline{m_3 m_4}$ and m_5 is in generic position—see Fig. 2. Then if $L'_3\{x\} := -L_3\{x\} - 2L_3\{1-x\}$, (m_0, \dots, m_5)

$$= \frac{1}{3} \sum_{i=0}^4 (-1)^i L'_3\{r(m_5 | m_0, \dots, \hat{m}_i, \dots, m_4)\} + \frac{1}{3} \eta_3.$$

Lemma. *In the group $B_3(F)$ we have*

$$(l_0, \dots, l_5) = (-1)^{|\sigma|} (l_{\sigma(0)}, \dots, l_{\sigma(5)}).$$

Remark. The configurations from (R1) are just the unstable ones in the sense of D. Mumford.

Theorem 4. *The homomorphism $L_3: \mathbf{Z}[P_F^1 \setminus \{0, 1, \infty\}] \rightarrow \mathbf{Z}[C_6(P_F^2)]$ induces an isomorphism modulo 6-torsion.*

$$L_3: \mathbf{Z}[P_F^1 \setminus \{0, 1, \infty\}] / R_3 \cong B_3(F) \otimes \mathbf{Z}.$$

(It is easy to check using (R2) and (R3) that L_3 is onto; the 7-term relation for a configuration $(x_1, x_2, x_3, y_1, y_2, y_3, z)$ then coincides with $(L_3(R_3(x_i, y_i, z)))$.)

Let us denote by M_3 the inverse homomorphism. Then the composition $L_3 \circ M_3: B_3(\mathbf{C}) \rightarrow \mathbf{Q}[P_{\mathbf{C}}^1 \setminus \{0, 1, \infty\}] \rightarrow \mathbf{R}$ defines a measurable function on configurations of six points in CP^2 , satisfying functional relations (R1) through (R3). So for $x \in P_{\mathbf{C}}^2$, $(L_3 \circ M_3)(x, g_1 x, \dots, g_5 x)$ is a measurable cocycle. Let us prove that its cohomology class lies in $\text{Im}(H_{\text{cis}}^5(GL_3(\mathbf{C}), \mathbf{R}) \rightarrow H^5(GL_3(\mathbf{C}), \mathbf{R}))$, where $H_{\text{cis}}^*(G, \mathbf{R})$ is continuous cohomology.

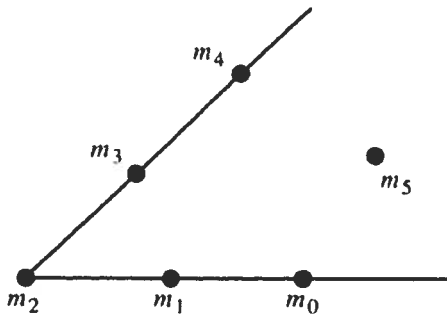


FIGURE 2

Consider the complex

$$\text{Meas } C_{2n-1}(CP^{n-1}) \xrightarrow{d_{2n-1}^*} \text{Meas } C_{2n}(CP^{n-1}) \xrightarrow{d_{2n}^*} \text{Meas } C_{2n+1}(CP^{n-1})$$

where $C_m(CP^n)$ is the space of all configurations of m points in CP^n , $\text{Meas}(X)$ is the space of all measurable functions on the space X , $d_m^*: (l_0, \dots, l_m) \mapsto \sum_{i=0}^m (-1)^i (l_0, \dots, \widehat{l}_i, \dots, l_m)$ and d_m^* is the induced map.

Theorem 5. *Ker $d_{2n}^* / \text{Im } d_{2n-1}^*$ is canonically isomorphic to the indecomposable part of $H_{\text{cts}}^{2n-1}(GL_n(\mathbb{C}), R)$.*

For $n = 2$ this was proved in [B1]. See also closely related work [HM].

Conjecture 3. There exists a canonical element in $\text{Ker } d_{2n}^*$ that can be expressed by classical n -logarithm $\mathcal{L}_n(z)$ and represents the Borel class in $H_{\text{cts}}^{2n-1}(GL_n(\mathbb{C}), R)$.

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